Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (1) (2024), 121 – 129
DOI : 10.15672/hujms.1185538

Research Article

On classification of 7−dimensional odd-nilpotent Leibniz algebras

Ismail Demir

Department of Mathematics, Faculty of Arts and Sciences, Usak University, Usak, Turkiye

Abstract

In this paper we extend the method of canonical form for congruence of bilinear forms to give the classification of some subclasses of 7−dimensional nilpotent Leibniz algebras. Odd-nilpotent Leibniz algebras are defined as that its even dimensional ideals in lower central series are all zero and the classification of 7−dimensional complex odd-nilpotent Leibniz algebras with one dimensional Leib ideal is obtained by applying the aforementioned method.

Mathematics Subject Classification (2020). 17A32, 17A60

Keywords. Leibniz algebra, nilpotency, classification

1. Introduction

As a nonantisymmetric generalization of Lie algebras, Leibniz algebras were first considered by Bloh who called them D-algebras emphasizing their connections with derivations [3]. Later these algebraic structures were restored by Loday [8]. A vector space L over $\mathbb{C}$ with a bilinear product $[\ ,\ ] : L \times L \longrightarrow L$ whose left multiplication is a derivation is called a Leibniz algebra. Define the ideals of L, $L^1 = L$ and $L^j = [L, L^{j-1}]$ for $j \in \mathbb{Z}_{\geq 2}$. A Leibniz algebra L is nilpotent of class c if $L^{c+1} = 0$ but $L^c \neq 0$ for some $c > 0$. L is called odd-nilpotent if its all terms of the lower central series are odd-dimensional. Another important ideal of L can be defined as $Leib(L) = \text{span}\{[a,a] \mid a \in L\}$. L is a Lie algebra if and only if the Leib ideal is zero. We define the center of L by $Z(L) = \{x \in L \mid [x,a] = 0 = [a,x] \text{ for all a} \in \text{L}\}$. Direct sum of two nonzero ideals of a Leibniz algebra is called split, otherwise it is called non-split. Throughout this paper, we consider only non-split and non-Lie complex Leibniz algebras.

It is always an intriguing problem to give the classification of any kind of algebras. To give the complete classification of nilpotent Lie algebras is considered to be a wild problem and it is still unsolved. Due to lack of antisymmetry property, classifying nilpotent Leibniz algebras is more problematic. The complete classification of complex nilpotent Leibniz algebras of dimension ≤ 4 has been given (see [1, 2], [4, 5], [8, 9]). 5−dimensional nilpotent Leibniz algebras classified in [6] with canonical forms for congruence technique. Recently, classification of some subsclasses 6−dimensional nilpotent Leibniz algebras is given in [7] with the same technique. In this paper, we apply this canonical forms for

Email address: ismail.demir@usak.edu.tr
Received: 07.10.2022; Accepted: 29.03.2023

congruence technique to give the classification of 7−dimensional odd-nilpotent Leibniz algebras with $\dim(Leib(L)) = 1$. This approach can be used to classify any $(2n-1)-$dimensional odd-nilpotent Leibniz algebras. We verify that the classes we obtained are pairwise nonisomorphic using the Mathematica program implementing Algorithm 2.6 given in [4].

2. Preliminaries

We give the following Lemmas which are very useful. They are given in [6, 7].

Lemma 2.1. *If L is a non-split Leibniz algebra then $Z(L) \subseteq L^2$.*

Lemma 2.2. *If L is a nilpotent Leibniz algebra then $Leib(L) \subseteq Z(L)$.*

Lemma 2.3. *Let L be a $n-$dimensional nilpotent Leibniz algebra with $\dim(Z(L)) = n-k$. If $\dim(Leib(L)) = 1$ then $\dim(L^2) \leq \frac{k^2-k+2}{2}$.*

Lemma 2.4. *Let L be a $n-$dimensional nilpotent Leibniz algebra with $\dim(L^2) = n-k, \dim(Leib(L)) = 1$ and $\dim(L^3) = t$. Then*

(i): $n \leq t + \frac{k^2+k+2}{2}$

(ii): $n \leq t + \frac{k^2+k}{2}$ *if $Leib(L) \subseteq L^3$*

Lemma 2.5. *If L is a $n-$dimensional nilpotent Leibniz algebra with $\dim(L^2) = n-k$ and $L^4 \neq 0$ then $\dim(Z(L)) < n-k-1$.*

Let L be a nilpotent Leibniz algebra. Denote $\chi(L) = (\dim(L), \dim(L^2), \dim(L^3), \ldots, \dim(L^c))$ where c is the class of nilpotency. Suppose $\dim(L^2) = 1$. Choose $L^2 = Leib(L) = \mathrm{span}\{w_n\}$. Let a subspace V be complementary to L^2 in L so that $L = L^2 \oplus V$. So $[x, y] = \alpha w_n$ for some $\alpha \in \mathbb{C}$, for any $x, y \in V$. Bilinear form $f(\ ,\ ) : V \times V \to \mathbb{C}$ can be defined by $f(x, y) = \alpha_n$ for all $x, y \in V$. The canonical forms for the congruence classes of matrices associated with any bilinear form on a complex vector space given in [10] is as follows. We denote

$$[A \backslash B] := \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}.$$

Theorem 2.6. [10] The matrix of a bilinear form is congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the following types:

$$(1)\quad A_{2k+1} = \left[\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \backslash \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \right]_{(2k+1)\times(2k+1)}$$

$$(2)\quad B_{2k}(c) = \left[\begin{bmatrix} 0 & & & c \\ & & c & 1 \\ & \iddots & \iddots & \\ c & 1 & & 0 \end{bmatrix} \backslash \begin{bmatrix} & & & 1 \\ & & 1 & c \\ & \iddots & \iddots & \\ 1 & c & & 0 \end{bmatrix} \right]_{2k\times 2k}, \quad c \neq \pm 1.$$

