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Categorification of Algebras: 2-Algebras

İbrahim İlker Akça [1], **Ummahan Ege Arslan**[2]

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Abstract — This paper introduces a categorification of k-algebras called 2-algebras, where k is a commutative ring. We define the 2-algebras as a 2-category with single object in which collections of all 1-morphisms and all 2-morphisms are k-algebras. It is shown that the category of 2-algebras is equivalent to the category of crossed modules in commutative k-algebras.

Subject Classification (2020): 18F99, 18G30, 18G55, 13C60.

1. Introduction

The term “categorification” coined by Louis Crane refers to the process of replacing set theoretic concepts by category-theoretic analogues in mathematics. A categorified version of a group is a 2-group. Internal categories in the category of groups are exactly the same as 2-groups. The Brown-Spencer theorem [3] thus constructs the associated 2-group of a crossed module given by Whitehead [11] to define an algebraic model for a “(connected) homotopy 2-type”. The fact that the composition in the internal category must be a group homomorphism implies that the “interchange law” must hold. This equation is in fact equivalent via the Brown-Spencer result to the Peiffer identity.

We will concerne in this paper exclusively with categorification of algebras. We will obtain analogous results in (commutative) algebras with regard to Porter’s work [9]. He states that there is an equivalence of categories between the category of internal categories in the category of k-algebras and the category of crossed modules of commutative k-algebras. Since the internal category in the category of k-algebras is a categorification of k-algebras, this internal category will be called as “strict 2-algebra” in this work. We define the strict 2-algebra by means of 2-module being a category in the category of modules as a 2-category with single object in which collections of 1-morphisms and 2-morphisms are k-algebras and we denote the category of strict 2-algebras by **2Alg** . Given a group G, it is known that automorphisms of G yield a 2-group. Analogous result in commutative algebras can be given that multiplications of C yield a strict 2-algebra where C is a commutative R-algebra and R is a commutative k-algebra.

A crossed module $\mathscr{A} = (\partial : C \longrightarrow R)$ of commutative algebras is given by an algebra morphism $\partial : C \longrightarrow R$

[1]iakca@ogu.edu.tr (Corresponding Author); [2]uege@ogu.edu.tr
[1,2]Department of Mathematics and Computer Sciences, Faculty of Science, Eskişehir Osmangazi University, Eskişehir, Turkey
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together with an action $\cdot$ of R on C such that the relations below hold for each $r \in R$ and each $c, c^{'} \in C$,

$$\begin{aligned} \partial(r \cdot c) &= r\partial(c) \\ \partial(c) \cdot c^{'} &= cc^{'}. \end{aligned}$$

In this paper we show that the category of strict 2-algebras is equivalent to the category of crossed modules in commutative algebras.

2. Internal Categories and 2-categories

We begin by recalling internal categories as well as 2-categories. Ehresmann defined internal categories in [5], and by now they are an important part of category theory [4].

2.1. Internal categories

Definition 2.1. Let **C** be any category. An internal category in **C**, say **A**, consists of:

• an object of objects $A_0 \in \mathbf{C}$

• an object of morphisms $A_1 \in \mathbf{C}$,

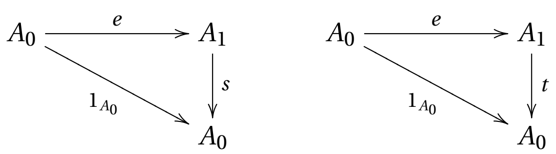
together with

• source and target morphisms $s, t : A_1 \longrightarrow A_0$,

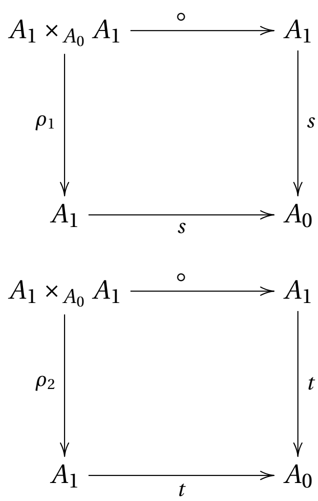
• an identity-assigning morphism $e : A_0 \longrightarrow A_1$,

• a composition morphism $\circ : A_1 \times_{A_0} A_1 \longrightarrow A_1$ such that the following diagrams commute, expressing the usual category laws:

• laws specifying the source and target of identity morphisms:



• laws specifying the source and target of composite morphisms:



• the associative law for composition of morphisms:

$$\begin{CD} A_1 \times_{A_0} A_1 \times_{A_0} A_1 @>{\circ}>> A_1 \times_{A_0} A_1 \\ @V{\rho_2}VV @VV{t}V \\ A_1 \times_{A_0} A_1 @>>{t}> A_0 \end{CD}$$

• the left and right unit laws for composition of morphisms:

$$\begin{array}{ccccc} A_0 \times_{A_0} A_1 & \xrightarrow{e \times_{A_0} 1_{A_1}} & A_1 \times_{A_0} A_1 & \xleftarrow{1_{A_1} \times_{A_0} e} & A_1 \times_{A_0} A_0 \\ & {\scriptstyle \rho_2} \searrow & \downarrow {\scriptstyle \circ} & \swarrow {\scriptstyle \rho_1} & \\ & & A_1 & & \end{array}$$

Here, the pullback $A_1 \times_{A_0} A_1$ is defined via the square:

$$\begin{CD} A_1 \times_{A_0} A_1 @>{\rho_2}>> A_1 \\ @V{\rho_1}VV @VV{s}V \\ A_1 @>>{t}> A_0. \end{CD}$$

We denote this internal category with $A = (A_0, A_1, s, t, e, \circ)$.

