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Finite coproducts in the category of quadratic modules of Lie algebras

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Quadratic modules, Quasi-quadratic modules, Coproduct objects

Abstract — In this study, we will construct finite coproduct objects in the category of quadratic modules of Lie algebras with a new approach using the idea of quasi-quadratic modules.

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1. Introduction

The concept of the crossed module is an algebraic model described by Whitehead for classifying homotopy 2-types [1]. It has attracted the attention of many researchers. This notion initially introduced in groups has also naturally appeared in various algebraic cases as commutative and associative algebras, Lie and Lie-Rinehart algebras, etc, [2–9]. Kassel and Loday studied the classification of central extensions of Lie algebras and crossed modules of Lie algebras in [10]. In [11], Casas and Ladra studied some properties of the category of crossed modules of Lie algebras. Ellis constructed the coproduct of crossed modules of Lie algebras with the same base of Lie algebras [12]. D. Conduché introduced one of the models beyond the algebraic 2-type and called 2-crossed modules [13] (For studies of homotopy, see [14–16]). In [17], Carrasco and Porter developed the coproduct of 2-crossed modules. Some of the related works for algebraic models associated with homotopy 3-type can be found in [18–21].

In this study, we focus on quadratic modules of Lie algebras, one of the algebraic 3-type model, developed by Baues for group case, and whose homotopy structure is defined [22]. The Lie algebra version of this model was introduced in the [23] studied by Ulualan and Uslu, while the studies in [24, 25] rely on quadratic module of commutative algebras. A different homotopy relation for quadratic modules of Lie algebras is constructed in [26, 27]. We will construct the finite coproduct objects in the category of quadratic modules suggested in Remark 3 given in [17]. For the construction of the coproduct of quadratic modules of Lie algebras with the same base of $nil(2)$-module, we will follow a construction technique similar to that used

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to built the coproduct structure of crossed modules of Lie algebras. The coproduct of two crossed modules with the same base object is the associated crossed module: see [28, Chapter 4]. The tool that we will use while upgrading the dimension of this construction will be the concept of quasi-quadratic modules given in [29]. Any quadratic module is a quasi-quadratic module. In more detail, the category of quadratic modules is a reflexive subcategory of the category of quasi-quadratic modules, and an associated quadratic module functor is defined as follows:

$$(-)^{cr} : \mathscr{Q}\mathscr{Q}\mathscr{M}_L \longrightarrow \mathbf{QM_L}$$

In our study, using the above functor, which is left adjoint to the inclusion functor, we will define the coproduct of two quadratic modules with the same base of $nil(2)$-module as the associated quadratic module to their coproduct in the category of quasi-quadratic modules.

2. Preliminaries

Let k be a commutative ring with unit and we will refer to a Lie algebra over k as a Lie algebra, and the Lie bracket multiplication will be denoted as $[-,-]$.

2.1. Lie Algebra Actions

Let Z and Y be Lie algebras over k, a k-bilinear map $Z \times Y \to Y, (z, y) \mapsto z * y$, is called a Lie algebra action of Z on Y, if the below equations are verified:

L1) $z * [y, y'] = [z * y, y'] + [y, z * y']$

L2) $[z, z'] * y = z * (z' * y) - z' * (z * y)$

for each $z, z' \in Z$ and $y, y' \in Y$.

2.2. Crossed Modules of Lie Algebras

A crossed module of Lie algebras, $\left(Y \xrightarrow{\partial} Z\right)$, consists of Lie algebras Y and Z with a left Lie algebra action "$*_1$" of Z on Y, and a Lie algebra homomorphism $\partial : Y \to Z$ satisfying the following conditions:

XMod$_L$1: $\partial(z *_1 y) = [z, \partial(y)]$, for all $z \in Z$ and $y \in Y$

XMod$_L$2: $\partial(y) *_1 y' = [y, y']$, for all $y, y' \in Y$

Note that "**XMod$_L$2**" is called the Peiffer identity, [10].