$$(3)\quad C_{2k+1} = \begin{bmatrix} 0 & & & & & & 1 \\ & & & & & 1 & 1 \\ & & & & \iddots & \iddots & \\ & & & 1 & 1 & & \\ & & 1 & -1 & & & \\ & \iddots & \iddots & & & & \\ 1 & -1 & & & & & 0 \end{bmatrix}_{(2k+1)\times(2k+1)}$$

$$(4)\quad D_{2k} = \left[\begin{bmatrix} 0 & & & 1 \\ & & 1 & -1 \\ & \iddots & \iddots & \\ 1 & -1 & & 0 \end{bmatrix} \backslash \begin{bmatrix} & & & 1 \\ & & 1 & 1 \\ & \iddots & \iddots & \\ 1 & 1 & & 0 \end{bmatrix} \right]_{2k\times 2k} \quad (k \text{ even})$$

$$(5)\quad E_{2k} = \begin{bmatrix} 0 & & & & & & 1 \\ & & & & & 1 & 1 \\ & & & & \iddots & \iddots & \\ & & & 1 & 1 & & \\ & & -1 & 1 & & & \\ & \iddots & \iddots & & & & \\ -1 & 1 & & & & & 0 \end{bmatrix}_{2k\times 2k}$$

$$(6)\quad F_{2k} = \left[\left[\begin{array}{cccc} 0 & & & -1 \\ & & -1 & 1 \\ & ⋰ & ⋰ & \\ -1 & 1 & & 0 \end{array}\right] \backslash \left[\begin{array}{cccc} & & & 1 \\ & & 1 & 1 \\ & ⋰ & ⋰ & \\ 1 & 1 & & 0 \end{array}\right]\right]_{2k\times 2k} \qquad (k \text{ odd})$$

Choosing a basis $\{w_1, w_2, \ldots, w_6\}$ for V and $L^2 = Leib(L) = \text{span}\{w_7\}$. We see that the matrix of the bilinear form $f(\ ,\ ): V \times V \to \mathbb{C}$ is one of the following (Matrices that yield split Leibniz algebras are omitted.)

Partition of 6	6×6 **matrices**
6	B_6, E_6, F_6
5+1	$A_5 \oplus 1, C_5 \oplus 1$
4+2	$B_4 \oplus F_2, B_4 \oplus E_2, B_4 \oplus B_2, D_4 \oplus F_2, D_4 \oplus E_2, D_4 \oplus B_2, E_4 \oplus F_2, E_4 \oplus E_2, E_4 \oplus B_2$
4+1+1	$B_4 \oplus 1 \oplus 1, D_4 \oplus 1 \oplus 1, E_4 \oplus 1 \oplus 1$
3+3	$A_3 \oplus A_3, A_3 \oplus C_3, C_3 \oplus C_3$
3+2+1	$A_3 \oplus F_2 \oplus 1, A_3 \oplus E_2 \oplus 1, A_3 \oplus B_2 \oplus 1, C_3 \oplus F_2 \oplus 1, C_3 \oplus E_2 \oplus 1, C_3 \oplus B_2 \oplus 1$
3+1+1+1	$A_3 \oplus 1 \oplus 1 \oplus 1, C_3 \oplus 1 \oplus 1 \oplus 1$
2+2+2	$F_2 \oplus F_2 \oplus F_2, F_2 \oplus F_2 \oplus E_2, F_2 \oplus F_2 \oplus B_2, F_2 \oplus E_2 \oplus E_2, F_2 \oplus E_2 \oplus B_2, F_2 \oplus B_2 \oplus B_2, E_2 \oplus E_2 \oplus E_2, E_2 \oplus E_2 \oplus B_2, E_2 \oplus B_2 \oplus B_2, B_2 \oplus B_2 \oplus B_2$
2+2+1+1	$F_2 \oplus F_2 \oplus 1 \oplus 1, F_2 \oplus E_2 \oplus 1 \oplus 1, F_2 \oplus B_2 \oplus 1 \oplus 1, E_2 \oplus E_2 \oplus 1 \oplus 1, E_2 \oplus B_2 \oplus 1 \oplus 1, B_2 \oplus B_2 \oplus 1 \oplus 1$
2+1+1+1+1	$F_2 \oplus 1 \oplus 1 \oplus 1 \oplus 1, E_2 \oplus 1 \oplus 1 \oplus 1 \oplus 1, B_2 \oplus 1 \oplus 1 \oplus 1 \oplus 1$
1+1+1+1+1+1	$1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1$

Table 1. Canonical form for congruence of 6×6 matrices.

Choose $\dim(L^2) = n - 2$ and $Leib(L) = \text{span}\{u_n\}$. We can extend it to a basis $\{u_3, u_4, \ldots, u_{n-1}, u_n\}$ for L^2 and let a subspace U be complementary to L^2 in L so that $L = L^2 \oplus U$. So $[u, v] = \alpha_3 u_3 + \alpha_4 u_4 + \alpha_{n-1} u_{n-1} + \alpha_n u_n$ for some $\alpha_i \in \mathbb{C},\ 3 \le i \le n$, for any $u, v \in U$. Bilinear form $f(\ ,\ ): U \times U \to \mathbb{C}$ can be defined by $f(u, v) = \alpha_n$ for all $u, v \in U$.

Choosing a basis $\{u_1, u_2\}$ for U and using Theorem 2.6 we see that the matrix of the bilinear form $f(\ ,\ ): U \times U \to \mathbb{C}$ is one of the following:

$$(i)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (ii)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (iii)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (iv)\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad (v)\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$$

where $c \neq 1, -1$. We can assume that N cannot be the matrix (i) because the resulting algebra is a Lie algebra. It is enough to consider the matrices (ii) and (iii) because others are isomorphic to one of these as showed in Lemma 2.1 in [6].