Definition 2.2. Let **C** be a category. Given internal categories A and A' in **C**, an **internal functor** between them, say $F : A \longrightarrow A'$, consists of

• a morphism $F_0 : A_0 \longrightarrow A'_0$,

• a morphism $F_1 : A_1 \longrightarrow A'_1$

such that the following diagrams commute, corresponding to the usual laws satisfied by a functor:

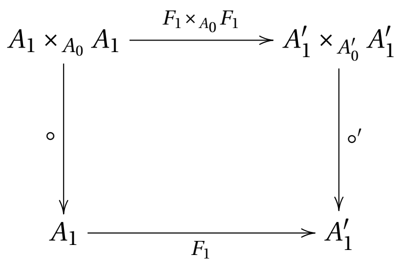
• preservation of source and target:

$$\begin{CD} A_1 @>{s}>> A_0 \\ @V{F_1}VV @VV{F_0}V \\ A'_1 @>>{s'}> A'_0 \end{CD} \qquad \begin{CD} A_1 @>{t}>> A_0 \\ @V{F_1}VV @VV{F_0}V \\ A'_1 @>>{t'}> A'_0 \end{CD}$$

• preservation of identity morphisms:

$$\begin{CD} A_0 @>{e}>> A_1 \\ @V{F_0}VV @VV{F_1}V \\ A'_0 @>>{e'}> A'_1 \end{CD}$$

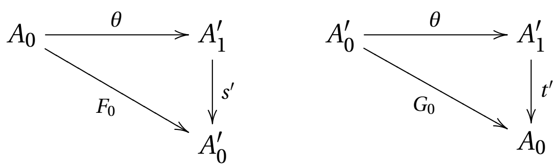
• preservation of composite morphisms:



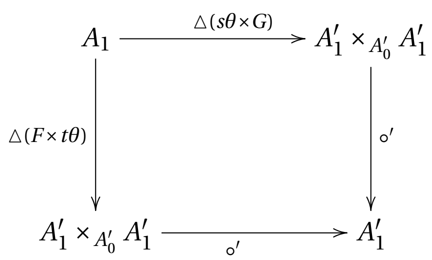
Given two internal functors $F : A \longrightarrow A'$ and $G : A' \longrightarrow A''$ in some category **C**, we define their composite $FG : A \longrightarrow A''$ by taking $(FG)_0 = F_0G_0$ and $(FG)_1 = F_1G_1$. Similarly, we define the identity internal functor in **C**, $1_A : A \longrightarrow A$ by taking $(1_A)_0 = 1_{A_0}$ and $(1_A)_1 = 1_{A_1}$.

Definition 2.3. Let **C** be a category. Given two internal functors $F, G : A \longrightarrow A'$ in **C**, an **internal natural transformation** in **C** between them, say $\theta : F \Longrightarrow G$, is a morphism $\theta : A_0 \longrightarrow A'_1$ for which the following diagrams commute, expressing the usual laws satisfied by a natural transformation:

• laws specifying the source and target of a natural transformation:



• the commutative square law:



Given an internal functor $F : A \longrightarrow A'$ in **C**, the identity internal natural transformation $1_F : F \Longrightarrow F$ in C is given by $1_F = F_0e$.

2.2. 2-categories

Definition 2.4. A 2-category $\mathscr{G}$ consists of a class of objects G_0 and for any pair of objects (A, B) a small category of morphisms $\mathscr{G}(A, B)$-with objects $G_1(A, B)$ and morphisms $G_2(A, B)$-, along with composition functors

$$\bullet \ : \ \mathscr{G}(A,B) \times \mathscr{G}(B,C) \ \longrightarrow \ \mathscr{G}(A,C)$$

for every triple (A, B, C) of objects and identity functors from the terminal category to $\mathscr{G}(A, A)$

$$iA : 1 \longrightarrow \mathscr{G}(A, A)$$

for all objects A such that $\bullet$ is associative and

$$F \bullet i_B = F = i_A \bullet F \quad \text{as well as} \quad \vartheta \bullet I_{i_B} = \vartheta = I_{i_A} \bullet \vartheta$$

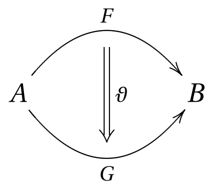
hold for all $F \in G_1(A,B)$ and $\vartheta \in G_2(A,B)$ where source and target morphisms are defined by

$$A \xrightarrow{F} B$$

$$\begin{array}{ccc} s : G_1(A,B) & \longrightarrow & G_0 \\ F & \longmapsto & s(F) = A \end{array}$$

$$\begin{array}{ccc} t : G_1(A,B) & \longrightarrow & G_0 \\ F & \longmapsto & t(F) = B \end{array}$$

for $F \in G_1(A,B)$ and

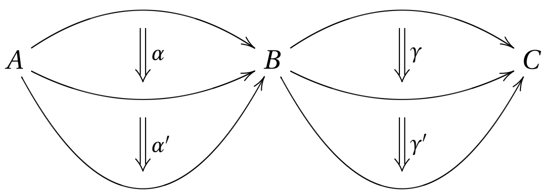


$$\begin{array}{ccc} s : G_2(A,B) & \longrightarrow & G_1 \\ \vartheta & \longmapsto & s(\vartheta) = F \end{array}$$

$$\begin{array}{ccc} t : G_2(A,B) & \longrightarrow & G_0 \\ \vartheta & \longmapsto & t(\vartheta) = G \end{array}$$

for $\vartheta : F \longrightarrow G \in G_2(A,B)$. For all pairs of objects (A,B) elements of $G_1(A,B)$ are called 1-morphisms or 1-cells of $\mathcal{G}$ and elements of $G_2(A,B)$ are called 2-morphisms or 2-cells of $\mathcal{G}$. We write G_1 and G_2 for the classes of all 1-morphisms and 2-morphisms respectively.

There are two ways of composing 2-morphisms: using the composition $\circ$ inside the categories $\mathcal{G}(A,B)$, called vertical composition, and using the morphism level of the functor $\bullet$, called horizontal composition. These compositions must be satisfy the following equation: for $\alpha, \alpha' \in G_2(A,B)$ with $t(\alpha) = s(\alpha')$ and $\gamma, \gamma' \in G_2(B,C)$ with $t(\gamma) = s(\gamma')$



$$(\alpha \circ \alpha') \bullet (\gamma \circ \gamma') = (\alpha \bullet \gamma) \circ (\alpha' \bullet \gamma')$$

which is called “interchange law”.

3. Constructions of Two-Algebras

In this section we will construct 2-algebras by categorification. We can categorify the notion of an algebra by replacing the equational laws by isomorphisms satisfying extra structure and properties we expect. In [2]

Baez and Crans introduce the Lie 2-algebra by means of the concept of 2-vector space defined as an internal category in the category of vector spaces by them. Obviously we get a new notion of “2-module”which can be considered as an internal category in the category of modules and we categorify the notion of an algebra.

3.1. 2-Modules

A categorified module or “2-module”should be a category with structure analogous to that of a $k-$module, with functors replacing the usual $k-$module operations. Here we instead define a 2-module to be an internal category in a category of $k-$modules **Mod** . Since the main component part of a $k-$algebra is a $k-$module, a 2-algebra will have an underlying 2-module of this sort. In this section we thus first define a category of these 2-modules.

