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On the Global of the Difference Equation $x_{n+1} = \frac{\alpha x_{n-m}+\eta x_{n-k}+\delta x_n}{\beta+\gamma x_{n-k}x_{n-l}(x_{n-k}+x_{n-l})}$

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Abstract
In this article, we consider and discuss some properties of the positive solutions to the following rational nonlinear DE $x_{n+1} = \frac{\alpha x_{n-m}+\eta x_{n-k}+\delta x_n}{\beta+\gamma x_{n-k}x_{n-l}(x_{n-k}+x_{n-l})}$, $n = 0,1,...,$ where the parameters $\alpha,\beta,\gamma,\delta,\eta \in (0,\infty)$, while m,k,l are positive integers, such that $m<k<l$ and the initial conditions $x_{-m},...,x_{-k},...,x_{-l},...,$
$x_{-1},...,x_0$ are arbitrary positive real numbers, we will give also, some numerical examples to illustrate our results.

Keywords: Difference equations, Equilibrium,Oscillates, Globally asymptotically stable, Prime period two solution, Rational difference equations, Qualitative properties of solutions of difference equations
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1. Introduction

The study of the solution of nonlinear rational sequence of high order is quite challenging and rewarding. Every dynamical system $b_{n+1} = f(b_n)$ determines DE and vice versa. An interesting class of nonlinear DE is the class of solvable DEs, and one of the interesting problems is to find equations that belong to this class and to solve them in closed form or in explicit form [1]-[14], [16]-[26]. Note that most of these Eq. often show increasingly complex behavior such as the existence of a bounded. The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. The applications of these difference equations can be found on the economy, biology and so on. It is known that nonlinear difference equations are capable of producing a complicated behavior regardless its order. The aim of this paper is to investigate some qualitative behavior of the solutions of the nonlinear DE

$$x_{n+1} = \frac{\alpha x_{n-m}+\eta x_{n-k}+\delta x_n}{\beta+\gamma x_{n-k}x_{n-l}(x_{n-k}+x_{n-l})}, \qquad n = 0,1,2,... \tag{1.1}$$

where the parameters $\alpha,\ \beta,\ \gamma,\ \delta,\ \eta \in (0,\infty)$, while $m,\ k,\ l,$ are positive integers, such that $m<k<l$ and the initial conditions $x_{-m},...,x_{-k},...,x_{-l},...,x_{-1},...,x_0$ are arbitrary positive real numbers. Equation (1.1) has been discussed in [15], when $m=1,\ k=2$ and $l=4$, and in [28], when $\delta=0$, where some global behavior of the more general nonlinear rational Eq. (1.1), we need the following well-known definitions and results [29]-[34].

Definition 1.1. *A difference equation of order $(k+1)$ is of the form*

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{-k}), \quad n = 0,1,2,..... \tag{1.2}$$

where F is a continuous function which maps some set J^{k+1} into J and J is a set of real numbers. An equilibrium point $\widetilde{x}$ of this equation is a point that satisfies the condition $\widetilde{x} = F(\widetilde{x},\widetilde{x},....,\widetilde{x})$. That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \widetilde{x}$ for all $n \geq -k$ is a solution of that equation.

Definition 1.2. *Let $\widetilde{x} \in (0,\infty)$ be an equilibrium point of the difference equation (1.2). Then*
(i) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if x_{-k}, ..., x_{-1},$x_0 \in (0,\infty)$ with $|x_{-k}-\widetilde{x}|+...+|x_{-1}-\widetilde{x}|+|x_0-\widetilde{x}| < \delta$, then $|x_n-\widetilde{x}| < \varepsilon$ for all $n \geq -k$.

(ii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if x_{-k}, ..., x_{-1}, $x_0 \in (0,\infty)$ with $|x_{-k}-\widetilde{x}|+...+|x_{-1}-\widetilde{x}|+|x_0-\widetilde{x}| < \gamma$, then

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iii) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called a global attractor if for every x_{-k}, ..., x_{-1}, $x_0 \in (0,\infty)$ we have

$$\lim_{n\to\infty} x_n = \widetilde{x}.$$

(iv) An equilibrium point $\widetilde{x}$ of the equation (1.2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{x}$ of the difference equation (1.2) is called unstable if it is not locally stable.