Example 2.1. Let I be a Lie ideal of a Lie algebra Z with $i : I \to Z$ the inclusion, in this case Z acts on the left I by conjugation and the inclusion Lie homomorphism i makes $\left(I \xrightarrow{i} Z\right)$, into a crossed module of Lie algebra.

Let $\left(Y \xrightarrow{\partial} Z\right)$ and $\left(Y' \xrightarrow{\partial'} Z'\right)$ are crossed modules of Lie algebras, a morphism, $f = (f_1, f_0) : \left(Y \xrightarrow{\partial} Z\right) \longrightarrow \left(Y \xrightarrow{\partial} Z\right)$ of crossed modules consists of Lie algebra homomorphisms $f_1 : Y \to Y'$ and $f_0 : Z \to Z'$ such that

- $\partial' f_1 = f_0 \partial$
- $f_1(z *_1 y) = f_0(z) *'_1 f_1(y)$

for all $Z \in Z$ and $y \in Y$. Thus, this means that the "f" morphism $*_1$ preserves the Lie algebra action, and the diagram below makes it commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\partial} & Z \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ Y' & \xrightarrow[\partial']{} & Z' \end{array}$$

Together with these definitions, we can define the category of crossed modules over Lie algebras by denoting it as $\mathbf{XMod_L}$. If we fix the base of the crossed module, the Z Lie algebra, then $\mathbf{XMod_L/Z}$ will be the category of crossed Z-modules, which is a subcategory of $\mathbf{XMod_L}$.

2.3. Quadratic Modules of Lie Algebras

A quadratic module of Lie algebras $\mathscr{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-]\otimes[-])\right)$ is a diagram:

$$\begin{array}{ccccc} & & C\otimes C & & \\ & {\scriptstyle\omega}\swarrow & \downarrow{\scriptstyle\Phi} & & \\ X & \xrightarrow[\delta]{} & Y & \xrightarrow[\partial]{} & Z \end{array}$$

of Lie algebra homomorphisms between Lie algebras such that $\mathbf{QM_L1}$, $\mathbf{QM_L2}$, $\mathbf{QM_L3}$, and $\mathbf{QM_L4}$ hold:

$\mathbf{QM_L1}$: The homomorphism $\partial : Y \to Z$ is a $nil(2)$-module and $Y \twoheadrightarrow C = Y^{cr}/[Y^{cr}, Y^{cr}]$ is defined by $y \mapsto [y]$ and Φ is defined by

$$\Phi([y_1]\otimes[y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

for $y_1, y_2 \in Y$,

$\mathbf{QM_L2}$: The boundary Lie homomorphisms composition of ∂ and δ satisfy $\partial\delta = 0$ and the quadratic map ω is a lift of the Peiffer commutator map Φ, that is $\delta\omega = \Phi$ or equivalently

$$\delta\omega = \Phi([y_1]\otimes[y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

for $y_1, y_2 \in Y$,

$\mathbf{QM_L3}$: X is a Lie Z-algebra, all of the homomorphisms in the diagram are Z-equivariant, and the action of Z on X also holds the following equality

$$\partial(y) *_3 x = \omega([\delta(x)]\otimes[y] + [y]\otimes[\delta(x)])$$

for $x \in X$ and $y \in Y$,

$\mathbf{QM_L4}$: For all $x_1, x_2 \in X$ commutators in X satisfy the formula

$$\omega([\delta(x_1)]\otimes[\delta(x_2)]) = [x_2, x_1]$$

Remark 2.2. It should be noted that $\left(X \xrightarrow{\delta} Y\right)$ is a crossed module, with

$$y *_2 x = \omega([\delta(x)] \otimes [y])$$

for each $y \in Y$ and $x \in X$. On the other hand, generally, $\left(Y \xrightarrow{\partial} Z\right)$ is only a $nil(2)$-module.

Remark 2.3. By $\mathbf{QM}_L\mathbf{3}$, we have:

$$\partial(y) *_3 x - y *_2 x = \omega([y] \otimes [\delta(x)])$$

where $*_2$ is a Lie action of Y on X.

