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One Parameter Commutative Octonions

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Abstract

Hyperbolic numbers had been developed in the 19th century. Octonions forms a noncommutative and nonassociative normed division algebra over reals. Octonions have many applications in fields of physics such as quantum logic and string theory. Cayley-Dickson process is applied to quaternions in order to construct octonions and in a sense, we follow a similar process. The aim of this study is to introduce the concept of commutative octonions. We construct this algebra by using some matrix methods. After construction, we give a number of properties of commutative octonions such as fundamental matrices and principal conjugates. We also obtain representation of a commutative octonion as decomposed form, holomorphic and analytic functions of commutative octonions.

Keywords: *Commutative octonions; Segre's quaternions; holomorphic functions*
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1. Introduction

Four-dimensional hyper complex numbers, real quaternions, were introduced by Hamilton in 1843 to extend complex numbers [7].The set of real quaternions are

$$\mathbb{H} = \{x_0 + x_1 i + x_2 j + x_3 k : i, j, k \notin \mathbb{R} \text{ and } x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where the multiplication rules of the elements of the ordered basis $\{1, i, j, k\}$ is

$$i^2 = j^2 = k^2 = -1, -ji = ij = k, -kj = jk = i \text{ and } -ik = ki = j.$$

One can see that quaternions are not commutative. The Hamilton quaternions form a division algebra and in this aspect, they can be regarded as an extension of complex numbers. Octonions have many applications in field. For instance, we can give M-theory cosmology [6], quantum theory [5, 12], cough monitoring [8], space time coding [13], electromagnetic and gravitational equations [16] as most striking examples. Segre gave a new type of quaternions whose multiplication rule has commutative property [11]. These numbers are called commutative quaternions or Segre's quaternions. The set of commutative quaternions is [3]

$$\mathrm{Q} = \{x_0 + x_1 i + x_2 j + x_3 k : i, j, k \notin \mathbb{R} \text{ and } x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where the versors satisfy

$$i^2 = k^2 = -1, j^2 = 1, ji = ij = k, -kj = jk = i \text{ and } ik = ki = -j.$$

A more general commutative quaternions were given by Catoni et al. [3]. They represented these numbers by

$$\mathbb{S} = \{x_0 + x_1 i + x_2 j + x_3 k : i, j, k \notin \mathrm{R} \text{ and } x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where the versors satisfy the following multiplication rules where δ is an arbitrary real number.

Table 1: Multiplication rules of the elements of $\{1, i, j, k\}$.

1	i	j	k
i	δ	k	δj
j	k	1	i
k	δj	i	δ

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They also examined algebraic properties of this type of quaternions. Following their work [2, 3], we construct the commutative octonions. This study can be regarded as an application of [2]. Kosal et al. [9] studied matrices for commutative quaternions and gave some interesting properties.
After discovery of quaternion algebra, Cayley and Graves gave octonion algebra independently. Octonions algebra is constructed by using the Cayley-Dickson method. An octonion o can be written as

$$o = p + p'e$$

where $p, p' \in \mathbb{Q}$ and e is a new imaginary unit, i.e. it is a square root of -1. Let $o_1 = p_1 + p'_1 e$ and $o_2 = p_2 + p'_2 e$ be two any octonions. Addition and multiplication of these two octonions are

$$o_1 + o_2 = p_1 + p_1 + (p'_1 + p'_2)e,$$
$$o_1 o_2 = (p_1 p_2 - \overline{p'_2} p'_1) + (p'_2 p_1 + p'_1 \overline{p'_2})$$

where $\overline{q}$ is the conjugate of the quaternion q. The ordered basis for octonion algebra over $\mathbb{R}$ consists of the elements

$$e_0 = 1,\ e_1 = i,\ e_2 = j,\ e_3 = k,\ e_4 = t,\ e_5 = it,\ e_6 = jt,\ e_7 = kt$$