In the rest of this paper, the terms a module and an algebra will always refer to a $k-$module and a $k-$algebra.

Definition 3.1. A 2-module is an internal category in **Mod** .

Thus, a 2-module M is a category with a module of objects M_0 and a module of morphisms M_1, such that the source and target maps $s,t: M_1 \longrightarrow M_0$, the identity assigning map $e: M_0 \longrightarrow M_1$, and the composition map $\circ: M_1 \times_{M_0} M_1 \longrightarrow M_1$ are all module morphisms. We write a morphism as $a: x \longrightarrow y$ when $s(a) = x$ and $t(a) = y$, and sometimes we write $e(x)$ as 1_x.

The following proposition is given for the **Vect** vector space category in [2]. But we rewrite this proposition for **Mod** .

Proposition 3.2. It is defined a 2-module by specifying the modules M_0 and M_1 along with the source, target and identity module morphisms and the composition morphism $\circ$, satisfying the conditions of Definition 2.1. The composition map is uniquely determined by

$$\begin{array}{rccl} \circ: & M_1 \times_{M_0} M_1 & \longrightarrow & M_1 \\ & (a,b) & \longmapsto & \circ(a,b) = a \circ b = a + b - (es)(b). \end{array}$$

Proof.

First given modules M_0, M_1 and module morphisms $s,t: M_1 \longrightarrow M_0$ and $e: M_0 \longrightarrow M_1$, we will define a composition operation that satisfies the laws in the definition of internal category, obtaining a 2-module.

Given $a, b \in M_1$ such that $t(a) = s(b)$, i.e.

$$a: x \longrightarrow y \text{ and } b: y \longrightarrow z$$

we define their composite $\circ$ by

$$\begin{array}{rccl} \circ: & M_1 \times_{M_0} M_1 & \longrightarrow & M_1 \\ & (a,b) & \longmapsto & \circ(a,b) = a \circ b = a + b - (es)(b). \end{array}$$

We will show that with this composition $\circ$ the diagrams of the definition of internal category commute. The triangles specifying the source and target of the identity-assigning morphism do not involve composition.

The second pair of diagrams commute since

$$\begin{aligned} s(a\circ b) &= s(a+b-(es)(b)) \\ &= s(a)+s(b)-(se)(s(b)) \\ &= s(a)+s(b)-s(b) \\ &= s(a)=x \end{aligned}$$

and since $t(a)=s(b)$,

$$\begin{aligned} t(a\circ b) &= t(a+b-(es)(b)) \\ &= t(a)+t(b)-(te)(s(b)) \\ &= t(a)+t(b)-s(b) \\ &= t(b)=z. \end{aligned}$$

The associative law holds for composition because module addition is associative. Finally the left and right unit laws are satisfied since given $a: x \longrightarrow y$,

$$\begin{aligned} e(x)\circ a &= e(x)+a-(es)(a) \\ &= e(x)+a-e(x) \\ &= a \end{aligned}$$

and

$$\begin{aligned} a\circ e(y) &= a+e(y)-(es)(e(y)) \\ &= a+e(y)-e(y) \\ &= a. \end{aligned}$$

We thus have a 2-module.

Given a 2-module M, we shall show that its composition must be defined by the formula given above. Suppose that (a,g) and (a',g') are composable pairs of morphisms in M_1, i.e.

$$a: x \longrightarrow y \text{ and } b: y \longrightarrow z$$

and

$$a': x' \longrightarrow y' \text{ and } b': y' \longrightarrow z'.$$

Since the source and target maps are module morphisms, $(a+a', b+b')$ also forms a composable pair, and since that the composition is module morphism

$$(a+a')\circ(b+b') = a\circ b + a'\circ b'.$$

Then if (a,b) is a composable pair, i.e, $t(a)=s(b)$, we have

$$\begin{aligned}
a\circ b &= (a+1_{M_1})\circ(1_{M_1}+b)\\
&= (a+e(s(b)-s(b)))\circ(e(s(b)-s(b))+b)\\
&= (a-e(s(b))+e(s(b)))\circ(e(s(b))-e(s(b))+b)\\
&= (a\circ e(s(b)))+(-e(s(b))+e(s(b)))\circ(-e(s(b))+b)\\
&= a\circ e(s(b))+(-e(s(b))\circ(-e(s(b))))+(e(s(b))\circ b)\\
&= a-e(s(b))+b\\
&= a+b-e(s(b)).
\end{aligned}$$

This show that we can define $\circ$ by

$$\begin{aligned}
\circ:\ M_1\times_{M_0}M_1 &\longrightarrow M_1\\
(a,b) &\longmapsto \circ(a,b)=a\circ b=a+b-e(s(b)).
\end{aligned}$$

Corollary 3.3. For $b\in\ker s$, we have

$$\begin{aligned}
a\circ b &= a+b-(es)(b)\\
&= a+b.
\end{aligned}$$

Definition 3.4. Let M and N be 2-modules, a 2-module functor $F: M\longrightarrow N$ is an internal functor in **Mod** from M to N. 2-modules and 2-module functors between them is called the category of 2-modules denoted by **2Mod**.

After we get the definition of a 2-module, we define the definition of a categorified algebra which is main concept of this paper.

3.2. Two-algebras

Definition 3.5. A weak 2-algebra consists of

· a 2-module A equipped with a functor $\bullet: A\times A\longrightarrow A$, which is defined by $(x,y)\mapsto x\bullet y$ and bilinear on objects and defined by $(f,g)\mapsto f\bullet g$ on morphisms satisfying interchange law, i.e.,

$$(f_1\bullet g_1)\circ(f_2\bullet g_2)=(f_1\circ f_2)\bullet(g_1\circ g_2)$$

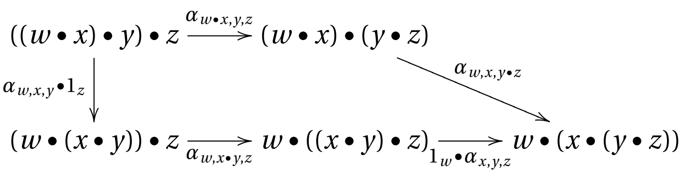
· $k-$bilinear natural isomorphisms

$$\alpha_{x,y,z}:(x\bullet y)\bullet z\longrightarrow x\bullet(y\bullet z)$$

$$l_x:1\bullet x\longrightarrow x$$

$$r_x:x\bullet 1\longrightarrow x$$

such that the following diagrams commute for all objects $w,x,y,z\in A_0$.