3. Classification of 7-dimensional odd-nilpotent Leibniz algebras with $\dim(Leib(L)) = 1$

Let L be a 7−dimensional complex nilpotent Leibniz algebra with $\dim(Leib(L)) = 1$. All possible odd-nilpotent Leibniz algebra characters are listed below:

- $\chi(L) = (7, 1, 0, 0, 0)$
- $\chi(L) = (7, 3, 0, 0, 0)$
- $\chi(L) = (7, 3, 1, 0, 0)$
- $\chi(L) = (7, 5, 0, 0, 0)$
- $\chi(L) = (7, 5, 1, 0, 0)$
- $\chi(L) = (7, 5, 3, 0, 0)$
- $\chi(L) = (7, 5, 3, 1, 0)$

If $\chi(L) = (7,5,0,0,0)$ or $\chi(L) = (7,5,1,0,0)$ then by Lemma 2.4 we see that no Leibniz algebra exists. If $\chi(L) = (7,5,3,0,0)$ then Lemma 2.1 requires $L^3 \subseteq Z(L) \subset L^2$. Hence $\dim(Z(L)) = 3$ or $\dim(Z(L)) = 4$. However $\dim(Z(L)) \neq 4$ since Lemma 2.3. Hence suppose $\dim(Z(L)) = 3$. Then $L^3 = Z(L)$ with Lemma 2.2 implies that $Leib(L) \subseteq L^3$ but from Lemma 2.4 (ii) we arrive a contradiction. Hence there is no Leibniz algebra for the case $\chi(L) = (7,5,3,0,0)$.

Theorem 3.1. *Let $\chi(L) = (7,1,0,0,0)$ and $\dim(Leib(L)) = 1$. Then, up to isomorphism, the nonzero multiplications in L is given by one of the following:*

L_1: $[w_1,w_6] = w_7, [w_2,w_5] = w_7, [w_2,w_6] = \alpha w_7, [w_3,w_4] = w_7, [w_3,w_5] = \alpha w_7, [w_4,w_3] = \alpha w_7, [w_5,w_2] = \alpha w_7, [w_5,w_3] = w_7, [w_6,w_1] = \alpha w_7, [w_6,w_2] = w_7, \quad \alpha \in \mathbb{C}\backslash\{1,-1\}$.

L_2: $[w_1,w_6] = w_7, [w_2,w_5] = w_7, [w_2,w_6] = w_7, [w_3,w_4] = w_7, [w_3,w_5] = w_7, [w_4,w_3] = -w_7, [w_4,w_4] = w_7, [w_5,w_2] = -w_7, [w_5,w_3] = w_7, [w_6,w_1] = -w_7, [w_6,w_2] = w_7$.

L_3: $[w_1,w_6] = w_7, [w_2,w_5] = w_7, [w_2,w_6] = w_7, [w_3,w_4] = w_7, [w_3,w_5] = w_7, [w_4,w_3] = -w_7, [w_5,w_2] = -w_7, [w_5,w_3] = w_7, [w_6,w_1] = -w_7, [w_6,w_2] = w_7$.

L_4: $[w_1,w_4] = w_7, [w_2,w_5] = w_7, [w_4,w_2] = w_7, [w_5,w_3] = w_7, [w_6,w_6] = w_7$.

L_5: $[w_1,w_5] = w_7, [w_2,w_4] = w_7, [w_2,w_5] = w_7, [w_3,w_3] = w_7, [w_3,w_4] = w_7, [w_4,w_2] = w_7, [w_4,w_3] = -w_7, [w_5,w_1] = w_7, [w_5,w_2] = -w_7, [w_6,w_6] = w_7$.

L_6: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = \alpha w_7, [w_3,w_2] = \alpha w_7, [w_4,w_1] = \alpha w_7, [w_4,w_2] = w_7, [w_5,w_6] = w_7, [w_6,w_5] = -w_7, \quad \alpha \in \mathbb{C}\backslash\{1,-1\}$.

L_7: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = \alpha w_7, [w_3,w_2] = \alpha w_7, [w_4,w_1] = \alpha w_7, [w_4,w_2] = w_7, [w_5,w_6] = w_7, [w_6,w_5] = -w_7, [w_6,w_6] = w_7, \quad \alpha \in \mathbb{C}\backslash\{1,-1\}$.

L_8: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = \alpha_1 w_7, [w_3,w_2] = \alpha_1 w_7, [w_4,w_1] = \alpha_1 w_7, [w_4,w_2] = w_7, [w_5,w_6] = w_7, [w_6,w_5] = \alpha_2 w_7, \quad \alpha_1,\alpha_2 \in \mathbb{C}\backslash\{1,-1\}$.

L_9: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = w_7, [w_4,w_1] = w_7, [w_4,w_2] = -w_7, [w_5,w_6] = w_7, [w_6,w_5] = -w_7$.

L_{10}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = w_7, [w_4,w_1] = w_7, [w_4,w_2] = -w_7, [w_5,w_6] = w_7, [w_6,w_5] = -w_7, [w_6,w_6] = w_7$.

L_{11}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = w_7, [w_4,w_1] = w_7, [w_4,w_2] = -w_7, [w_5,w_6] = w_7, [w_6,w_5] = \alpha w_7, \quad \alpha \in \mathbb{C}\backslash\{1,-1\}$.

L_{12}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = -w_7, [w_3,w_3] = w_7, [w_4,w_1] = -w_7, [w_4,w_2] = w_7, [w_5,w_6] = w_7, [w_6,w_5] = -w_7$.