where t is another versor different from $\{1, i, j, k\}$, and any octonion o can be expressed as

$$o = \sum_{i=0}^{7} a_i e_i,\ \ a_i \in \mathbb{R}.$$

Thus, there are eight objects $e_i \ \ (i = 0, \dots, 7)$ in the ordered basis of octonion algebra. The multiplication rules of the elements of standard basis $\{e_0, e_1, e_2, \cdots, e_7\}$ for octonions algebra can be found in [10]. The octonions division algebra over the real numbers $\mathbb{R}$ is a non-commutative and non-associative algebra.
There are some studies on octonions whose coefficients are well-known integer sequences. We can refer to [1, 14, 15] for this type of studies.

2. Commutative Octonions

By following Catoni et al [3], for any real number α, we define the commutative octonions with the help of the multiplication rules as follows.

Table 2: Multiplication rules of elements of standard basis $\{e_0, e_1, e_2, \cdots, e_7\}$ for commutative octonions algebra.

$\cdot$	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	α	e_3	$\alpha\, e_2$	e_5	$\alpha\, e_4$	e_7	$\alpha\, e_6$
e_2	e_2	e_3	1	e_1	e_6	e_7	e_4	e_5
e_3	e_3	$\alpha\, e_2$	e_1	α	e_7	$\alpha\, e_6$	e_5	$\alpha\, e_4$
e_4	e_4	e_5	e_6	e_7	1	e_1	e_2	e_3
e_5	e_5	$\alpha\, e_4$	e_7	$\alpha\, e_6$	e_1	α	e_3	$\alpha\, e_2$
e_6	e_6	e_7	e_4	e_5	e_2	e_3	1	e_1
e_7	e_7	$\alpha\, e_6$	e_5	$\alpha\, e_4$	e_3	$\alpha\, e_2$	e_1	α

These multiplication rules can be obtained by the similar way to the octonion algebra mentioned above. Let $\mathbb{O}$ be the set of commutative octonions, i.e.

$$\mathbb{O} = \left\{ o = \sum_{i=0}^{7} a_i e_i : a_0, a_1, \dots, a_7 \in \mathbb{R}, e_0 = 1, e_1, e_2, \dots, e_7 \notin \mathbb{R} \right\}$$

where the versors $e_0, e_1, \dots, e_7$ satisfy the multiplication rules in Table 2.
Let $o = \sum_{i=0}^{7} c_i e_i \in \mathbb{O}$, then the characteristic matrix of o is

$$\mathrm{N} = \begin{bmatrix} c_0 & \alpha c_1 & c_2 & \alpha c_3 & c_4 & \alpha c_5 & c_6 & \alpha c_7 \\ c_1 & c_0 & c_3 & c_2 & c_5 & c_4 & c_7 & c_6 \\ c_2 & \alpha c_3 & c_0 & \alpha c_1 & c_6 & \alpha c_7 & c_4 & \alpha c_5 \\ c_3 & c_2 & c_1 & c_0 & c_7 & c_6 & c_5 & c_4 \\ c_4 & \alpha c_5 & c_6 & \alpha c_7 & c_0 & \alpha c_1 & c_2 & \alpha c_3 \\ c_5 & c_4 & c_7 & c_6 & c_1 & c_0 & c_3 & c_2 \\ c_6 & \alpha c_7 & c_4 & \alpha c_5 & c_2 & \alpha c_3 & c_0 & \alpha c_1 \\ c_7 & c_6 & c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \end{bmatrix} \equiv \left[\begin{array}{c|c} \Psi & \Omega \\ \hline \Omega & \Psi \end{array} \right] \tag{2.1}$$

where Ψ and Ω are the following 4×4 matrices:

$$\Psi = \begin{bmatrix} c_0 & \alpha c_1 & c_2 & \alpha c_3 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & \alpha c_3 & c_0 & \alpha c_1 \\ c_3 & c_2 & c_1 & c_0 \end{bmatrix} \text{ and } \Omega = \begin{bmatrix} c_4 & \alpha c_5 & c_6 & \alpha c_7 \\ c_5 & c_4 & c_7 & c_6 \\ c_6 & \alpha c_7 & c_4 & \alpha c_5 \\ c_7 & c_6 & c_5 & c_4 \end{bmatrix}.$$