$$\begin{array}{ccc} (x \bullet 1) \bullet y & \xrightarrow{\alpha_{x,1,y}} & x \bullet (1 \bullet y) \\ & \searrow^{r_x \bullet 1_y} & \downarrow{\scriptstyle 1_x \bullet l_y} \\ & & x \bullet y \end{array}$$

A strict 2-algebra is the special case where $\alpha_{x,y,z}$, l_x, r_x are all identity morphisms. In this case we have

$$(x \bullet y) \bullet z = x \bullet (y \bullet z)$$

$$1 \bullet x = x, x \bullet 1 = x$$

Strict 2-algebra is called commutative strict 2-algebra if $x \bullet y = y \bullet x$ for all objects $x, y \in A_0$ and $f \bullet g = g \bullet f$ for all morphisms $f, g \in A_1$.

In the rest of this paper, the term 2-algebra will always refer to a commutative strict 2-algebra. A homomorphism between 2-algebras should preserve both the 2-module structure and the $\bullet$ functor.

Definition 3.6. Given 2-algebras A and A', a homomorphism

$$F : A \longrightarrow A'$$

consists of

· a linear functor F from the underlying 2-module of A to that of A', and

· a bilinear natural transformation

$$F_2(x, y) : F_0(x) \bullet F_0(y) \longrightarrow F_0(x \bullet y)$$

· an isomorphism $F : 1' \longrightarrow F_0(1)$ where 1 is the identity object of A and $1'$ is the identity object of A',

such that the following diagrams commute for $x, y, z \in A_0$,

$$\begin{array}{ccccc} (F(x) \bullet F(y)) \bullet F(z) & \xrightarrow{F_2 \bullet 1} & F(x \bullet y) \bullet F(z) & \xrightarrow{F_2} & F((x \bullet y) \bullet z) \\ {\scriptstyle \alpha_{F(x),F(y),F(z)}}\downarrow & & & & \downarrow{\scriptstyle F(\alpha_{x,y,z})} \\ F(x) \bullet (F(y) \bullet F(z)) & \xrightarrow[1 \bullet F_2]{} & F(x) \bullet F(y \bullet z) & \xrightarrow[F_2]{} & F(x \bullet (y \bullet z)). \end{array}$$

$$\begin{array}{ccc} 1' \bullet F(x) & \xrightarrow{l'_{F(x)}} & F(x) \\ {\scriptstyle F_0 \bullet 1}\downarrow & & \uparrow{\scriptstyle F(l_x)} \\ F(1) \bullet F(x) & \xrightarrow[F_2]{} & F(1 \bullet x). \end{array}$$

$$\begin{array}{ccc} F(x) \bullet 1' & \xrightarrow{r'_{F(x)}} & F(x) \\ {\scriptstyle 1 \bullet F_0}\downarrow & & \uparrow{\scriptstyle F(r_x)} \\ F(x) \bullet F(1) & \xrightarrow[F_2]{} & F(x \bullet 1). \end{array}$$

Definition 3.7. 2-algebras and homomorphisms between them give the category of 2-algebras denoted by

2Alg .

Therefore if $A = (A_0, A_1, s, t, e, \circ, \bullet)$ is a 2-algebra, A_0 and A_1 are algebras with this $\bullet$ bilinear functor. Thus we can take that 2-algebra is a 2-category with a single object say $*$, and A_0 collections of its 1-morphisms and A_1 collections of its 2-morphisms are algebras with identity.

3.3. Multiplication Algebras yield a 2-algebra

In [8] Norrie developed Lue's work [6] and introduced the notion of an actor of crossed modules of groups where it is shown to be the analogue of the automorphism group of a group. In the category of commutative algebras the appropriate replacement for automorphism groups is the multiplication algebra $\mathscr{M}(C)$ of an algebra C which is defined by MacLane [7].

Let C be an associative (not necessarily unitary or commutative) R-algebra. We recall Mac Lane's construction of the R-algebra $\mathrm{Bim}(C)$ of bimultipliers of C [7].

An element of $\mathrm{Bim}(C)$ is a pair (γ, δ) of R-linear mappings from C to C such that

$$\gamma(cc') = \gamma(c)c'$$

$$\delta(cc') = c\delta\left(c'\right)$$

and

$$c\gamma\left(c'\right) = \delta(c)c'.$$

$\mathrm{Bim}(C)$ has an obvious R-module structure and a product

$$(\gamma, \delta)(\gamma', \delta') = (\gamma\gamma', \delta'\delta),$$

the value of which is still in $\mathrm{Bim}(C)$.

Suppose that $\mathrm{Ann}(C) = 0$ or $C^2 = C$. Then $\mathrm{Bim}(C)$ acts on C by

$$\begin{array}{rcll} \mathrm{Bim}(C) \times C & \to & C; & ((\gamma, \delta), c) \mapsto \gamma(c), \\ C \times \mathrm{Bim}(C) & \to & C; & (c, (\gamma, \delta)) \mapsto \delta(c) \end{array}$$

and there is a

$$\begin{array}{rccl} \mu: & C & \longrightarrow & \mathrm{Bim}(C) \\ & c & \longmapsto & (\gamma_c, \delta_c) \end{array}$$

with

$$\gamma_c(x) = cx \qquad \text{and} \qquad \delta_c(x) = xc.$$

Commutative case: we still assume $\mathrm{Ann}(C) = 0$ or $C^2 = C$. If C is a commutative R-algebra and $(\gamma, \delta) \in \mathrm{Bim}(C)$, then $\gamma = \delta$. This is because for every $x \in C$:

$$\begin{array}{rcl} x\delta(c) & = & \delta(c)x = c\gamma(x) = \gamma(x)c \\ & = & \gamma(xc) = \gamma(cx) = \gamma(c)x = x\gamma(c). \end{array}$$

Thus $\mathrm{Bim}(C)$ may be identified with the R-algebra $\mathscr{M}(C)$ of multipliers of C. Recall that a multiplier of C is a linear mapping $\lambda : C \longrightarrow C$ such that for all $c, c' \in C$

$$\lambda(cc') = \lambda(c)c'.$$

Also $\mathscr{M}(C)$ is commutative as

$$\lambda'\lambda(xc) = \lambda'(\lambda(x)c) = \lambda(x)\lambda'(c) = \lambda'(c)\lambda(x) = \lambda\lambda'(cx) = \lambda\lambda'(xc)$$

for any $x \in C$. Thus $\mathscr{M}(C)$ is the set of all multipliers λ such that $\lambda\gamma = \gamma\lambda$ for every multiplier γ.