Definition 1.3. *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.*

Definition 1.4. *We say that a sequence $\{x_n\}_{n=-l}^{\infty}$ is bounded and persisting if , there exists positive constants m and M such that*

$$m \leq x_n \leq M, \quad for\ all \quad n \geq -k.$$

Definition 1.5. *A positive semicycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $x_l, x_{l+1}, ..., x_m$ all greater than or equal to $\tilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that*

$$either \quad l = -k \quad or \quad l > -k \quad and \quad x_{l-1} < \tilde{x},$$

and

$$either \quad m = \infty \quad or \quad m < \infty \quad and \quad x_{m+1} < \tilde{x}.$$

A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ consists of "a string" of terms $x_l, x_{l+1}, ..., x_m$ all less than $\tilde{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$either \quad l = -k \quad or \quad l > -k \quad and \quad x_{l-1} \geq \tilde{x},$$

and

$$either \quad m = \infty \quad or \quad m < \infty \quad and \quad x_{m+1} \geq \tilde{x}.$$

Definition 1.6. *The linearized Eq. of Eq. (1.2) about the equilibrium point $\widetilde{x}$ is the linear Eq.*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\widetilde{x},\widetilde{x},...,\widetilde{x})}{\partial x_{n-i}} y_{n-i}. \tag{1.3}$$

Now, assume that the characteristic Eq. associated with Eq. (1.3) is

$$p(\lambda) = p_0\lambda^k + p_1\lambda^{k-1} + ... + p_{k-1}\lambda + p_k = 0, \tag{1.4}$$

where

$$p_i = \partial F(\widetilde{x},\widetilde{x},...,\widetilde{x}) / \partial x_{n-i}.$$

Theorem 1.7. *Let $p_i \in R,\ i = 1,2,...,$ and $k \in \{0,1,2,...\}$, then*

$$\sum_{i=1}^{k} |p_i| < 1,$$

is sufficient condition for asymptotic stability of difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0,1,2,...$$

Theorem 1.8 (The Linearized Stability Theorem)**.**
Suppose that F is a continuously differentiable function defined on an open neighbourhood of the equilibrium $\widetilde{x}$. Then the following statements are true.
(i) If all roots of the characteristic equation (1.4) of the linearized equation (1.3) have an absolute value less than one, then the equilibrium point $\widetilde{x}$ is locally asymptotically stable.
(ii) If at least one root of Eq.(1.4) has an absolute value greater than one, then the equilibrium point $\widetilde{x}$ is unstable.

2. Change of Variables

By using the change of variables $x_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{3}} y_n$, the equation (1.1) reduces to the following difference equation

$$y_{n+1} = \frac{r y_{n-m} + t y_{n-k} + s y_n}{1 + y_{n-k} y_{n-l} (y_{n-k} + y_{n-l})}, \qquad n = 0,1,2,... \tag{2.1}$$

where $r = \frac{\alpha}{\beta} > 0$, $s = \frac{\delta}{\beta} > 0$, $t = \frac{\eta}{\beta} > 0$, and the initial conditions $y_{-l},...,y_{-k},...,y_{-m},...,y_{-l},y_0 \in (0,\infty)$. In the next section, we shall study the global behavior of Eq. (2.1).

3. The Dynamics of Eq. (2.1)

The equilibrium points $\tilde{y}$ of Eq. (2.1) are the positive solutions of equation

$$\tilde{y} = \frac{[r+s+t]\tilde{y}}{1+2\tilde{y}^3}. \tag{3.1}$$

Thus $\tilde{y}_1 = 0$, is always an equilibrium point of the equation (2.5). If $(r+s+t) > 1$, then the only positive equilibrium point $\tilde{y}_2$ of equation (2.1) is given by

$$\tilde{y}_2 = \left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}. \tag{3.2}$$

Let us introduce a continuous function $F : (0,\infty)^4 \to (0,\infty)$, which is defined by

$$F(v_0,v_1,v_2,v_3) = \frac{r v_0 + s v_1 + t v_2}{1 + v_2^2 v_3 + v_2 v_3^2}. \tag{3.3}$$

Consequently, we get

$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_0} = \frac{r}{1 + v_2^2 v_3 + v_2 v_3^2},$$
$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_1} = \frac{s}{1 + v_2^2 v_3 + v_2 v_3^2},$$