Similarly, we can express matrices Ψ and Ω as

$$\Psi=\begin{bmatrix}\Psi' & \Psi'' \\ \Psi'' & \Psi'\end{bmatrix} \text{ and } \Omega=\begin{bmatrix}\Omega' & \Omega'' \\ \Omega'' & \Omega'\end{bmatrix}$$

where

$$\Psi'=\begin{bmatrix}c_0 & \alpha c_1 \\ c_1 & c_0\end{bmatrix},\ \Psi''=\begin{bmatrix}c_2 & \alpha c_3 \\ c_3 & c_2\end{bmatrix}$$

and

$$\Omega'=\begin{bmatrix}c_4 & \alpha c_5 \\ c_5 & c_4\end{bmatrix},\ \Omega''=\begin{bmatrix}c_6 & \alpha c_7 \\ c_7 & c_6\end{bmatrix}.$$

Determinant of the matrix N is given in the following theorem.

Theorem 2.1. *Determinant of the matrix* N *is*

$$\begin{aligned} det(\mathrm{N}) = &\left[\varepsilon(0,2,4,6)^2-\alpha\varepsilon(1,3,5,7)^2\right]\times\left[\varepsilon(0,-2,4,-6)^2-\alpha\varepsilon(1,-3,5,-7)^2\right] \\ &\times\left[\varepsilon(0,2,-4,-6)^2-\alpha\varepsilon(1,3,-5,-7)^2\right]\times\left[\varepsilon(0,-2,-4,6)^2-\alpha\varepsilon(1,-3,-5,7)^2\right]. \end{aligned} \tag{2.2}$$

where $\varepsilon(p,q,r,s)=c_p+c_q+c_r+c_s$ *and* $\varepsilon(-p)=-c_p$.

Proof. If we evaluate the determinant of the matrix N in Eq. (2.1), we obtain

$$\begin{aligned} det(\mathrm{N}) &= det(\Psi^2-\Omega^2) \\ &= det\left((\Psi'+\Omega')^2-(\Psi''+\Omega'')^2\right)\times det\left((\Psi'-\Omega')^2-(\Psi''-\Omega'')^2\right) \\ &= det\left(\Psi'+\Omega'+\Psi''+\Omega''\right)\times det\left(\Psi'+\Omega'-\Psi''-\Omega''\right)\times det\left(\Psi'-\Omega'+\Psi''-\Omega''\right)\times det\left(\Psi'-\Omega'-\Psi''+\Omega''\right). \end{aligned}$$

Substituting determinants of these matrices into the last equation completes the proof. □

Eq. (2.1) also gives the principal conjugations of the commutative octonion $o=\sum_{i=0}^{7}c_ie_i$ as follows

$$o_1=c_0+c_1e_1+c_2e_2+c_3e_3-(c_4e_4+c_5e_5+c_6e_6+c_7e_7), \tag{2.3}$$
$$o_2=c_0+c_1e_1+c_4e_4+c_5e_5-(c_2e_2+c_3e_3+c_6e_6+c_7e_7), \tag{2.4}$$
$$o_3=c_0+c_1e_1+c_6e_6+c_7e_7-(c_2e_2+c_3e_3+c_4e_4+c_5e_5), \tag{2.5}$$
$$o_4=c_0+c_2e_2+c_4e_4+c_6e_6-(c_1e_1+c_3e_3+c_5e_5+c_7e_7), \tag{2.6}$$
$$o_5=c_0+c_2e_2+c_5e_5+c_7e_7-(c_1e_1+c_3e_3+c_4e_4+c_6e_6), \tag{2.7}$$
$$o_6=c_0+c_3e_3+c_4e_4+c_7e_7-(c_1e_1+c_2e_2+c_5e_5+c_6e_6), \tag{2.8}$$