L_{13}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = -w_7, [w_3,w_3] = w_7, [w_4,w_1] = -w_7, [w_4,w_2] = w_7, [w_5,w_6] = w_7, [w_6,w_5] = -w_7, [w_6,w_6] = w_7$.

L_{14}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = -w_7, [w_3,w_3] = w_7, [w_4,w_1] = -w_7, [w_4,w_2] = w_7, [w_5,w_6] = w_7, [w_6,w_5] = \alpha w_7, \quad \alpha \in \mathbb{C}\backslash\{1,-1\}$.

L_{15}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = \alpha w_7, [w_3,w_2] = \alpha w_7, [w_4,w_1] = \alpha w_7, [w_4,w_2] = w_7, [w_5,w_5] = w_7, [w_6,w_6] = w_7 \quad \alpha \in \mathbb{C}\backslash\{1,-1\}$.

L_{16}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = w_7, [w_4,w_1] = w_7, [w_4,w_2] = -w_7, [w_5,w_5] = w_7, [w_6,w_6] = w_7$.

L_{17}: $[w_1,w_4] = w_7, [w_2,w_3] = w_7, [w_2,w_4] = w_7, [w_3,w_2] = -w_7, [w_3,w_3] = w_7, [w_4,w_1] = -w_7, [w_4,w_2] = w_7, [w_5,w_5] = w_7, [w_6,w_6] = w_7$.

L_{18}: $[w_1,w_3] = w_7, [w_3,w_2] = w_7, [w_4,w_6] = w_7, [w_6,w_5] = w_7$.

L_{19}: $[w_1,w_3] = w_7, [w_3,w_2] = w_7, [w_4,w_6] = w_7, [w_5,w_5] = w_7, [w_5,w_6] = w_7, [w_6,w_4] = w_7, [w_6,w_5] = -w_7$.

L_{20}: $[w_1,w_3] = w_7, [w_2,w_2] = w_7, [w_2,w_3] = w_7, [w_3,w_1] = w_7, [w_3,w_2] = -w_7, [w_4,w_6] = w_7, [w_5,w_5] = w_7, [w_5,w_6] = w_7, [w_6,w_4] = w_7, [w_6,w_5] = -w_7$.

L_{21}: $[w_1,w_3] = w_7, [w_3,w_2] = w_7, [w_4,w_5] = w_7, [w_5,w_4] = -w_7, [w_6,w_6] = w_7$.

L_{22}: $[w_1,w_3] = w_7, [w_3,w_2] = w_7, [w_4,w_5] = w_7, [w_5,w_4] = -w_7, [w_5,w_5] = w_7, [w_6,w_6] = w_7$.

L_{23}: $[w_1,w_3] = w_7, [w_3,w_2] = w_7, [w_4,w_5] = w_7, [w_5,w_4] = \alpha w_7, [w_6,w_6] = w_7, \quad \alpha \in \mathbb{C}\backslash\{1,-1\}$.

L_{24}: $[w_1, w_3] = w_7, [w_2, w_2] = w_7, [w_2, w_3] = w_7, [w_3, w_1] = w_7, [w_3, w_2] = -w_7, [w_4, w_5] = w_7, [w_5, w_4] = -w_7, [w_6, w_6] = w_7.$

L_{25}: $[w_1, w_3] = w_7, [w_2, w_2] = w_7, [w_2, w_3] = w_7, [w_3, w_1] = w_7, [w_3, w_2] = -w_7, [w_4, w_5] = w_7, [w_5, w_4] = -w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{26}: $[w_1, w_3] = w_7, [w_2, w_2] = w_7, [w_2, w_3] = w_7, [w_3, w_1] = w_7, [w_3, w_2] = -w_7, [w_4, w_5] = w_7, [w_5, w_4] = \alpha w_7, [w_6, w_6] = w_7, \quad \alpha \in \mathbb{C}\backslash\{1, -1\}.$

L_{27}: $[w_1, w_3] = w_7, [w_3, w_2] = w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{28}: $[w_1, w_3] = w_7, [w_2, w_2] = w_7, [w_2, w_3] = w_7, [w_3, w_1] = w_7, [w_3, w_2] = -w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{29}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_5, w_6] = w_7, [w_6, w_5] = -w_7, [w_6, w_6] = w_7.$

L_{30}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_5, w_6] = w_7, [w_6, w_5] = \alpha w_7, \quad \alpha \in \mathbb{C}\backslash\{1, -1\}.$

L_{31}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_4, w_4] = w_7, [w_5, w_6] = w_7, [w_6, w_5] = -w_7, [w_6, w_6] = w_7.$

L_{32}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_4, w_4] = w_7, [w_5, w_6] = w_7, [w_6, w_5] = \alpha w_7, \quad \alpha \in \mathbb{C}\backslash\{1, -1\}.$

L_{33}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = \alpha_1 w_7, [w_5, w_6] = w_7, [w_6, w_5] = \alpha_2 w_7, \quad \alpha_1, \alpha_2 \in \mathbb{C}\backslash\{1, -1\}.$

L_{34}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_2, w_2] = w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_4, w_4] = w_7, [w_5, w_6] = w_7, [w_6, w_5] = -w_7, [w_6, w_6] = w_7.$

L_{35}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_2, w_2] = w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_4, w_4] = w_7, [w_5, w_6] = w_7, [w_6, w_5] = \alpha w_7, \quad \alpha \in \mathbb{C}\backslash\{1, -1\}.$

L_{36}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_2, w_2] = w_7, [w_3, w_4] = w_7, [w_4, w_3] = \alpha_1 w_7, [w_5, w_6] = w_7, [w_6, w_5] = \alpha_2 w_7, \quad \alpha_1, \alpha_2 \in \mathbb{C}\backslash\{1, -1\}.$