Lemma 2.4. Let $\mathscr{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-])\right)$ be a quadratic module of Lie algebras and consider the "$*_2$" and "$*_3$" Lie algebra actions. Then for all $z \in Z$ and $y_1, y_2, y_3 \in Y$, we have:

$$z *_3 \omega([y_1] \otimes [y_2]) = \omega([z *_1 y_1] \otimes [y_2]) + \omega([y_1] \otimes [z *_1 y_2]) \quad (2.1)$$

$$\begin{aligned} \omega([[y_1, y_2]] \otimes [y_3]) &= \partial(y_1) *_3 \omega([y_2] \otimes [y_3]) + \omega([y_1] \otimes [[y_2, y_3]]) \\ &\quad - \partial(y_2) *_3 \omega([y_1] \otimes [y_3]) - \omega([y_2] \otimes [[y_1, y_3]]) \end{aligned} \quad (2.2)$$

$$\omega([y_1] \otimes [[y_2, y_3]]) = y_2 *_2 \omega([y_1] \otimes [y_3]) - y_3 *_2 \omega([y_1] \otimes [y_2]) \quad (2.3)$$

Example 2.5. Thanks to any $nil(2)$-module $\left(Y \xrightarrow{\partial} Z\right)$, we define a quadratic module as follows:

$$\begin{array}{ccccc} & & C \otimes C & & \\ & \overset{1}{\swarrow} & \downarrow \Phi & & \\ X & \xrightarrow[\Phi]{} & Y & \xrightarrow[\partial]{} & Z \end{array}$$

Example 2.6. If $\mathscr{L} = (X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-]))$ is a quadratic module, then $Im\delta$ is a Lie ideal of Y and we have there is an induced crossed module structure on

$$Y / Im\delta \xrightarrow{\quad\partial\quad} Z.$$

Let $\mathscr{L} = \left(X \xrightarrow{\delta} Y \xrightarrow{\partial} Z, \omega([-] \otimes [-])\right)$ and $\mathscr{L}' = \left(X' \xrightarrow{\delta'} Y' \xrightarrow{\partial'} Z', \omega'([-] \otimes [-])\right)$ be two quadratic modules, a morphism of quadratic modules of Lie algebras given by a diagram

$$\begin{array}{ccccccc} C \otimes C & \xrightarrow{\omega} & X & \xrightarrow{\delta} & Y & \xrightarrow{\partial} & Z \\ {\scriptstyle \varphi_* \otimes \varphi_*}\downarrow & & \downarrow {\scriptstyle f_2} & & \downarrow {\scriptstyle f_1} & & \downarrow {\scriptstyle f_0} \\ C' \otimes C' & \xrightarrow[\omega']{} & X' & \xrightarrow[\delta']{} & Y' & \xrightarrow[\partial']{} & Z' \end{array}$$

where (f_1, f_0) is a morphism between $nil(2)$-modules which induced $\varphi_* : C \to C'$ and where

$$\begin{aligned} f_1(z *_1 y) &= f_0(z) *'_1 f_1(y) \\ f_2(z *_3 x) &= f_0(z) *'_3 f_2(x) \\ f_2(\omega([y_1] \otimes [y_2])) &= \omega'([f_1(y_1)] \otimes [f_1(y_2)]) \end{aligned}$$

for all $z \in Z$, $y, y_1, y_2 \in Y$ and $x \in X$. We will denote by $\mathbf{QM_L}$ the category of quadratic modules of Lie algebras and by $\mathbf{QM_L}/(\mathbf{Y} \xrightarrow{\partial} \mathbf{Z})$ the subcategory of quadratic modules of fixed $nil(2)$-module $\left(Y \xrightarrow{\partial} Z\right)$.