and

$$o_7=c_0+c_3e_3+c_5e_5+c_6e_6-(c_1e_1+c_2e_2+c_4e_4+c_7e_7). \tag{2.9}$$

From these conjugations, for $\alpha\neq 0$, we can construct the bijective mapping between $c_0,c_1,\ldots,c_7\to o_0=o,o_1,\ldots,o_7$ as

$$\begin{aligned} c_0 &= \frac{\zeta(0,1,2,3,4,5,6,7)}{8}, \\ c_1 &= e_1\frac{\zeta(0,1,2,3,-4,-5,-6,-7)}{8\alpha}, \\ c_2 &= e_2\frac{\zeta(0,1,-2,-3,4,5,-6,-7)}{8}, \\ c_3 &= e_3\frac{\zeta(0,1,-2,-3,-4,-5,6,7)}{8\alpha}, \\ c_4 &= e_4\frac{\zeta(0,-1,2,-3,4,-5,6,-7)}{8}, \\ c_5 &= e_5\frac{\zeta(0,-1,2,-3,-4,5,-6,7)}{8\alpha}, \\ c_6 &= e_6\frac{\zeta(0,-1,-2,3,4,-5,-6,7)}{8} \end{aligned}$$

$$c_7=e_7\frac{\zeta(0,-1,-2,3,-4,5,6,-7)}{8\alpha}$$

where $\zeta(r_0,r_1,\cdots,r_7)=o_{r_0}+o_{r_1}+\cdots+o_{r_7}$ and $\zeta(-r)=-o_r$.
From the eigenvalues of the characteristic matrix (2.1), we obtain the norm of a commutative octonion $o=\sum_{i=0}^{7}c_ie_i$ as

$$\|o\|=o\cdot o_1\cdot o_2\cdots o_7=det(\mathrm{N}). \tag{2.10}$$

By using Eq.(2.1), we can obtain the following matrix expressions of all the versors:

$$
\mathrm{e}_0 = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad
\mathrm{e}_1 = \begin{bmatrix} 0&\alpha&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&\alpha&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&\alpha&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&\alpha\\ 0&0&0&0&0&0&1&0 \end{bmatrix},
$$

$$
\mathrm{e}_2 = \begin{bmatrix} 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0 \end{bmatrix}, \quad
\mathrm{e}_3 = \begin{bmatrix} 0&0&0&\alpha&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&\alpha&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&\alpha\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&\alpha&0&0\\ 0&0&0&0&1&0&0&0 \end{bmatrix},
$$

$$
\mathrm{e}_4 = \begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0 \end{bmatrix}, \quad
\mathrm{e}_5 = \begin{bmatrix} 0&0&0&0&0&\alpha&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&\alpha\\ 0&0&0&0&0&0&1&0\\ 0&\alpha&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&\alpha&0&0&0&0\\ 0&0&1&0&0&0&0&0 \end{bmatrix},
$$

$$
\mathrm{e}_6 = \begin{bmatrix} 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0 \end{bmatrix}, \quad
\mathrm{e}_7 = \begin{bmatrix} 0&0&0&0&0&0&0&\alpha\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&\alpha&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&\alpha&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&\alpha&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0 \end{bmatrix}.
$$