$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_2} = \frac{t(1 + v_2^2 v_3 + v_2 v_3^2) - (r v_0 + s v_1 + t v_2)(2 v_2 v_3 + v_3^2)}{(1 + v_2^2 v_3 + v_2 v_3^2)^2},$$

$$\frac{\partial F(v_0,v_1,v_2,v_3)}{\partial v_3}=\frac{-(rv_0+sv_1+tv_2)(v_2^2+2v_2v_3)}{(1+v_2^2v_3+v_2v_3^2)^2}.$$

At $\tilde{y}_1=0$, we have $\frac{\partial F(0,0,0,0)}{\partial v_0}=r,\ \frac{\partial F(0,0,0,0)}{\partial v_1}=s,\ \frac{\partial F(0,0,0,0)}{\partial v_2}=t,\ \frac{\partial F(0,0,0,0)}{\partial v_3}=0$, and the linearized equation of Eq. (2.1) about $\tilde{y}_1=0$, is the equation

$$z_{n+1}-\rho_0 z_n-\rho_1 z_{n-m}-\rho_2 z_{n-k}=0, \tag{3.4}$$

where $\rho_0=s,\ \ \rho_1=r,\ \ \rho_2=t$. At $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$, we have

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_0}=\frac{r}{1+2\tilde{y}_2^3}=\frac{r}{1+([r+s+t]-1)}=\frac{r}{[r+s+t]},$$

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_1}=\frac{s}{1+2\tilde{y}_2^3}=\frac{s}{1+([r+s+t]-1)}=\frac{s}{[r+s+t]},$$

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_2}=\frac{2t-3([r+s+t]-1)}{2[r+s+t]},$$

$$\frac{\partial F(\tilde{y}_2,\tilde{y}_2,\tilde{y}_2,\tilde{y}_2)}{\partial v_3}=\frac{-3([r+s+t]-1)}{2[r+s+t]}.$$

And the linearized equation of Eq. (2.1) about $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ is the equation

$$z_{n+1}-\rho_0 z_n-\rho_1 z_{n-m}-\rho_2 z_{n-k}-\rho_3 z_{n-l}=0, \tag{3.5}$$

where $\rho_0=\frac{s}{[r+s+t]},\ \rho_1=\frac{r}{[r+s+t]},\ \rho_2=\frac{2t-3([r+s+t]-1)}{2[r+s+t]},\ \rho_3=\frac{-3([r+s+t]-1)}{2[r+s+t]}$.

Theorem 3.1. *(i) If $[r+s+t]<1$, then the equilibrium point $\tilde{y}_1=0$ is locally asymptotically stable.*
(ii) If $[r+s+t]>1$, then the equilibrium point $\tilde{y}_1=0$ is unstable.
(iii) If $[r+s+t]>1,\ 2t>3([r+s+t]-1)$, then the equilibrium point $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ is unstable.

Proof. With reference to Theorem 1.1, we deduce from Eq. (3.4) that $|\rho_0|+|\rho_1|+|\rho_2|=[r+s+t]<1$, and then the proof of parts (i), (ii) follow. Also, from Eq. (3.5) we deduce for $[r+s+t]>1$ that $|\rho_0|+|\rho_1|+|\rho_2|+|\rho_3|=1+\frac{3([r+s+t]-1)}{[r+s+t]}>1$, and hence the proof of part (iii) follows. □

Theorem 3.2. *Assume that $[r+s+t]>1$, and let $\{y_n\}_{n=-l}^{\infty}$ be a solution of Eq. (2.1) such that*

$$\begin{aligned}&y_{-l},y_{-l+2},\ldots,y_{-l+2n},\ldots,y_{-k},y_{-k+2},\ldots,y_{-k+2n},\ldots,\\&y_{-m+1},y_{-m+3},\ldots,y_{-m+2n+1},\ldots,y_0\geq\tilde{y}_2\\&\textit{and}\\&y_{-l+1},y_{-l+3},\ldots,y_{-l+2n+1},\ldots,y_{-k+1},y_{-k+3},\ldots,\\&y_{-k+2n+1},\ldots,y_{-m},y_{-m+2},\ldots,y_{-m+2n},\ldots,y_{-1}<\tilde{y}_2.\end{aligned} \tag{3.6}$$

Then $\{y_n\}_{n=-l}^{\infty}$ oscillates about $\tilde{y}_2=\left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ with a semicycle of length one.