2.4. Quasi-Quadratic Modules of Lie Algebras

A quasi-quadratic module of Lie algebras is a semiexact sequence

$$X \xrightarrow{\ \delta\ } Y \xrightarrow{\ \partial\ } Z$$

of Z-Lie algebras together with an ω quadratic map

$$\omega([-] \otimes [-]) : C \otimes C \longrightarrow X$$

such that $\mathcal{QQM}_L 1$, $\mathcal{QQM}_L 2$, $\mathcal{QQM}_L 3$, and $\mathcal{QQM}_L 4$ hold:

$\mathcal{QQM}_L 1$**:** For all $y_1, y_2, y_3 \in Y$,

$$\delta\omega([y_1] \otimes [y_2]) = \Phi([y_1] \otimes [y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

$\mathcal{QQM}_L 2$**:**

$$\begin{aligned} \omega([[y_1, y_2]] \otimes [y_3]) &= \partial(y_1) *_3 \omega([y_2] \otimes [y_3]) + \omega([y_1] \otimes [[y_2, y_3]])| \\ &\quad -\partial(y_2) *_3 \omega([y_1] \otimes [y_3]) - \omega([y_2] \otimes [[y_1, y_3]]) \end{aligned}$$

$\mathcal{QQM}_L 3$**:**

$$\omega([y_1] \otimes [[y_2, y_3]]) = y_2 *_2 \omega([y_1] \otimes [y_3]) - y_3 *_2 \omega([y_1] \otimes [y_2])$$

$\mathcal{QQM}_L 4$**:**

$$[\omega([y_1] \otimes [y_2]), \partial(y_1) *_3 (y_2 *_2 x)] = \omega([\partial(y_1) *_1 [y_2, \delta x]] \otimes [\delta\omega([y_1] \otimes [y_2])])$$

Quasi-quadratic module morphisms are defined in the same way as quadratic module morphisms. We will denote the category of quasi-quadratic module of Lie algebras by $\mathcal{QQM}_L$.

Furthermore, any quadratic module of Lie algebras is a quasi-quadratic module of Lie algebras, and we can construct a quadratic module of Lie algebras associated with a quasi-quadratic module of Lie algebras (see in [29, Lemma 3.2 and Lemma 3.4]). There exists an adjunction as below:

$$\mathcal{QQM}_L \underset{\hookleftarrow}{\overset{(-)^{cr}}{\rightleftarrows}} \mathbf{QM_L}$$

3. Finite Coproducts in the Category of Quadratic Modules of Lie Algebras

3.1. The Coproduct of Crossed Modules of Lie Algebras

In this section, we recall the definition of a coproduct object of crossed modules of Lie algebras given by [12].

Let $\left(Y \xrightarrow{\partial_Y} Z, *_Y\right)$ and $\left(X \xrightarrow{\partial_X} Z, *_X\right)$ be two crossed Z-modules of Lie algebras. As follows, Y has a left Lie algebra action on X:

$$\begin{aligned} Y \times X &\longrightarrow X \\ (y,x) &\longmapsto y \star x = \partial_Y(y) *_X x \end{aligned} \tag{3.1}$$

Thus, we can define $X \rtimes Y$ a semidirect product Lie algebra. Considering the left Lie algebra action of Z on $X \rtimes Y$, we define the following Lie algebra homomorphism:

$$\begin{aligned} \partial : X \rtimes Y &\longrightarrow Z \\ (x,y) &\longmapsto \partial((x,y)) = \partial_x(x) + \partial_y(y) \end{aligned}$$

for $(x,y) \in X \rtimes Y$.

Naturally, $\left(X \rtimes Y \xrightarrow{\partial} Z, *\right)$ is a pre-crossed Z-module:

$$\begin{aligned} \partial(z*(x,y)) &= \partial((z *_X x, z *_Y y)) \\ &= \partial(z *_X x, z *_Y y) \\ &= [z, \partial_X(x)] + [z, \partial_Y(y)] \\ &= [z, \partial_X + \partial_Y(y)] \\ &= [z, \partial((x,y))] \end{aligned}$$

for all $z \in Z$ and $(x,y) \in X \rtimes Y$. Let I be an ideal of $X \rtimes Y$ generated by the elements below

$$[(x,y),(x',y')] - \partial((x,y)) * (x',y')$$

for all $(x,y),(x',y') \in X \rtimes Y$.