In [10] Porter states that automorphisms of a group G yield a 2-group. The appropriate analogue of this result in algebra case can be given. We claim that multiplications of an R-algebra C give a 2-algebra which is called a multiplication 2-algebra.

Let k be a commutative ring, R be a k-algebra with identity and C be a commutative R-algebra with $Ann(C) = 0$ or $C^2 = C$. Take $A_0 = \mathscr{M}(C)$ and say 1-morphisms to the elements of A_0. We define the action of $\mathscr{M}(C)$ on C as follows:

$$\begin{array}{rcl} \mathscr{M}(C) \times C & \longrightarrow & C \\ (f, x) & \longmapsto & f \blacktriangleright x = f(x). \end{array}$$

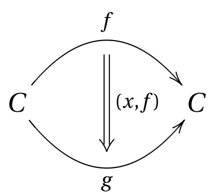
Using the action of $\mathscr{M}(C)$ on C, we can form the semidirect product

$$C \rtimes \mathscr{M}(C) = \{(x, f) | x \in C,\ f \in \mathscr{M}(C)\}$$

with multiplication

$$(x, f)(x', f') = (f \blacktriangleright x' + f' \blacktriangleright x + x'x, f'f).$$

Take $A_1 = C \rtimes \mathscr{M}(C)$ and say 2-morphisms to the elements of A_1. Therefore we get the following diagram for $(x, f) \in C \rtimes \mathscr{M}(C)$,



and we define the source, target and identity assigning maps as follows;

$$\begin{array}{rrcl} s: & C \rtimes \mathscr{M}(C) & \longrightarrow & \mathscr{M}(C) \\ & (x, f) & \longmapsto & s(x, f) = f \end{array} \qquad \begin{array}{rrcl} t: & C \rtimes \mathscr{M}(C) & \longrightarrow & \mathscr{M}(C) \\ & (x, f) & \longmapsto & t(x, f) = M_x \cdot f \end{array}$$

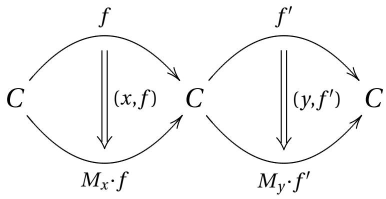
and

$$\begin{array}{rrcl} e: & \mathscr{M}(C) & \longrightarrow & C \rtimes \mathscr{M}(C) \\ & f & \longmapsto & e(f) = (0, f) \end{array}$$

where $M_x \cdot f$ is defined by $(M_x \cdot f)(u) = xu + f(u)$ for $u \in C$.

There are two ways of composing 2-morphisms: vertical and horizontal composition. Now we define this compositions.

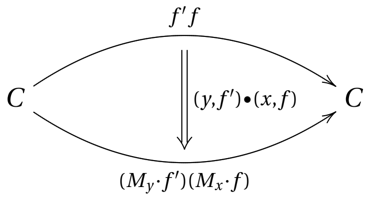
For $(x,f),(y,f') \in C \rtimes \mathscr{M}(C)$



the horizontal composition is defined by

$$(x,f) \bullet (y,f') = (f'(x) + f(y) + xy, f'f),$$

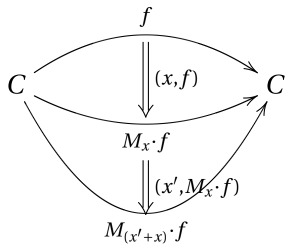
thus we have



and

$$\begin{aligned} t(f'(x) + f(y) + xy, f'f) &= M_{f'(x)+f(y)+xy} \cdot f'f \\ &= (M_y \cdot f')(M_x \cdot f) \end{aligned}$$

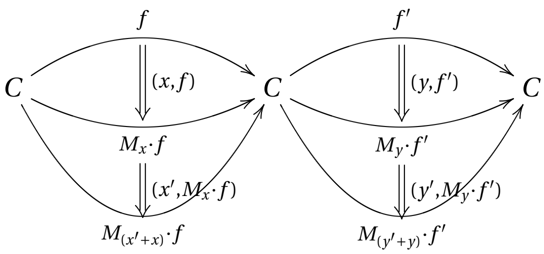
The vertical composition is defined by



$$(x,f) \circ (x', M_x \cdot f) = (x' + x, f)$$

for $(x,f),(x', M_x \cdot f) \in C \rtimes \mathscr{M}(C)$ with $t(x,f) = s(x', M_x \cdot f) = M_x \cdot f$.

It remains to satisfy the interchange law, i.e.



$$\begin{aligned} [(x,f) \circ (x', M_x \cdot f)] \bullet [(y,f') \circ (y', M_y \cdot f')] &= [(x,f) \bullet (y,f')] \\ &\quad \circ [(x', M_x \cdot f) \bullet (y', M_y \cdot f')]. \end{aligned}$$

Evaluating the two sides separately, we get

$$\begin{aligned}\textbf{LHS} &= (x'+x,f)\bullet(y'+y,f')\\ &= (f'(x'+x)+f(y'+y)+(x'+x)(y'+y),f'f)\\ &= (f'(x')+f'(x)+f(y')+f(y)+x'y'+x'y+xy'+xy,f'f)\end{aligned}$$

and

$$\begin{aligned}\textbf{RHS} &= (f'(x)+f(y)+xy,f'f)\circ((M_y\cdot f')(x')\\ &\quad +(M_x\cdot f)(y')+x'y',(M_y\cdot f')(M_x\cdot f))\\ &= (f'(x)+f(y)+xy+(M_y\cdot f')(x')+(M_x\cdot f)(y')+x'y',f'f)\\ &= (f'(x)+f(y)+xy+yx'+f'(x')+xy'+f(y')+x'y',f'f)\end{aligned}$$

LHS and RHS are equal, thus interchange law is satisfied. Therefore we get a 2-algebra consists of the R-algebra C as single object and the R-algebra A_0 of 1-morphisms and the R-algebra A_1 of 2-morphisms.