3. Some Properties of Commutative Octonions

Except two systems, any system which satisfies associativity and distributivity with respect to the sum, it can satisfy either commutativity or it does not have divisors of zero [2]. One can see that the commutative octonions satisfy the commutative properties from the multiplication rules easily. For the first two properties (distributive and associative), the characteristic matrix in Eq.(2.1) can be used.
A commutative octonion can be represented by four complex numbers with four linearly independent bases. Let $o = \sum_{i=0}^{7} c_i e_i$ be a commutative octonion. We can derive

$$
\begin{aligned}
o &= \mathrm{c}_0 + \mathrm{c}_1 e_1 + \cdots + \mathrm{c}_7 e_7 \\
&= \mathrm{c}_0 + \mathrm{c}_1 e_1 + \mathrm{c}_2 e_2 + \mathrm{c}_3 e_3 + (\mathrm{c}_4 + \mathrm{c}_5 e_1 + \mathrm{c}_6 e_2 + \mathrm{c}_7 e_3) e_4 \\
&= (\mathrm{c}_0 + \mathrm{c}_1 e_1 + \mathrm{c}_2 e_2 + \mathrm{c}_3 e_3)\left(\frac{1+e_4}{2} + \frac{1-e_4}{2}\right) + (\mathrm{c}_4 + \mathrm{c}_5 e_1 + \mathrm{c}_6 e_2 + \mathrm{c}_7 e_3)\left(\frac{1+e_4}{2} - \frac{1-e_4}{2}\right) \\
&= q_1 e_1' + q_2 e_2' \qquad (3.1)
\end{aligned}
$$

where

$$
\begin{aligned}
q_1 &= (\mathrm{c}_0 + \mathrm{c}_4) + (\mathrm{c}_1 + \mathrm{c}_5) e_1 + (\mathrm{c}_2 + \mathrm{c}_6) e_2 + (\mathrm{c}_3 + \mathrm{c}_7) e_3, \\
q_2 &= (\mathrm{c}_0 - \mathrm{c}_4) + (\mathrm{c}_1 - \mathrm{c}_5) e_1 + (\mathrm{c}_2 - \mathrm{c}_6) e_2 + (\mathrm{c}_3 - \mathrm{c}_7) e_3,
\end{aligned}
$$

and

$$
e_1' = \frac{1+e_4}{2}, \quad e_2' = \frac{1-e_4}{2}.
$$

Thus we can represent a commutative octonion by two commutative quaternions. Here e_1' and e_2' are idempotent basis and satisfy

$$
(e_1')^2 = e_1', \quad (e_2')^2 = e_2', \quad e_1' e_2' = 0.
$$

From [3], we know that the commutative quaternions q_1 and q_2 can be represented as follows

$$
q_1 = x^{(1)} e_1'' + x^{(2)} e_2'', \quad q_2 = x^{(3)} e_1'' + x^{(4)} e_2'' \qquad (3.2)
$$

where

$$
e_1'' = \frac{1+e_2}{2}, \quad e_2'' = \frac{1-e_2}{2}
$$

and

$$x^{(1)} = (x_0 + x_2) + (x_1 + x_3)e_1,$$
$$x^{(2)} = (x_0 - x_2) + (x_1 - x_3)e_1,$$
$$x^{(3)} = (y_0 + y_2) + (y_1 + y_3)e_1,$$
$$x^{(4)} = (y_0 - y_2) + (y_1 - y_3)e_1.$$

Here we set

$$x_i = c_i + c_{i+4} \text{ and } y_i = c_i - c_{i+4} \quad (0 \le i \le 3).$$

By using Eqs.(3.1) and (3.2), we obtain

$$\begin{aligned} o &= q_1 e_1' + q_2 e_2' \\ &= \left[x^{(1)} e_1'' + x^{(2)} e_2''\right] e_1' + \left[x^{(3)} e_1'' + x^{(4)} e_2''\right] e_2' \\ &= x^{(1)} i_1 + x^{(2)} i_2 + x^{(3)} i_3 + x^{(4)} i_4 \end{aligned} \tag{3.3}$$

where

$$i_1 = e_1'' e_1' = \frac{1 + e_2 + e_4 + e_6}{4}, \quad i_2 = e_2'' e_1' = \frac{1 - e_2 + e_4 - e_6}{4}, i_3 = e_1'' e_2' = \frac{1 + e_2 - e_4 - e_6}{4}, \quad i_4 = e_2'' e_2' = \frac{1 - e_2 - e_4 + e_6}{4}.$$