Proof. Assume that (3.6) holds. Then

$$y_1=\frac{ry_{-m}+sy_0+ty_{-k}}{1+y_{-k}y_{-l}(y_{-k}+y_{-l})}<\frac{ry_{-m}+sy_0+ty_{-k}}{1+2\tilde{y}_2^3}<\frac{[r+s+t]\,\tilde{y}_2}{1+([r+s+t]-1)}=\tilde{y}_2,$$

and

$$y_2=\frac{ry_{-m+1}+sy_1+ty_{-k+1}}{1+y_{-k+1}y_{-l+1}(y_{-k+1}+y_{-l+1})}\geq\frac{ry_{-m+1}+sy_1+ty_{-k+1}}{1+2\tilde{y}_2^3}\geq\frac{[r+s+t]\,\tilde{y}_2}{1+([r+s+t]-1)}=\tilde{y}_2,$$

and hence the proof follows by induction.

Theorem 3.3. *Assume that $[r+s+t]<1$, then the equilibrium point $\tilde{y}_1=0$ of Eq. (2.1) is globally asymptotically stable.*

Proof. We have shown in Theorem 3 that if $[r+s+t]<1$ then the equilibrium point $\tilde{y}_1=0$ is locally asymptotically stable. It remains to show that $\tilde{y}_1=0$ is a global attractor. To this end, let $\{y_n\}_{n=-l}^{\infty}$ be a solution of Eq. (2.1). It suffics to show that $\lim_{n\to\infty} y_n=0$. Since

$$0\le y_{n+1}=\frac{ry_{n-m}+sy_n+ty_{n-k}}{1+y_{n-k}y_{n-l}(y_{n-k}+y_{n-l})}\le ry_{n-m}+sy_n+ty_{n-k}<y_{n-k}.$$

Then we have $\lim_{n\to\infty} y_n=0$. This completes the proof. □

Theorem 3.4. *Assume that $[r+s+t]>1$, then Eq. (2.1) possesses an unbounded solution.*

Proof. With the aid of Theorem 3.3, we have

$$\begin{aligned} y_{2n+2} &= \frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{1+y_{-k+2n+1}\,y_{-l+2n+1}(y_{-k+2n+1}+y_{-l+2n+1})}>\frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{1+2\tilde{y}_2^3}\\ &> \frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{1+([r+s+t]-1)}=\frac{ry_{-m+2n+1}+sy_{2n+1}+ty_{-k+2n+1}}{[r+s+t]}, \end{aligned}$$

and

$$\begin{aligned} y_{2n+3} &= \frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{1+y_{-k+2n+2}\,y_{-l+2n+2}(y_{-k+2n+2}+y_{-l+2n+2})}\le\frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{1+2\tilde{y}_2^3}\\ &\le \frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{1+([r+s+t]-1)}=\frac{ry_{-m+2n+2}+sy_{2n+2}+ty_{-k+2n+2}}{[r+s+t]}. \end{aligned}$$

From which it follows that

$$\lim_{n\to\infty} y_{2n}=\infty \quad \textit{and} \quad \lim_{n\to\infty} y_{2n+1}=0.$$

Hence, the proof of Theorem 3.4 is now completed. □

Theorem 3.5. *(1) If m is odd, and k,l are even, Eq. (2.1) has prime period two solution if $(r-[s+t])<1$ and has not prime period two solution if $(r-[s+t])\ge 1$.*
(2) If m is even and k,l are odd, Eq. (2.1) has not prime period two solution.
(3) If all m,k,l are even, Eq. (2.1) has prime period two solution.
(4) If all m,k,l are odd, Eq. (2.1) has prime period two solution if $(r-[s+t])>1$, and has not prime period two solution if $(r-[s+t])\le 1$.
(5) If m,k are even and l is odd, Eq. (2.1) has not prime period two solution.
(6) If m,k are odd and l is even, Eq. (2.1) has prime period two solution if $(r-[s+t])>1$, and has not prime period two solution if $(r-[s+t])\le 1$.
(7) If m,l are odd and k is even, Eq. (2.1) has prime period two solution if $(r-[s+t])>1$, and has not prime period two solution if $(r-[s+t])\le 1$.
(8) If m,l are even and k is odd, Eq. (2.1) has not prime period two solution.