L_{37}: $[w_1, w_2] = w_7, [w_2, w_1] = \alpha_1 w_7, [w_3, w_4] = w_7, [w_4, w_3] = \alpha_2 w_7, [w_5, w_6] = w_7, [w_6, w_5] = \alpha_3 w_7, \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}\backslash\{1, -1\}.$

L_{38}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{39}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{40}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_4] = w_7, [w_4, w_3] = \alpha w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7, \quad \alpha \in \mathbb{C}\backslash\{1, -1\}.$

L_{41}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_2, w_2] = w_7, [w_3, w_4] = w_7, [w_4, w_3] = -w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{42}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_2, w_2] = w_7, [w_3, w_4] = w_7, [w_4, w_3] = \alpha w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7, \quad \alpha \in \mathbb{C}\backslash\{1, -1\}.$

L_{43}: $[w_1, w_2] = w_7, [w_2, w_1] = \alpha_1 w_7, [w_3, w_4] = w_7, [w_4, w_3] = \alpha_2 w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7, \quad \alpha_1, \alpha_2 \in \mathbb{C}\backslash\{1, -1\}.$

L_{44}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_3, w_3] = w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{45}: $[w_1, w_2] = w_7, [w_2, w_1] = -w_7, [w_2, w_2] = w_7, [w_3, w_3] = w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

L_{46}: $[w_1, w_2] = w_7, [w_2, w_1] = \alpha w_7, [w_3, w_3] = w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7, \quad \alpha \in \mathbb{C}\backslash\{1, -1\}.$

L_{47}: $[w_1, w_1] = w_7, [w_2, w_2] = w_7, [w_3, w_3] = w_7, [w_4, w_4] = w_7, [w_5, w_5] = w_7, [w_6, w_6] = w_7.$

Proof. Let $L^2 = Leib(L) = \text{span}\{w_7\}$. Then there exists an ordered basis $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ of L and the matrices listed in Table 1 results 47 pairwise nonisomorphic Leibniz algebras. □

Now let $\chi(L) = (7,5,3,1,0)$. Using Lemma 2.2 we get $Leib(L) \subseteq Z(L)$. Also from Lemma 2.4 we obtain $Leib(L) \nsubseteq L^3$. Lemma 2.1 requires $L^4 \subseteq Z(L) \subset L^2$. From Lemma 2.5 we have $\dim(Z(L)) < 4$. If $\dim(Z(L)) = 1$ then $Leib(L) = Z(L) = L^4 \subseteq L^3$ leads a contradiction. Hence $\dim(Z(L)) = 2$ or $\dim(Z(L)) = 3$.

Theorem 3.2. *Let $\chi(L) = (7,5,3,1,0), \dim(Leib(L)) = 1$ and $\dim(Z(L)) = 3$. Then, up to isomorphism, the nonzero multiplications in L is given by one of the following:*

- L_1**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_1,w_3] = w_4 = -[w_3,w_1], [w_2,w_3] = w_5 = -[w_3,w_2], [w_1,w_4] = w_6 = -[w_4,w_1]$.
- L_2**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_1,w_3] = w_5 = -[w_3,w_1], [w_2,w_3] = w_4 = -[w_3,w_2], [w_2,w_4] = w_6 = -[w_4,w_2]$.
- L_3**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_2,w_2] = w_7, [w_1,w_3] = w_4 = -[w_3,w_1], [w_2,w_3] = w_5 = -[w_3,w_2], [w_1,w_4] = w_6 = -[w_4,w_1]$.
- L_4**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_2,w_2] = w_7, [w_1,w_3] = w_4 = -[w_3,w_1], [w_2,w_3] = iw_4+w_5 = -[w_3,w_2], [w_1,w_4] = w_6 = -[w_4,w_1], [w_2,w_4] = iw_6 = -[w_4,w_2], i = \sqrt{-1}$.

Proof. Let us take a complementary subspace W to L^3 in L^2. Since $L^4 \neq 0$ we have $L^3 \neq Z(L)$. Also if $\dim(L^3 \cap Z(L)) = 1$ then $W \subseteq Z(L)$ and since

$$L^3 = [L, L^2] = [L, L^3 \oplus W] = L^4$$

we arrive a contradiction. Therefore $\dim(L^3 \cap Z(L)) = 2$. Using $Leib(L) \nsubseteq L^3, L^4$, choose $Leib(L) = \text{span}\{e_7\}, L^4 = \text{span}\{e_6\}$ and $L^3 = \text{span}\{e_4,e_5,e_6\}$. Then $Z(L) = \text{span}\{e_5,e_6,e_7\}$ and $L^2 = \text{span}\{e_3,e_4,e_5,e_6,e_7\}$. Take $V = \text{span}\{\mathrm{e}_1,\mathrm{e}_2\}$.

Case 1: If the matrix $N = (ii)$, then the nontrivial multiplications in L given as follows: $[e_1,e_1] = e_7, [e_1,e_2] = \alpha_1e_3 + \alpha_2e_4 + \alpha_3e_5 + \alpha_4e_6 = -[e_2,e_1], [e_1,e_3] = \beta_1e_4 + \beta_2e_5 + \beta_3e_6 = -[e_3,e_1], [e_2,e_3] = \beta_4e_4 + \beta_5e_5 + \beta_6e_6 = -[e_3,e_2], [e_1,e_4] = \gamma_1e_6 = -[e_4,e_1], [e_2,e_4] = \gamma_2e_6 = -[e_4,e_2], [e_3,e_4] = \gamma_3e_6 = -[e_4,e_3]$.