Moreover, we have:

$$\begin{aligned} [(x,y),(x',y')] - \partial((x,y)) * (x',y') &= ([x,x'] + y \star x' - y' \star x, [y,y']) - ((\partial_X(x) + \partial_Y(y)) * (x',y')) \\ &= ([x,x'] + \partial_Y(y) *_X x' - \partial_Y(y') *_X x, [y,y']) \\ &\quad -(\partial_X(x) *_X x' + \partial_Y(y) *_X x', \partial_X(x) *_X y' + \partial_Y(y) *_Y y') \\ &= ([x,x'] + \partial_Y(y) *_X x' - \partial_Y(y') *_X x - \partial_X(x) *_X x' - \partial_Y(y) *_X x', [y,y'] \\ &\quad -\partial_X(x) *_Y y' - \partial_Y(y) *_Y y') \\ &= (-\partial_Y(y') *_X x, -\partial_X(x) *_Y y') \end{aligned}$$

This means I is generated by the elements:

$$(-\partial_Y(y') *_X x, -\partial_X(x) *_Y y')$$

Consider $\partial(I) = 0$. Then, for each $(x, y) \in X \rtimes Y$, we get the following induced morphism:

$$\begin{aligned} \bar{\partial} : (X \rtimes Y)/I &\longrightarrow Z \\ ((x, y) + I) &\longmapsto \bar{\partial}((x, y) + I) = \partial_x(x) + \partial_y(y) \end{aligned}$$

For all $(x, y), (x', y') \in X \rtimes Y$, this structure gives us the crossed module definition:

$$\begin{aligned} \bar{\partial}((x, y) + I) * ((x', y') + I) &= (\partial_X(x) + \partial_Y(y)) * ((x', y') + I) \\ &= (\partial_X(x) *_X x' + \partial_Y(y) *_X x', \partial_X(x) *_Y y' \\ &\quad + \partial_Y(y) *_Y y') + I \\ &= ([x, x'] + \partial_Y(y) *_Y x' - \partial_Y(y') *_X x, [y, y']) + I \\ &= [(x, y) + I, (x', y') + I] \quad (\because \mathbf{XMod_L 2}) \end{aligned}$$

With these structures, we have the crossed module of Lie algebra $\left((X \rtimes Y)/I \xrightarrow{\bar{\partial}} Z\right)$, which is the coproduct object in $\mathbf{XMod_L}/\mathbf{Z}$.

3.2. The Coproduct of Quadratic Modules of Lie Algebras

Let

$$\mathcal{L}_1 = \left(\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{\omega_1} & \downarrow \Phi & & \\ X_1 & \xrightarrow[\delta_1]{} & Y & \xrightarrow[\partial]{} & Z \end{array} \right) \text{and } \mathcal{L}_2 = \left(\begin{array}{ccccc} & & C \otimes C & & \\ & \swarrow^{\omega_2} & \downarrow \Phi & & \\ X_2 & \xrightarrow[\delta_2]{} & Y & \xrightarrow[\partial]{} & Z \end{array} \right)$$

be two quasi-quadratic module over $(Y \xrightarrow{\partial} Z)$. Thus we have

$$\delta_1\omega_1([y_1] \otimes [y_2]) = \delta_2\omega_2([y_1] \otimes [y_2]) = \partial(y_1) *_1 y_2 - [y_1, y_2]$$

for all $y_1, y_2 \in Y$. Then let I be the Lie ideal of $X_1 \rtimes X_2$ generated by the elements:

$$\left(q_1\omega_1([y_1] \otimes [y_2]), q_2\omega_2([y_1] \otimes [y_2])\right)$$

where $q_i = \pm 1$ and $q_1 \neq q_2$.

Now let us define quadratic map.

$$\begin{aligned} \bar{\omega}([-] \otimes [-]) : C \otimes C &\longrightarrow (X_1 \rtimes X_2)/I \\ (y_1, y_2) &\longmapsto \bar{\omega}([y_1] \otimes [y_2]) = (\omega_1([y_1] \otimes [y_2]), 0) + I \\ &\qquad = (0, \omega_2([y_1] \otimes [y_2])) + I \end{aligned}$$

by considering $(\omega_1([y_1] \otimes [y_2]), -\omega_2([y_1] \otimes [y_2])) \in I$.