Proof. Assume that there exists distinct positive solutions

$$\ldots,\phi,\psi,\phi,\psi,\ldots$$

of prime period two of Eq. (2.1).
(1) If m is odd, and k,l are even, then $y_{n+1}=y_{n-m}$ and $y_n=y_{n-k}=y_{n-l}$. It follows from Eq. (2.1) that

$$\phi=\frac{r\phi+[s+t]\,\psi}{1+2\psi^3},\qquad \psi=\frac{r\psi+[s+t]\,\phi}{1+2\phi^3}.$$

Consequently, we have

$$0<2\phi\psi(\phi+\psi)=1-(r-[s+t]). \tag{3.7}$$

We deduce that (3.7) is always true if $(r-[s+t]) < 1$ and hence Eq. (2.1) has prime period two solution. If $(r-[s+t]) \geq 1$, we have a contradiction, and hence Eq. (2.1) has not prime period two solution.
(2) If m is even, and k,l are odd, then $y_n = y_{n-m}$,and $y_{n+1} = y_{n-k} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s+t]\,\psi}{1+2\phi^3}, \quad \psi = \frac{[r+s+t]\,\phi}{1+2\psi^3}.$$

Consequently, we have

$$0 < 2\,(\phi+\psi)\,(\phi^2+\psi^2) = -\left([r+s+t]+1\right). \tag{3.8}$$

Since $[r+s+t] > 0$, we have a contradiction. Hence Eq. (2.1) has not prime period two solution.
(3) If all m,k,l are even, then $y_n = y_{n-m} = y_{n-k} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s+t]\,\psi}{1+2\psi^3}, \quad \psi = \frac{[r+s+t]\,\phi}{1+2\phi^3}.$$

Consequently, we get

$$0 < 2\phi\psi(\phi+\psi) = [r+s+t]+1. \tag{3.9}$$

Since $[r+s+t] > 0$, the formula (3.14) is always true. Hence Eq. (2.1) has prime period two solution.
(4) If all m,k,l are odd, then $y_{n+1} = y_{n-m} = y_{n-k} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{r\phi+s\psi}{1+2\phi^3}, \quad \psi = \frac{r\psi+s\phi}{1+2\psi^3}.$$

Consequently, we get

$$0 < 2(\phi+\psi)(\phi^2+\psi^2) = (r-[s+t])-1. \tag{3.10}$$

If $(r-[s+t]) > 1$, the formula (15) is always true, and hence Eq. (2.1) has prime period two solution. If $(r-[s+t]) \leq 1$, we have a contradiction and hence Eq. (2.1) has not prime period two solution.
(5) If m,k are even, and l is odd, then $y_n = y_{n-k} = y_{n-m}$, and $y_{n+1} = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s+t]\,\psi}{1+\psi^2\phi+\psi\phi^2}, \quad \psi = \frac{[r+s+t]\,\phi}{1+\phi^2\psi+\phi\psi^2}.$$

Consequently, we have

$$0 < \phi\psi(\phi+\psi) = -([r+s+t]+1). \tag{3.11}$$

Since $[r+s+t] > 0$, we have a contradiction. Hence Eq. (2.1) has not a prime period two solution.
(6) If m,k are odd, and l is even, then $y_{n+1} = y_{n-m} = y_{n-k}$, and $y_n = y_{n-l}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+t]\,\phi+s\psi}{1+\phi^2\psi+\phi\psi^2}, \quad \psi = \frac{[r+t]\,\psi+s\phi}{1+\psi^2\phi+\psi\phi^2}.$$

Consequently, we have

$$0 < \phi\psi(\phi+\psi) = ([r+t]-s)-1. \tag{3.12}$$

If $([r+t]-s) > 1$,the formula (3.17) is always true, and hence Eq. (2.1) has prime period two solution. If $([r+t]-s) \leq 1$, we have a contradiction. Hence Eq.(2.5) has not a prime period two solution.
(7) If m,l are odd, and k is even, then $y_{n+1} = y_{n-m} = y_{n-l}$, and $y_n = y_{n-k}$. It follows from Eq. (2.1) that

$$\phi = \frac{r\phi+[s+t]\,\psi}{1+\psi^2\phi+\psi\phi^2}, \quad \psi = \frac{r\psi+[s+t]\,\phi}{1+\phi^2\psi+\phi\psi^2},$$

which give the same results of case (6).
(8) If m,l are even, and k is odd, then $y_n = y_{n-m} = y_{n-l}$, and $y_{n+1} = y_{n-k}$. It follows from Eq. (2.1) that

$$\phi = \frac{[r+s]\,\psi+t\phi}{1+\psi^2\phi+\psi\phi^2}, \quad \psi = \frac{[r+s]\,\phi+t\psi}{1+\phi^2\psi+\phi\psi^2},$$

which give the same results of case (5). Hence the proof of Theorem 3.5 is now completed. □

4. Numerical Examples

In order to illustrate the results of the previous section and to support our theoretical discussions, we consider some numerical examples in this section. These examples represent different types of qualitative behavior of solutions of Eq. (2.1).