From Leibniz identities we get the following equations:

$$\begin{cases}\gamma_3 = 0\\ \beta_4\gamma_1 - \beta_1\gamma_2 = 0\end{cases} \tag{3.1}$$

First suppose $\gamma_2 = 0$. Then $\gamma_1 \neq 0$ and from the second equation in (3.1) we have $\beta_4 = 0$. Using $\dim(L^3) = 3$ we can see that $\beta_1,\beta_5 \neq 0$. Then the base change $w_1 = e_1, w_2 = e_2, w_3 = \alpha_1e_3 + \alpha_2e_4 + \alpha_3e_5 + \alpha_4e_6, w_4 = \alpha_1(\beta_1e_4 + \beta_2e_5 + \beta_3e_6) + \alpha_2\gamma_1e_6, w_5 = \alpha_1(\beta_5e_5 + \beta_6e_6), w_6 = \alpha_1\beta_1\gamma_1e_6, w_7 = e_7$ shows L is isomorphic to L_1. Now suppose $\gamma_2 \neq 0$. Then with the base change $w_1 = \gamma_2e_1 - \gamma_1e_2, w_2 = e_2, w_3 = e_3, w_4 = e_4, w_5 = e_5, w_6 = e_6, w_7 = \gamma_2^2e_7$ we can force $\gamma_1 = 0$. Then from the second equation in (3.1) we get $\beta_1 = 0$. So $\beta_2 \neq 0$ since $\dim(L^3) = 3$. Then the base change $w_1 = e_1, w_2 = e_2, w_3 = \alpha_1e_3+\alpha_2e_4+\alpha_3e_5+\alpha_4e_6, w_4 = \alpha_1(\beta_1e_4 + \beta_2e_5 + \beta_3e_6) + \alpha_2\gamma_2e_6, w_5 = \alpha_1(\beta_2e_5 + \beta_3e_6), w_6 = \alpha_1\beta_4\gamma_2e_6, w_7 = e_7$ shows L is isomorphic to L_2.

Case 2: If the matrix $N = (iii)$, then the nontrivial multiplications in L given as follows: $[e_1,e_1] = e_7, [e_1,e_2] = \alpha_1e_3+\alpha_2e_4+\alpha_3e_5+\alpha_4e_6 = -[e_2,e_1], [e_2,e_2] = e_7, [e_1,e_3] = \beta_1e_4+\beta_2e_5+\beta_3e_6 = -[e_3,e_1], [e_2,e_3] = \beta_4e_4 + \beta_5e_5 + \beta_6e_6 = -[e_3,e_2], [e_1,e_4] = \gamma_1e_6 = -[e_4,e_1], [e_2,e_4] = \gamma_2e_6 = -[e_4,e_2], [e_3,e_4] = \gamma_3e_6 = -[e_4,e_3]$.

Then again Leibniz identities yield the equations in (3.1). Let $\gamma_2 = 0$. Then $\gamma_1 \neq 0$ since $\dim(Z(L)) = 3$. From (3.1) we have $\beta_4 = 0$. Using $\dim(L^3) = 3$ we obtain $\beta_1,\beta_5 \neq 0$. Then the base change $w_1 = e_1, w_2 = e_2, w_3 = \alpha_1e_3+\alpha_2e_4+\alpha_3e_5+\alpha_4e_6, w_4 = \alpha_1(\beta_1e_4+\beta_2e_5+\beta_3e_6)+\alpha_2\gamma_1e_6, w_5 = \alpha_1(\beta_5e_5+\beta_6e_6), w_6 = \alpha_1\beta_1\gamma_1e_6, w_7 = e_7$ shows L is isomorphic to L_3. Now take $\gamma_2 \neq 0$. If $\gamma_1 = 0$ then the base change $w_1 = e_2, w_2 = e_1, w_3 = e_3, w_4 = e_4, w_5 = e_5, w_6 = e_6, w_7 = e_7$ forces $\gamma_2 = 0$ and therefore L is isomorphic to L_3. So let $\gamma_1 \neq 0$. Suppose $\gamma_1^2 + \gamma_2^2 \neq 0$. Then the base change $w_1 = \gamma_1e_1 + \gamma_2e_2, w_2 = \gamma_2e_1 - \gamma_1e_2, w_3 = e_3, w_4 = e_4, w_5 = e_5, w_6 = e_6, w_7 = (\gamma_1^2+\gamma_2^2)e_7$ forces $\gamma_2 = 0$ and therefore L is isomorphic to L_3. Now take $\gamma_1^2+\gamma_2^2 = 0$.

Then from the second equation in (3.1) we obtain $\beta_1^2 + \beta_4^2 = 0$. Then the base change $w_1 = e_1, w_2 = e_2, w_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 + \alpha_4 e_6, w_4 = \alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \alpha_2\gamma_1 e_6, w_5 = \alpha_1[(\beta_5 - \beta_2 i)e_5 + (\beta_6 - \beta_3 i)e_6], w_6 = \alpha_1\beta_1\gamma_1 e_6, w_7 = e_7$ shows L is isomorphic to L_4. □

Theorem 3.3. *Let* $\chi(L) = (7,5,3,1,0), \dim(Leib(L)) = 1$ *and* $\dim(Z(L)) = 2$. *Then, up to isomorphism, the nonzero multiplications in L is given by one of the following:*