4. Crossed modules and 2-algebras

Crossed modules have been used widely and in various contexts since their definition by Whitehead [11] in his investigations of the algebraic structure of relative homotopy groups. We recalled the definition of crossed modules of commutative algebras given by Porter [10].

Let R be a k-algebra with identity. A pre-crossed module of commutative algebras is an R-algebra C together with a commutative action of R on C and a morphism

$$\partial : C \longrightarrow R$$

such that for all $c\in C$, $r\in R$

$$\text{CM1) } \partial(r\blacktriangleright c)=r\partial c.$$

This is a crossed R-module if in addition for all $c,c'\in C$

$$\text{CM2) } \partial c\blacktriangleright c' = cc'.$$

The last condition is called the Peiffer identity. We denote such a crossed module by (C,R,∂).

A morphism of crossed modules from (C,R,∂) to (C',R',∂') is a pair of k-algebra morphisms $\phi : C\longrightarrow C'$, $\psi : R\longrightarrow R'$ such that

$$\partial'\phi=\psi\partial \qquad \text{and} \qquad \phi(r\blacktriangleright c)=\psi(r)\blacktriangleright\phi(c).$$

Thus we get a category $\mathbf{XMod}_k$ of crossed modules (for fixed k).

Examples of Crossed Modules

1. Any ideal I in R gives an inclusion map, $inc : I\longrightarrow R$ which is a crossed module. Conversely given an arbitrary R-module $\partial : C\longrightarrow R$ one easily sees that the Peiffer identity implies that ∂C is an ideal in R.

2. Any R-module M can be considered as an R-algebra with zero multiplication and hence the zero morphism $0 : M \to R$ sending everything in M to the zero element of R is a crossed module. Conversely: If (C, R, ∂) is a crossed module, $\partial(C)$ acts trivially on $\ker\partial$, hence $\ker\partial$ has a natural $R/\partial(C)$-module structure.

As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

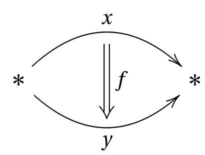
3. Let be $\mathscr{M}(C)$ multiplication algebra. Then $\left(C, \mathscr{M}(C), \mu\right)$ is multiplication crossed module. $\mu : C \to \mathscr{M}(C)$ is defined by $\mu(r) = \delta_r$ with $\delta_r\left(r'\right) = rr'$ for all $r, r' \in C$, where δ is multiplier $\delta : C \to C$ such that for all $r, r' \in C$, $\delta\left(rr'\right) = \delta(r)\, r'$. Also $\mathscr{M}(C)$ acts on C by $\delta \blacktriangleright r = \delta(r)$.(See [1] for details).

In [10] Porter states that there is an equivalence of categories between the category of internal categories in the category of k-algebras and the category of crossed modules of commutative k-algebras. In the following theorem, we will give a categorical presentation of this equivalence.

Theorem 4.1. The category of crossed modules $\mathbf{XMod}_k$ is equivalent to that of 2-algebras, **2Alg**.

Proof.

Let $A = (A_0, A_1, s, t, e, \circ, \bullet)$ be a 2-algebra consisting of a single object say $*$ and an algebra A_0 of 1-morphisms and an algebra A_1 of 2-morphisms. For $x, y \in A_0$ and $f : x \to y \in A_1$, we get the following diagram



We define s, t morphisms $s : A_1 \longrightarrow A_0, s(f) = x, t : A_1 \longrightarrow A_0, t(f) = y$ and e morphism $e : A_0 \longrightarrow A_1$ for $x \in A_0$, $e(x) : x \longrightarrow x \in A_1$.

The s, t and e morphisms are algebra morphisms and we have

$$\begin{aligned} se(x) &= s(e(x)) = x = Id_{A_0}(x) \\ te(x) &= t(e(x)) = x = Id_{A_0}(x) \end{aligned}$$

We define

$$\text{Ker } s = H = \{f \in A_1 \mid s(f) = Id_{A_0}\} \subseteq A_1$$

and $\partial = t\,|_H$ algebra homomorphism by $\partial : H \longrightarrow A_0, \partial(h) = t(h)$. We have semidirect product Ker $s \rtimes A_0 = \{(h, x) \mid h \in \text{Ker}s, x \in A_0\}$ with multiplication $(h, x) \bullet (h', x') = (x \blacktriangleright h' + x' \blacktriangleright h + h' \bullet h, x \bullet x')$ where action of A_0 on Ker$s$ is defined by $x \blacktriangleright h = e(x) \bullet h$. For each $f \in A_1$, we can write $f = n + e(x)$ where $n = f - es(f) \in$Ker$s$ and $x = s(f)$. Suppose $f' = n' + e(x')$. Then

$$\begin{aligned} f \bullet f' &= (n + e(x)) \bullet (n' + e(x')) \\ &= n \bullet n' + n \bullet e(x') + e(x) \bullet n' + e(x) \bullet e(x') \\ &= e(x') \bullet n + e(x) \bullet n' + n \bullet n' + e(x \bullet x') \\ &= x' \blacktriangleright n + x \blacktriangleright n' + n \bullet n' + e(x \bullet x'). \end{aligned}$$

There is a map

$$\begin{aligned} \phi : \quad A_1 &\longrightarrow \text{Ker}s \rtimes A_0 \\ n + e(x) &\longmapsto \phi(n + e(x)) = (n, x). \end{aligned}$$

Now

$$\begin{aligned}\phi(f\bullet f') &= \phi(x'\blacktriangleright n + x\blacktriangleright n' + n\bullet n' + e(x\bullet x'))\\ &= (x'\blacktriangleright n + x\blacktriangleright n' + n\bullet n', x\bullet x')\\ &= (n,x)\bullet(n',x')\\ &= \phi(f)\bullet\phi(f')\end{aligned}$$

so ϕ is a homomorphism. Also, there is an obvious inverse

$$\begin{aligned}\phi^{-1}: \quad \mathrm{Ker}s\rtimes A_0 &\longrightarrow A_1\\ (n,x) &\longmapsto \phi^{-1}(n,x) = n + e(x)\end{aligned}$$

which is also a homomorphism. Hence ϕ is an isomorphism and we have established that Ker $s\rtimes A_0 \simeq A_1$. Since A is a 2-algebra and Ker $s\rtimes A_0 \simeq A_1$, we can define algebra morphisms