Here $i_k \ (k = 1, \dots, 4)$ satisfy

$$i_k^2 = i_k \text{ and } i_k i_l = 0 \ (\text{for } k \neq l). \tag{3.4}$$

Finally, to summarize above transformations, we can say that a commutative octonion $o = \sum_{i=0}^{7} c_i e_i$ can be represented as the form in Eq.(3.3) where

$$x^{(1)} = \varepsilon(0,2,4,6) + \varepsilon(1,3,5,7)e_1, \tag{3.5}$$
$$x^{(2)} = \varepsilon(0,-2,4,-6) + \varepsilon(1,-3,5,-7)e_1 \tag{3.6}$$
$$x^{(3)} = \varepsilon(0,2,-4,-6) + \varepsilon(1,3,-5,-7)e_1 \tag{3.7}$$
$$x^{(4)} = \varepsilon(0,-2,-4,6) + \varepsilon(1,-3,-5,7)e_1. \tag{3.8}$$

Thus i_1, i_2, i_3 and i_4 are linearly independent and the set of commutative octonions can be shown as a direct sum of four complex number fields. For any positive integer n, Eqs. (3.3) and (3.4) give

$$o^n = \left(x^{(1)}\right)^n i_1 + \left(x^{(2)}\right)^n i_2 + \left(x^{(3)}\right)^n i_3 + \left(x^{(4)}\right)^n i_4. \tag{3.9}$$

Let $o' = x_1^{(1)} i_1 + x_1^{(2)} i_2 + x_1^{(3)} i_3 + x_1^{(4)} i_4$ and $o'' = x_2^{(1)} i_1 + x_2^{(2)} i_2 + x_2^{(3)} i_3 + x_2^{(4)} i_4$ be two commutative octonions. Then by using Eqs. (3.3) and (3.4) again, we have

$$o' o'' = x_1^{(1)} x_2^{(1)} i_1 + x_1^{(2)} x_2^{(2)} i_2 + x_1^{(3)} x_2^{(3)} i_3 + x_1^{(4)} x_2^{(4)} i_4 \tag{3.10}$$

and

$$\frac{o'}{o''} = \frac{x_1^{(1)}}{x_2^{(1)}} i_1 + \frac{x_1^{(2)}}{x_2^{(2)}} i_2 + \frac{x_1^{(3)}}{x_2^{(3)}} i_3 + \frac{x_1^{(4)}}{x_2^{(4)}} i_4. \tag{3.11}$$

Seven conjugates of a commutative octonion in Eqs.(2.3) - (2.9) can be represented according to the decomposed form (3.3) as follows

Theorem 3.1. *For a commutative octonion $o = \sum_{i=0}^{7} c_i e_i = x^{(1)} i_1 + x^{(2)} i_2 + x^{(3)} i_3 + x^{(4)} i_4$, we have*

$$o_1 = x^{(3)} i_1 + x^{(4)} i_2 + x^{(1)} i_3 + x^{(2)} i_4, \tag{3.12}$$
$$o_2 = x^{(2)} i_1 + x^{(1)} i_2 + x^{(4)} i_3 + x^{(3)} i_4, \tag{3.13}$$
$$o_3 = x^{(4)} i_1 + x^{(3)} i_2 + x^{(2)} i_3 + x^{(1)} i_4, \tag{3.14}$$
$$o_4 = \overline{x^{(1)}} i_1 + \overline{x^{(2)}} i_2 + \overline{x^{(3)}} i_3 + \overline{x^{(4)}} i_4, \tag{3.15}$$
$$o_5 = \overline{x^{(3)}} i_1 + \overline{x^{(4)}} i_2 + \overline{x^{(1)}} i_3 + \overline{x^{(2)}} i_4, \tag{3.16}$$
$$o_6 = \overline{x^{(2)}} i_1 + \overline{x^{(1)}} i_2 + \overline{x^{(4)}} i_3 + \overline{x^{(3)}} i_4, \tag{3.17}$$
$$o_7 = \overline{x^{(4)}} i_1 + \overline{x^{(3)}} i_2 + \overline{x^{(2)}} i_3 + \overline{x^{(1)}} i_4 \tag{3.18}$$