- L_5**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_1,w_3] = w_4 = -[w_3,w_1], [w_2,w_3] = w_5 = -[w_3,w_2], [w_2,w_4] = w_6 = -[w_4,w_2], [w_1,w_5] = w_6 = -[w_5,w_1]$.
- L_6**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_1,w_3] = w_5 = -[w_3,w_1], [w_2,w_3] = w_4 = -[w_3,w_2], [w_1,w_4] = \alpha w_6 = -[w_4,w_1], [w_2,w_4] = w_6 = -[w_4,w_2], [w_1,w_5] = w_6 = -[w_5,w_1], \quad \alpha \in \mathbb{C}$.
- L_7**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_2,w_2] = w_7, [w_1,w_3] = w_4 = -[w_3,w_1], [w_2,w_3] = w_5 = -[w_3,w_2], [w_1,w_4] = \alpha w_6 = -[w_4,w_1], [w_2,w_4] = w_6 = -[w_4,w_2], [w_1,w_5] = w_6 = -[w_5,w_1], \quad \alpha \in \mathbb{C}$.
- L_8**:** $[w_1,w_1] = w_7, [w_1,w_2] = w_3 = -[w_2,w_1], [w_2,w_2] = w_7, [w_1,w_3] = w_5 = -[w_3,w_1], [w_2,w_3] = w_4 = -[w_3,w_2], [w_1,w_4] = \alpha_1 w_6 = -[w_4,w_1], [w_2,w_4] = w_6 = -[w_4,w_2], [w_1,w_5] = \alpha_2 w_6 = -[w_5,w_1], [w_2,w_5] = \alpha_1 w_6 = -[w_5,w_2], \quad \alpha_1,\alpha_2 \in \mathbb{C}$.

Proof. Choose $Leib(L) = \text{span}\{e_7\}, L^4 = \text{span}\{e_6\}$. Then $Z(L) = \text{span}\{e_6,e_7\}$ and $L^3 = \text{span}\{e_4,e_5,e_6\}$,
$L^2 = \text{span}\{e_3,e_4,e_5,e_6,e_7\}$. Take $V = \text{span}\{\mathrm{e}_1,\mathrm{e}_2\}$.
Case 1: If the matrix $N = (ii)$, then the nontrivial multiplications in L given as follows: $[e_1,e_1] = e_7, [e_1,e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 + \alpha_4 e_6 = -[e_2,e_1], [e_1,e_3] = \beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6 = -[e_3,e_1], [e_2,e_3] = \beta_4 e_4 + \beta_5 e_5 + \beta_6 e_6 = -[e_3,e_2], [e_1,e_4] = \gamma_1 e_6 = -[e_4,e_1], [e_2,e_4] = \gamma_2 e_6 = -[e_4,e_2], [e_3,e_4] = \gamma_3 e_6 = -[e_4,e_3], [e_1,e_5] = \theta_1 e_6 = -[e_5,e_1], [e_2,e_5] = \theta_2 e_6 = -[e_5,e_2], [e_3,e_5] = \theta_3 e_6 = -[e_5,e_3], [e_4,e_5] = \theta_4 e_6 = -[e_5,e_4]$.
From Leibniz identities we get the following equations:

$$\begin{cases} \theta_4 = \gamma_3 = \theta_3 = 0 \\ \beta_4\gamma_1 + \beta_5\theta_1 - \beta_1\gamma_2 - \beta_2\theta_2 = 0 \end{cases} \tag{3.2}$$

The base change $w_1 = e_1, w_2 = e_2, w_3 = e_3, w_4 = e_4, w_5 = \theta_2 e_4 - \gamma_2 e_5, w_6 = e_6, w_7 = e_7$ forces $\theta_2 = 0$. Hence let $\theta_2 = 0$. If $\gamma_2 = 0$ then $\theta_1 e_4 - \gamma_1 e_5 \in Z(L)$, which contradicts with the fact that $\dim(Z(L)) = 2$. Let $\gamma_2 \neq 0$. Then with the base change $w_1 = \gamma_2 e_1 - \gamma_1 e_2, w_2 = e_2, w_3 = e_3, w_4 = e_4, w_5 = e_5, w_6 = e_6, w_7 = \gamma_2^2 e_7$ we can make $\gamma_1 = 0$. If $\beta_4 = 0$ then the base change $w_1 = e_1, w_2 = e_2, w_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 + \alpha_4 e_6, w_4 = \alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \alpha_3\theta_1 e_6, w_5 = \alpha_1(\beta_5 e_5 + \beta_6 e_6) + \alpha_2\gamma_2 e_6, w_6 = \alpha_1\beta_1\gamma_2 e_6, w_7 = e_7$ shows L is isomorphic to L_5. If $\beta_4 \neq 0$ then the base change $w_1 = e_1, w_2 = \sqrt{\frac{\beta_2\theta_1}{\beta_4\gamma_2}} e_2, w_3 = \sqrt{\frac{\beta_2\theta_1}{\beta_4\gamma_2}}(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 + \alpha_4 e_6), w_4 = \frac{\beta_2\theta_1}{\beta_4\gamma_2}[\alpha_1(\beta_4 e_4 + \beta_5 e_5 + \beta_6 e_6) + \alpha_2\gamma_2 e_6], w_5 = \sqrt{\frac{\beta_2\theta_1}{\beta_4\gamma_2}}[\alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \alpha_3\theta_1 e_6], w_6 = \sqrt{\frac{\beta_2\theta_1}{\beta_4\gamma_2}}\alpha_1\beta_2\theta_1 e_6, w_7 = e_7$ shows L is isomorphic to $L_6(\alpha)$.
Case 2: If the matrix $N = (iii)$, then the nontrivial multiplications in L given as follows: $[e_1,e_1] = e_7, [e_1,e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 + \alpha_4 e_6 = -[e_2,e_1], [e_2,e_2] = e_7, [e_1,e_3] = \beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6 = -[e_3,e_1], [e_2,e_3] = \beta_4 e_4 + \beta_5 e_5 + \beta_6 e_6 = -[e_3,e_2], [e_1,e_4] = \gamma_1 e_6 = -[e_4,e_1], [e_2,e_4] = \gamma_2 e_6 = -[e_4,e_2], [e_3,e_4] = \gamma_3 e_6 = -[e_4,e_3], [e_1,e_5] = \theta_1 e_6 = -[e_5,e_1]$,
$[e_2,e_5] = \theta_2 e_6 = -[e_5,e_2], [e_3,e_5] = \theta_3 e_6 = -[e_5,e_3], [e_4,e_5] = \theta_4 e_6 = -[e_5,e_4]$.
Then again Leibniz identities yield the equations in (3.2). The base change $w_1 = e_1, w_2 = e_2, w_3 = e_3, w_4 = e_4, w_5 = \theta_2 e_4 - \gamma_2 e_5, w_6 = e_6, w_7 = e_7$ forces $\theta_2 = 0$. So let $\theta_2 = 0$. Then $\theta_1, \gamma_2 \neq 0$ since $\dim(Z(L)) = 2$. If $\beta_4 = 0$ then the base change $w_1 = e_1, w_2 = e_2, w_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 + \alpha_4 e_6, w_4 = \alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + (\alpha_2\gamma_1 + \alpha_3\theta_1)e_6, w_5 = \alpha_1(\beta_5 e_5 + \beta_6 e_6) + \alpha_2\gamma_2 e_6, w_6 = \alpha_1\beta_5\theta_1 e_6, w_7 = e_7$ shows L is isomorphic to $L_7(\alpha)$. If $\beta_4 \neq 0$ then the base change $w_1 = e_1, w_2 = e_2, w_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 + \alpha_4 e_6, w_4 = \alpha_1(\beta_4 e_4 + \beta_5 e_5 + \beta_6 e_6) +$