Moreover, Y acts on $X_1 \rtimes X_2/I$ as follows:

$$\diamond : Y \times (X_1 \rtimes X_2/I) \longrightarrow X_1 \rtimes X_2/I$$

$$(y,((x,x')+I)) \longmapsto y \diamond ((x,x')+I) = (y *_2 x, y *_2 x') + I$$

for all $y \in Y$ and $(x,x') \in X_1 \rtimes X_2$. Additionally, using $\bar{\delta}(I)=0$, we have induced morphism as follows:

$$\bar{\delta} : (X_1 \rtimes X_2)/I \longrightarrow Y$$

$$((x_1,x_2)+I) \longrightarrow \bar{\delta}((x_1,x_2)+I) = \delta_1(x_1)+\delta_2(x_2)$$

Theorem 3.1. The pair of $\bar{\mathscr{L}} = \big((X_1 \rtimes X_2)/I \xrightarrow{\bar{\delta}} Y \xrightarrow{\partial} Z, \bar{\omega}([-]\otimes[-]), \partial, \diamond\big)$ is a quasi-quadratic module which is the coproduct object of $\mathscr{L}_1$ and $\mathscr{L}_2$ in the category of $\mathscr{QQM}_L/(Y \xrightarrow{\partial} Z)$.

Proof.
First we need to show that $\bar{\mathscr{L}}$ is a quasi-quadratic module. For this we need to provide $\mathscr{QQM}_L1$, $\mathscr{QQM}_L2$, $\mathscr{QQM}_L3$, and $\mathscr{QQM}_L4$ axioms:

$\mathscr{QQM}_L1$**:**

$$\begin{aligned}
\bar{\delta}\bar{\omega}([y_1]\otimes[y_2]) &= \bar{\delta}((\omega_1([y_1]\otimes[y_2]),0)+I) \\
&= \bar{\delta}(\omega_1([y_1]\otimes[y_2]),0)+\bar{\delta}(I) \\
&= \delta_1\omega_1([y_1]\otimes[y_2]) \\
&= \partial(y_1) *_1 y_2 - [y_1,y_2]
\end{aligned}$$

$\mathscr{QQM}_L2$**:**

$$\begin{aligned}
\bar{\omega}([[y_1,y_2]]\otimes[y_3]) &= (\omega_1([[y_1,y_2]]\otimes[y_3]),0)+I \\
&= (\partial(y_1) *_3 \omega_1([y_2]\otimes[y_3] + \omega_1([y_1]\otimes[[y_2,y_3]]) \\
&\quad -\partial(y_2) *_3 \omega_1([y_1]\otimes[y_3]) - \omega([y_2]\otimes[[y_1,y_3]]),0)+I \\
&= (\partial(y_1) *_3 \omega_1([y_2]\otimes[y_3]),0)+I+(\omega_1([y_1])\otimes[[y_2,y_3]]),0)+I \\
&\quad +(-\partial(y_2) *_3 \omega_1([y_1]\otimes[y_3]),0)+I+(-\omega_1([y_2])\otimes[[y_1,y_3]]),0)+I \\
&= \partial(y_1) *_3 \bar{\omega}([y_2]\otimes[y_3]) + \bar{\omega}([y_1]\otimes[[y_2,y_3]]) \\
&\quad -\partial(y_2) *_3 \bar{\omega}([y_1]\otimes[y_3]) - \bar{\omega}([y_2]\otimes[[y_1,y_3]])
\end{aligned}$$