where $\bar{x}$ is the complex conjugate of a complex number x.

Proof. The proof can be done easily by using Eqs. (2.3) − (2.9), Eqs.(3.5) − (3.8) and the multiplication rule (3.10). □

We can give the following results immediately.
Corollary For a commutative octonion $o=\sum_{i=0}^{7} c_i e_i = x^{(1)} i_1 + x^{(2)} i_2 + x^{(3)} i_3 + x^{(4)} i_4$, we have

$$oo_4 = \|x^{(1)}\|^2 i_1 + \|x^{(2)}\|^2 i_2 + \|x^{(3)}\|^2 i_3 + \|x^{(4)}\|^2 i_4, \tag{3.19}$$

$$o_1 o_5 = \|x^{(3)}\|^2 i_1 + \|x^{(4)}\|^2 i_2 + \|x^{(1)}\|^2 i_3 + \|x^{(2)}\|^2 i_4, \tag{3.20}$$

$$o_2 o_6 = \|x^{(2)}\|^2 i_1 + \|x^{(1)}\|^2 i_2 + \|x^{(4)}\|^2 i_3 + \|x^{(3)}\|^2 i_4, \tag{3.21}$$

$$o_3 o_7 = \|x^{(4)}\|^2 i_1 + \|x^{(3)}\|^2 i_2 + \|x^{(2)}\|^2 i_3 + \|x^{(1)}\|^2 i_4 \tag{3.22}$$

where $\|x\|$ is the absolute square of a complex number x, i.e. $\|x\|^2 = x\bar{x}$.

3.1. Holomorphic and Analytic Functions

We show an octonion function by

$$G \equiv \sum_{i=0}^{7} G_i(\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_7) e_i \tag{3.23}$$

where G_i $(i=0,1,\cdots,7)$ are real functions with partial derivatives for the variables, $\mathrm{c}_0, \mathrm{c}_1, \cdots, \mathrm{c}_7$. Catoni et al. [2] introduced the Generalized Cauchy – Riemann – like (GCR) conditions. While there are some methods for calculation of the GCR conditions, Catoni et al. [4] give the following theorem.

Theorem 3.2. *[4, p.91] The Jacobian matrix of a hypercomplex function's components has the same form of the characteristic matrix.*

By combining this method and the study of Catoni et al. [3] for Segre's commutative quaternions we have the following theorem for holomorphic functions of octonion.

Theorem 3.3. *G is called a holomorphic functions of octonion if*
1) G is differentiable with non-zero derivatives and not a zero divisor,
2) The GCR conditions for the partial derivatives of components of G are