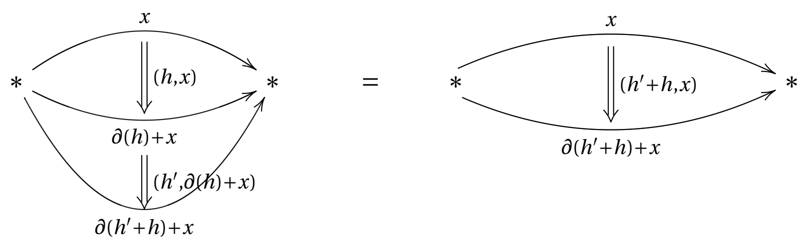
$$\begin{aligned}s: \quad \mathrm{Ker}s\rtimes A_0 &\longrightarrow A_0\\ (h,x) &\longmapsto s(h,x) = x\end{aligned} \qquad \begin{aligned}t: \quad \mathrm{Ker}s\rtimes A_0 &\longrightarrow A_0\\ (h,x) &\longmapsto t(h,x) = \partial(h) + x\end{aligned}$$

and

$$\begin{aligned}e: \quad A_0 &\longrightarrow \mathrm{Ker}s\rtimes A_0\\ x &\longmapsto e(x) = (0,x)\end{aligned}$$

and for $t(h,x) = s(h',\partial(h)+x) = \partial(h)+x$ we define

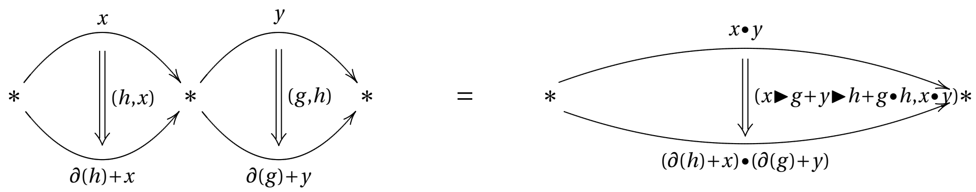
$$\begin{aligned}\circ: \quad \mathrm{Ker}s\rtimes A_0\ {}_t\!\times_s \mathrm{Ker}s\rtimes A_0 &\longrightarrow \mathrm{Ker}s\rtimes A_0\\ \big((h,x),(h',\partial(h)+x)\big) &\longmapsto (h'+h,x)\end{aligned}$$



which is vertical composition;

$$(h,x)\circ(h',\partial(h)+x) = (h'+h,x).$$

For $(h,x),(g,y)\in \mathrm{Ker}s\rtimes A_0$, horizontal composition is defined by



$$\begin{aligned}(h,x)\bullet(g,y) &= (x\blacktriangleright g + y\blacktriangleright h + g\bullet h, x\bullet y)\\ &= (e(x)\bullet g + e(y)\bullet h + g\bullet h, x\bullet y).\end{aligned}$$

Thus we have

CM1)

$$\begin{aligned}\partial(x \blacktriangleright h) &= \partial(e(x) \bullet h)\\ &= \partial(e(x)) \bullet \partial(h)\\ &= (te)(x) \bullet \partial(h)\\ &= x \bullet \partial(h).\end{aligned}$$

Also by interchange law we have

$$\begin{aligned}\left[(h,x) \bullet (g,y)\right] \circ \left[(h',\partial(h)+x) \bullet (g',\partial(g)+y)\right] &= \left[(h,x) \circ (h',\partial(h)+x)\right]\\ &\quad \bullet \left[(g,y) \circ (g',\partial(g)+y)\right].\end{aligned}$$

Therefore, evaluating the two sides of this equation gives:

$$\begin{aligned}LHS &= (x \blacktriangleright g + y \blacktriangleright h + g \bullet h, x \bullet y)\\ &\quad \circ ((\partial(h)+x) \blacktriangleright g' + \left(\partial(g)+y\right) \blacktriangleright h' + g' \bullet h', (\partial(h)+x) \bullet \left(\partial(g)+y\right))\\ &= ((\partial(h)+x) \blacktriangleright g' + \left(\partial(g)+y\right) \blacktriangleright h' + g' \bullet h' + x \blacktriangleright g + y \blacktriangleright h + g \bullet h, x \bullet y)\\ &= (\partial(h) \blacktriangleright g' + e(x) \bullet g' + \partial(g) \blacktriangleright h'\\ &\quad + e(y) \bullet h' + g' \bullet h' + e(x) \bullet g + e(y) \bullet h + g \bullet h, x \bullet y)\\ RHS &= (h'+h, x) \bullet (g'+g, y)\\ &= (x \blacktriangleright \left(g'+g\right) + y \blacktriangleright (h'+h) + \left(g'+g\right) \bullet (h'+h), x \bullet y)\\ &= \left(e(x) \bullet g' + e(x) \bullet g + e(y) \bullet h' + e(y) \bullet h + g' \bullet h' + g' \bullet h + g \bullet h' + g \bullet h, x \bullet y\right).\end{aligned}$$

Since the two sides are equal, we know that their first components must be equal, so we have

$$\partial(h) \blacktriangleright g' + \partial(g) \blacktriangleright h' = h \bullet g' + g \bullet h'$$

and

$$\begin{aligned}h \bullet g' + g \bullet h' &= \partial(h) \blacktriangleright g' + \partial(g) \blacktriangleright h'\\ &= \partial(h+g) \blacktriangleright (g'+h') - \partial(h) \blacktriangleright h' - \partial(g) \blacktriangleright g'\\ &= \partial(h+g) \blacktriangleright (g'+h') - \left(h \bullet h' + g \bullet g'\right),\end{aligned}$$

thus

$$\begin{aligned}\partial(h+g) \blacktriangleright (g'+h') &= h \bullet g' + g \bullet h' + \left(h \bullet h' + g \bullet g'\right)\\ &= \left(h+g\right) \bullet \left(h'+g'\right)\end{aligned}$$

and writing $(h+g) = l, \left(h'+g'\right) = l' \in Ker s$, we get

$$\partial(l) \blacktriangleright l' = l \bullet l'$$

which is the Peiffer identity as required. Hence $(Ker s, A_0, \partial)$ is a crossed module.