Example 4.1. *Figure 4.1, shows that the solution of Eq. (2.1) is bounded if* $x_{-3} = 1,\ x_{-2} = 2,\ x_{-1} = 3,\ x_0 = 4,\ m = 1,\ k = 2,\ l = 3,\ r = 0.1,\ s = 0.2,\ t = 0.3,\ i.e\ [r+s+t] < 1.$

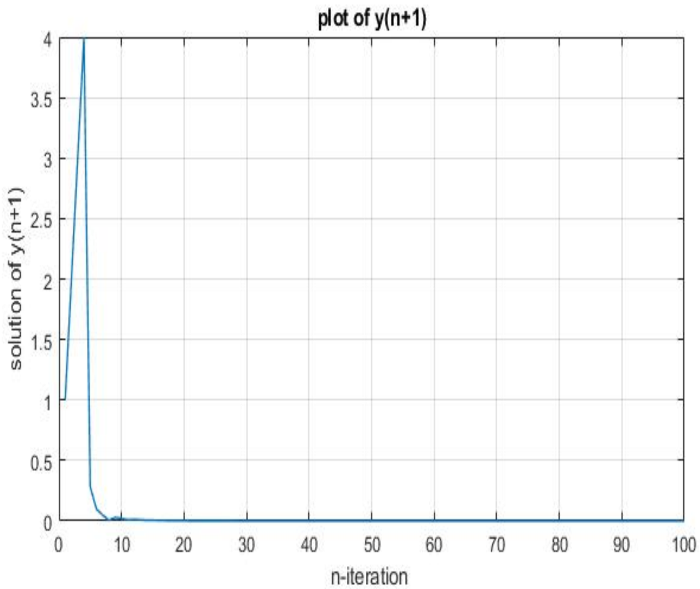

Figure 4.1. The solution of Eq. (2.1) is bounded.

Example 4.2. *Figure 4.2, shows that the solution of Eq. (2.1) is unbounded if* $x_{-3} = 1,\ x_{-2} = 2,\ x_{-1} = 3,\ x_0 = 4,\ m = 1,\ k = 2,\ l = 3,\ r = 1,\ s = 2,\ t = 3,\ i.e\ [r+s+t] > 1.$

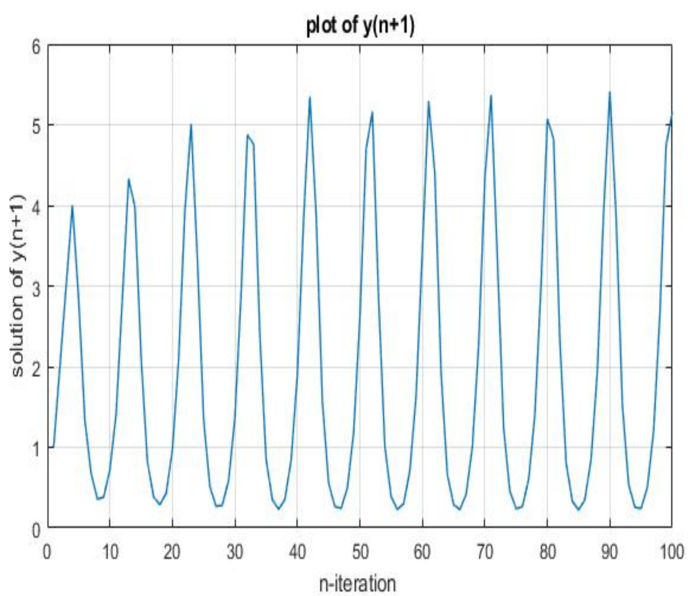

Figure 4.2. The solution of Eq. (2.1) is unbounded.