$$\begin{aligned}
&G_{0,\mathrm{c}_0} = G_{1,\mathrm{c}_1} = G_{2,\mathrm{c}_2} = G_{3,\mathrm{c}_3} = G_{4,\mathrm{c}_4} = G_{5,\mathrm{c}_5} = G_{6,\mathrm{c}_6} = G_{7,\mathrm{c}_7}\\
&G_{0,\mathrm{c}_1} = \alpha G_{1,\mathrm{c}_0} = G_{2,\mathrm{c}_3} = \alpha G_{3,\mathrm{c}_2} = G_{4,\mathrm{c}_5} = \alpha G_{5,\mathrm{c}_4} = G_{6,\mathrm{c}_7} = \alpha G_{7,\mathrm{c}_6}\\
&G_{0,\mathrm{c}_2} = G_{1,\mathrm{c}_3} = G_{2,\mathrm{c}_0} = G_{3,\mathrm{c}_1} = G_{4,\mathrm{c}_6} = G_{5,\mathrm{c}_7} = G_{6,\mathrm{c}_4} = G_{7,\mathrm{c}_5}\\
&G_{0,\mathrm{c}_3} = \alpha G_{1,\mathrm{c}_2} = G_{2,\mathrm{c}_1} = \alpha G_{3,\mathrm{c}_0} = G_{4,\mathrm{c}_7} = \alpha G_{5,\mathrm{c}_6} = G_{6,\mathrm{c}_5} = \alpha G_{7,\mathrm{c}_4}\\
&G_{0,\mathrm{c}_4} = G_{1,\mathrm{c}_5} = G_{2,\mathrm{c}_6} = G_{3,\mathrm{c}_7} = G_{4,\mathrm{c}_0} = G_{5,\mathrm{c}_1} = G_{6,\mathrm{c}_2} = G_{7,\mathrm{c}_3}\\
&G_{0,\mathrm{c}_5} = \alpha G_{1,\mathrm{c}_4} = G_{2,\mathrm{c}_7} = \alpha G_{3,\mathrm{c}_6} = G_{4,\mathrm{c}_1} = \alpha G_{5,\mathrm{c}_0} = G_{6,\mathrm{c}_3} = \alpha G_{7,\mathrm{c}_2}\\
&G_{0,\mathrm{c}_6} = G_{1,\mathrm{c}_7} = G_{2,\mathrm{c}_4} = G_{3,\mathrm{c}_5} = G_{4,\mathrm{c}_2} = G_{5,\mathrm{c}_3} = G_{6,\mathrm{c}_0} = G_{7,\mathrm{c}_1}\\
&G_{0,\mathrm{c}_7} = \alpha G_{1,\mathrm{c}_6} = G_{2,\mathrm{c}_5} = \alpha G_{3,\mathrm{c}_4} = G_{4,\mathrm{c}_3} = \alpha G_{5,\mathrm{c}_2} = G_{6,\mathrm{c}_1} = \alpha G_{7,\mathrm{c}_0}.
\end{aligned}$$

Let $G(o)$ be an octonion holomorphic function and its power series in q about 0 be

$$G(o) = \sum_{r=0}^{\infty} t_r o^r \tag{3.24}$$

where $t_r \in \mathbb{O}$. From Eq.(3.1), we write

$$t_r = f_r e_1' + g_r e_2'$$

where $f_r, g_r \in \mathbb{H}$. Then we obtain

$$\begin{aligned}
G(q) &= \sum_{r=0}^{\infty} (f_r e_1' + g_r e_2')(q_1 e_1' + q_2 e_2')^r\\
&= \sum_{r=0}^{\infty} (f_r e_1' + g_r e_2')(q_1^r e_1' + q_2^r e_2')\\
&= \sum_{r=0}^{\infty} (f_r q_1^r e_1' + g_r q_2^r e_2')\\
&= e_1' \sum_{r=0}^{\infty} f_r q_1^r + e_2' \sum_{r=0}^{\infty} g_r q_2^r
\end{aligned}$$

where q_1 and q_2 are commutative quaternions given in Eq.(3.1).

4. Conclusion

Octonions have many applications in field. We mentioned some of them such as cosmology, quantum theory, etc. Octonions form a non–commutative and non–associative algebra. These hyper–complex numbers are constructed by Cayley–Dickson process over Hamilton quaternions. The aim of this study is to introduce commutative octonions constructed by Cayley–Dickson process over commutative quaternions. We think that several applications in field will be applied by other researchers. It will be very interesting that investigation of commutative octonions whose coefficients are well-known integer sequences such as Fibonacci, Lucas, Pell, etc.

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