$(\alpha_2\gamma_2)e_6, w_5 = \alpha_1(\beta_4 e_4 + \beta_5 e_5 + \beta_6 e_6) + (\alpha_2\gamma_1 + \alpha_3\theta_1)e_6, w_6 = \alpha_1\beta_4\gamma_2 e_6, w_7 = e_7$ shows L is isomorphic to $L_8(\alpha, \beta)$. □

For the remaining cases $\chi(L) = (7,3,0,0,0)$ and $\chi(L) = (7,3,1,0,0)$ the same technique can be applied. Notice that the 4×4 matrices from Theorem 2.6 will yield the desired algebras.

4. Conclusion

The number of isomorphism classes for some subclasses odd-nilpotent Leibniz algebras of dimension ≤ 7 given in Table 2. Considering there are only 6 non-split complex nilpotent Lie algebras of dimension 5, we can claim that the classification problem for Leibniz algebras is indeed wild.

Characteristic of L	**Number of isomorphism classes of complex odd-nilpotent Leibniz algebras with** $\dim(Leib(L)) = 1$
$\chi(L) = (3,1,0,0,0)$	2 single algebras, 1 one-parameter infinite family.
$\chi(L) = (5,1,0,0,0)$	9 single algebras, 4 one-parameter infinite families, 1 two-parameter infinite family
$\chi(L) = (5,3,0,0,0)$	No Leibniz algebra.
$\chi(L) = (5,3,1,0,0)$	4 single algebras.
$\chi(L) = (7,1,0,0,0)$	30 single algebras, 13 one-parameter infinite families, 3 two-parameter infinite families, 1 three-parameter infinite family
$\chi(L) = (7,5,0,0,0)$	No Leibniz algebra.
$\chi(L) = (7,5,1,0,0)$	No Leibniz algebra.
$\chi(L) = (7,5,3,0,0)$	No Leibniz algebra.
$\chi(L) = (7,5,3,1,0)$	5 single algebras, 2 one-parameter infinite families, 1 two-parameter infinite family

Table 2. Number of isomorphism classes of odd-nilpotent Leibniz algebras.

As a future work, we can extend the canonical forms for the congruence technique to higher dimensions to obtain complete classification of complex odd-nilpotent Leibniz algebras.

References

[1] S. Albeverio, B. A. Omirov and I. S. Rakhimov, *Classification of 4-dimensional nilpotent complex Leibniz algebras*, Extracta Mathematicae **21**(3), 197-210, 2006.

[2] S. A. Ayupov and B. A. Omirov, *On 3-dimensional Leibniz algebras*, Uzbek. Math. Zh. **1**, 9-14, 1999.

[3] A. Bloh, *On a generalization of Lie algebra notion*, Math. in USSR Doklady **165**(3), 471-473, 1965.

[4] J.M. Casas, M.A. Insua, M. Ladra and S. Ladra, *An algorithm for the classification of 3-dimensional complex Leibniz algebras*, Linear Algebra Appl. **9**, 3747-3756, 2012. DOI: 10.1016/j.laa.2011.11.039

[5] I. Demir, K. C. Misra and E. Stitzinger, *On classification of four-dimensional nilpotent Leibniz Algebras*, Comm. Algebra **45**(3), 1012-1018, 2017. DOI: 10.1080/00927872.2016.1172626.

[6] I. Demir, *Classification of* 5−*dimensional complex nilpotent Leibniz algebras*, in Representations of Lie Algebras, Quantum Groups and Related Topics, 95-120, Contemp. Math. **713**, Amer. Math. Soc., Providence, RI, 2018.

[7] I. Demir, *Classification of some subclasses of 6-dimensional nilpotent Leibniz algebras.*, Turk J. Math. **44**, 1925-1940, 2020.

[8] J. L. Loday, *Une version non commutative des algebres de Lie: les algebres de Leibniz*, Enseign. Math. (2) **39**(3-4), 269-293, 1993.

[9] I. M. Rikhsiboev and I. S. Rakhimov, *Classification of three dimensional complex Leibniz algebras*, AIP Conference Proc. **1450**, 358-362, 2012. DOI: 10.1063/1.4724168.

[10] F. D. Teran, *Canonical forms for congruence of matrices: a tribute to H. W. Turnbull and A. C. Aitken*, 2nd meeting on Linear Algebra, Matrix Analysis and Applications, (ALAMA 2010), Valencia(Spain), 2010.