$\mathscr{QQM}_L3$**:**

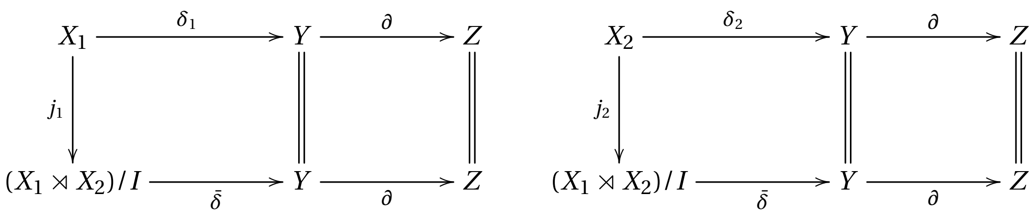
$$\begin{aligned}
\bar{\omega}([y_1]\otimes[[y_2,y_3]]) &= (\omega_1([y_1]\otimes[[y_2,y_3]]),0)+I \\
&= (y_2 *_2 \omega_1([y_1]\otimes[y_3]) - y_3 *_2 \omega_1([y_1]\otimes[y_2]),0)+I \\
&= (y_2 *_2 \omega_1([y_1]\otimes[y_3],0)+I-(y_3 *_2 \omega_1([y_1]\otimes[y_2]),0)+I \\
&= y_2 *_2 (\omega_1([y_1]\otimes[y_3],0)+I - y_3 *_2 (\omega_1([y_1]\otimes[y_2]),0)+I \\
&= y_2 \diamond \bar{\omega}([y_1]\otimes[y_3]) - y_3 \diamond \bar{\omega}([y_1]\otimes[y_2])
\end{aligned}$$

$\mathcal{QQM}_L4$**:**

$$
\begin{aligned}
[\bar{\omega}([y_1]\otimes[y_2]), \partial(y_1) *_3 (y_2 \diamond (x_1,x_2)+I)] &= [(\omega_1([y_1]\otimes[y_2]),0)+I, (\partial(y_1) *_3 (y_2 *_2 x_1), \partial(y_1) *_3 (y_2 *_2 x_2))+I] \\
&= ([\omega_1([y_1]\otimes[y_2]), \partial(y_1) *_3 (y_2 *_2 x_1)] + 0 \star (\partial(y_1) *_3 (y_2 *_2 x_1)) \\
&\quad -(\partial(y_1) *_3 (y_2 *_2 x_2)) \star \omega_1([y_1]\otimes[y_2]), [0, \partial(y_1) *_3 (y_2 *_2 x_2)]) + I \\
&= (\omega_1([\partial(y_1) *_1 [y_2, \delta_1(x_1)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]) \\
&\quad -\delta_2(\partial(y_1) *_3 (y_2 *_3 x_2)) *_2 \omega_1([y_1]\otimes[y_2]),0)+I \\
&= (\omega_1([\partial(y_1) *_1 [y_2, \delta_1(x_1)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]) \\
&\quad +\omega_1([\partial(y_1) *_1 [y_2, \delta_2(x_2)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]),0)+I \\
&= (\omega_1([\partial(y_1) *_1 [y_2, \delta_1(x_1)+\delta_2(x_2)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]]),0)+I \\
&= \bar{\omega}([\partial(y_1) *_1 [y_2, \bar{\delta}((x_1,x_2)+I)]] \otimes [\partial(y_1) *_1 y_2 - [y_1,y_2]])
\end{aligned}
$$

for all $y_1, y_2, y_3 \in Y$ and $(x_1,x_2)+I \in (X_1 \rtimes X_2)/I$.

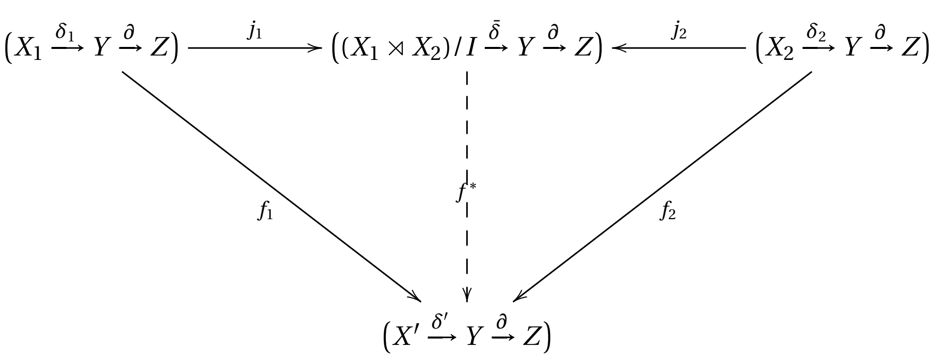
Furthermore, the canonical morphisms are given by