Example 4.3. *Figure 4.3, shows that Eq. (2.1) is globally asymptotically stable if* $x_{-4} = 1,\ x_{-3} = 2,\ x_{-2} = 3,\ x_{-1} = 4,\ x_0 = 5,\ m = 2,\ k = 3,\ l = 4,\ r = 0.1,\ s = 0.5,\ t = 0.2,\ i.e\ [r+s+t] < 1.$

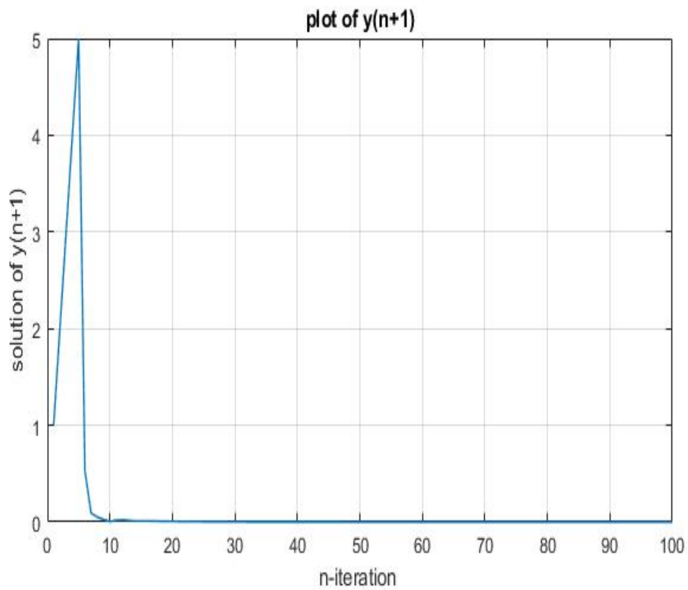

Figure 4.3. The solution of Eq. (2.1) is globally asymptotically stable.

Example 4.4. *Figure 4.4, shows that Eq. (2.1) has no positive prime period two solutions if* $x_{-3} = 1,\ x_{-2} = 2,\ x_{-1} = 3,\ x_0 = 4,\ m = 2,\ k = 1,\ l = 3,\ r = 100,\ s = 300,\ t = 400.$

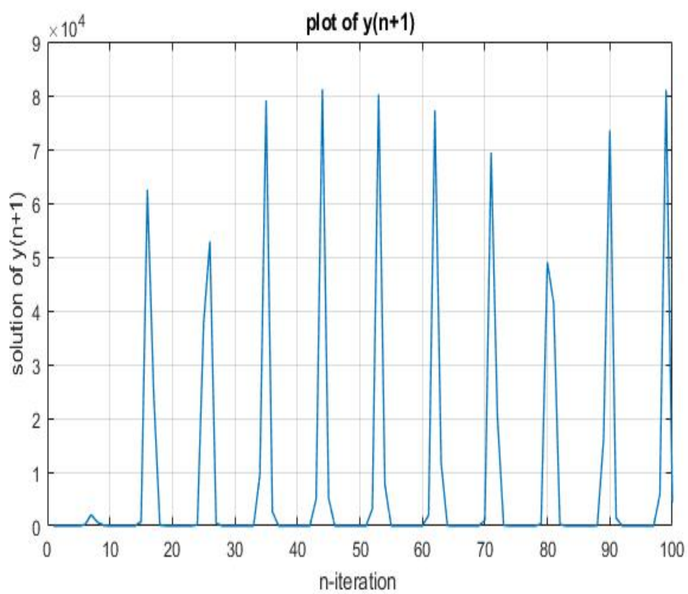

Figure 4.4. The solution of Eq. (2.1) is globally asymptotically stable.

5. Conclusions

In this article, we have shown that Eq. (2.1) has two equilibrium points $\tilde{y}_1 = 0$ and $\tilde{y}_2 = \left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$. If $[r+s+t] < 1$, we have proved that $\tilde{y}_1 = 0$ is globally asymptotically stable, while if $[r+s+t] > 1$, the solution of Eq. (2.1) oscillates about the point $\tilde{y}_2 = \left(\frac{[r+s+t]-1}{2}\right)^{\frac{1}{3}}$ with a semicycle of length one. When $[r+s+t] > 1$, we have proved that the solution of Eq. (2.1) is unbounded. The periodicity of the solution of Eq. (2.1) has been discussed in details in Theorem 3.5.

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