Let $\mathscr{A} = (A_0, A_1, s, t, e, \circ, \bullet)$ and $\mathscr{A}' = (A_0', A_1', s', t', e', \circ', \bullet')$ be 2-algebras and $F = (F_0, F_1) : \mathscr{A} \longrightarrow \mathscr{A}'$ be a 2-algebra morphism. Then $F_0 : A_0 \longrightarrow A_0'$ and $F_1 : A_1 \longrightarrow A_1'$ are the k-algebra morphisms. We define $f_1 =$

$F_1|_{Kers} : Kers \longrightarrow Kers^{'}$ and $f_0 = F_0 : A_0 \longrightarrow A_0^{'}$. For all $a \in Kers$ and $x \in A_0$,

$$\begin{aligned} f_0\partial(a) &= F_0 t(a) \\ &= t^{'} F_1(a) \\ &= \partial^{'} f_1(a) \end{aligned}$$

and

$$\begin{aligned} f_1(x \blacktriangleright a) &= F_1(e(x)a) \\ &= F_1(e(x))F_1(a) \\ &= e^{'} F_0(x)F_1(a) \\ &= e^{'} f_0(x) f_1(a) \\ &= f_0(x) \blacktriangleright f_1(a). \end{aligned}$$

Thus (f_1, f_0) map is a crossed module morphism $(Kers, A_0, \partial) \longrightarrow (Kers^{'}, A_0^{'}, \partial^{'})$. So we have a functor

$$\Gamma : \mathbf{2Alg} \longrightarrow \mathbf{XMod}_k.$$

Conversely, let (G, C, ∂) be a crossed module of algebras. Therefore there is an algebra morphism $\partial : G \to C$ and an action of C on G such that

CM1) $\partial(x \blacktriangleright g) = x\partial(g)$,

CM2) $\partial(g) \blacktriangleright g' = gg'$.

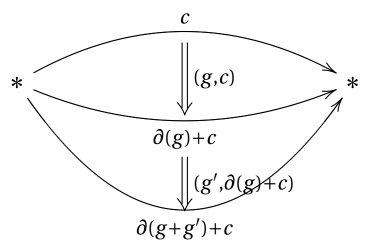
Since C acts on G, we can form the semidirect product $G \rtimes C$ as defined by

$$G \rtimes C = \{(g, c) \mid g \in G, c \in C\}$$

with multiplication

$$(g, c)(g', c') = (c \blacktriangleright g' + c' \blacktriangleright g + g'g, cc')$$

and define maps $s, t : G \rtimes C \to C$ and $e : C \to G \rtimes C$ by $s(g, c) = c$, $t(g, c) = \partial(g) + c$ and $e(c) = (0, c)$. These maps are clearly algebra morphisms.



For $t(g, c) = s(g', \partial(g) + c) = \partial(g) + c$, we define composition

$$\begin{aligned} \circ : \quad (G \rtimes C)_t \times_s (G \rtimes C) &\longrightarrow (G \rtimes C) \\ ((g, c), (g', \partial(g) + c)) &\longmapsto (g + g', c), \end{aligned}$$

for $(g, c), (h, d) \in G \rtimes C$ and $(g, c), (g', \partial(g) + c) \in G \rtimes C$, following equations give horizontal and vertical composition respectively.

$$(g, c) \bullet (h, d) = (c \blacktriangleright h + d \blacktriangleright g + gh, cd)$$

$$(g,c)\circ(g',\partial(g)+c)=(g+g',c)$$

Finally, since it must be that $\circ$ is an algebra morphism and by the crossed module conditions, interchange law is satisfied. Therefore we have constructed a 2-algebra $\mathscr{A}=(C,G\rtimes C,s,t,e,\circ,\bullet)$ consists of the single object say $*$ and the k-algebra C of 1-morphisms and the k-algebra $G\rtimes C$ of 2-morphisms. Let (G,C,∂) and $(G^{'},C^{'},\partial^{'})$ be crossed modules and $f=(f_1,f_0):(G,C,\partial)\longrightarrow(G^{'},C^{'},\partial^{'})$ be a crossed module morphism. We define

$$\begin{array}{rrcl} F_1: & G\rtimes C & \longrightarrow & G^{'}\rtimes C^{'} \\ & (g,c) & \longmapsto & F_1(g,c)=(f_1(g),f_0(c)) \end{array}$$

and

$$\begin{array}{rrcl} F_0: & C & \longrightarrow & C^{'} \\ & c & \longmapsto & F_0(c)=f_0(c). \end{array}$$

Then

$$\begin{aligned} s^{'}F_1(g,c) &= s^{'}(f_1(g),f_0(c)) \\ &= f_0(c) \\ &= F_0(c) \\ &= F_0s(g,c), \end{aligned}$$

$$\begin{aligned} t^{'}F_1(g,c) &= t^{'}(f_1(g),f_0(c)) \\ &= \partial^{'}f_1(g)+f_0(c) \\ &= f_0\partial(g)+f_0(c) \\ &= F_0(\partial(g)+c) \\ &= F_0t(g,c), \end{aligned}$$

$$\begin{aligned} e^{'}F_0(c) &= (0,f_0(c)) \\ &= F_1(0,c) \\ &= F_1e(c), \end{aligned}$$

$$\begin{aligned} F_1((g,c)\circ(g^{'},c^{'})) &= F_1(g+g^{'},c) \\ &= (f_1(g+g^{'}),f_0(c)) \\ &= (f_1(g)+f_1(g^{'}),f_0(c)) \\ &= (f_1(g),f_0(c))\circ(f_1(g^{'}),f_0(c^{'})) \\ &= F_1(g,c)\circ F_1(g^{'},c^{'}), \end{aligned}$$

$$\begin{aligned} F_1((g,c)\bullet(h,d)) &= F_1(c\blacktriangleright h+d\blacktriangleright g+gh,cd) \\ &= (f_1(c\blacktriangleright h)+f_1(d\blacktriangleright g)+f_1(gh),f_0(cd)) \\ &= (f_0(c)\blacktriangleright f_1(h)+f_0(d)\blacktriangleright f_1(g)+f_1(g)f_1(h),f_0(c)f_0(d)) \\ &= (f_1(g),f_0(c))\bullet(f_1(h),f_0(d)) \\ &= F_1(g,c)\bullet F_1(h,d) \end{aligned}$$

for all $(g,c)\in G\rtimes C$ and $c\in C$. Therefore $F=(F_1,F_0)$ is a 2-algebra morphism from $(C,G\rtimes C,s,t,e,\circ,\bullet)$ to $(C^{'},G^{'}\rtimes C^{'},s^{'},t^{'},e^{'},\circ^{'},\bullet^{'})$. Thus we get a functor

$$\Psi:\mathbf{XMod}_k\longrightarrow\mathbf{2Alg}.$$

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