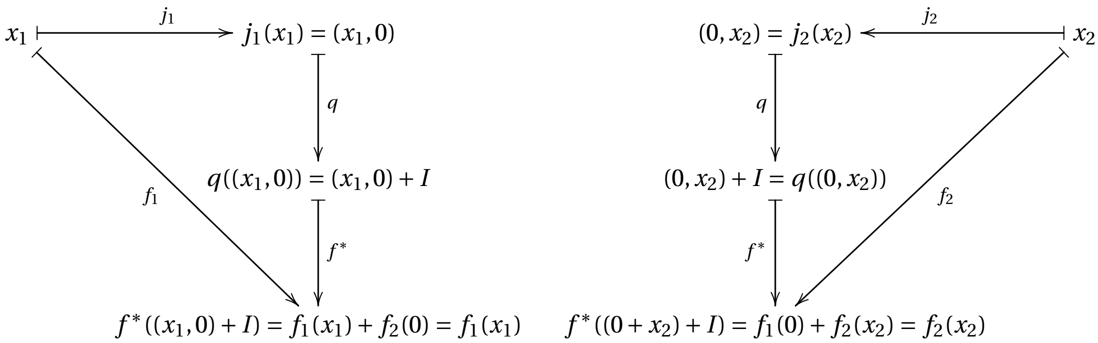


where $j_t, t=1,2$, composition $X_t \xrightarrow{i_t} (X_1 \rtimes X_2) \to (X_1 \rtimes X_2)/I$, i_t is canonical inclusion in the semidirect product and second morphism is quotient homomorhism:

$$
\begin{aligned}
f^* : (X_1 \rtimes X_2)/I &\longrightarrow X' \\
(x_1,x_2)+I &\longmapsto f^*((x_1,x_2)+I) = f_1(x_1)+f_2(x_2)
\end{aligned}
$$

which satisfies the universal property of coproduct object with the following commutative diagram completes the proof.





Corollary 3.2. Consider the "$(-)^{cr}$" functor and adjunction given in [29]. If $\mathscr{L}_1 = (X_1 \xrightarrow{\delta_1} Y \xrightarrow{\partial} Z, \omega_1([-]\otimes[-]))$ and $\mathscr{L}_2 = (X_2 \xrightarrow{\delta_2} Y \xrightarrow{\partial} Z, \omega_2([-]\otimes[-]))$ are $\mathbf{QM_L}/(\mathbf{Y} \xrightarrow{\partial} \mathbf{Z})$, then applying functor $(-)^{cr}$ to $((X_1 \rtimes X_2)/I \xrightarrow{\bar{\delta}} Y \xrightarrow{\partial} Z, \bar{\omega}([-]\otimes[-]))$ with the morphism $u_t : X_t \xrightarrow{j_t} (X_1 \rtimes X_2)/I \to ((X_1 \rtimes X_2)/I)^{cr}$, gives the coproduct object of $\mathscr{L}_1$ and $\mathscr{L}_2$ in the category of $\mathbf{QM_L}/(\mathbf{Y} \xrightarrow{\partial} \mathbf{Z})$.

We denote the coproduct object of $\mathscr{L}_1$ and $\mathscr{L}_2$ by

$$\left(\mathscr{L}_1 \rtimes_* \mathscr{L}_2\right) = X_1 \rtimes_* X_2 \xrightarrow{(\delta_1 \rtimes_* \delta_2)} Y \xrightarrow{\partial} Z$$

4. Conclusion

In this paper, we constructed finite coproduct objects in the category of quadratic modules of Lie algebras with the same base as the $nil(2)$-module. This structure can be generalised by changing the base. This construction can be defined for other algebraic cases and thus may reveal important structures for nonabelian algebraic topology or categorification.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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