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A STRONGER FORM OF LOCALLY CLOSED SET AND ITS HOMEOMORPHIC IMAGE

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Abstract. Through this paper, via the operators $(\cdot)^\star$ and Ψ, we presented notion of $\star$-Locally set in an ideal topological space $\zeta_{\mathbb{I}}$ as a new stronger form of locally closed set, and considered relations with various existing weak form of locally closed set. Preservations of direct images as well as inverse images of $(\cdot)^\star$, Ψ, $\star$-perfect and various weak forms of locally closed set including $\star$-Locally closed set are important investigating part. Besides, we pointed out that consideration of 'bijectivity' in Lemma 3.1 of [24] is sufficient, and the Lemma 3.3 of [24] is wrong. We demonstrated two modifications of the last one.

1. Introduction

Locally closed set and its study is not a new idea in topology. This notion was disclosed by Bourbaki [3], and after that it has been extensively studied by a good number of mathematicians (see [7,12,20,21]). This study has been interesting because it generalizes both open and closed sets. But the study of a locally closed set relative to an ideal (see [13]) is a new idea, and this has been introduced through this paper. The authors Jeyanthi *et al.* [12] and the author Dontchev [6] have studied locally closed sets in terms of ideal, but these locally closed sets differ somewhat from the current one.

We now consider some preliminary concepts from literature for developing the paper.

Consider a topological space $(\mathbf{Z}, \mathbb{T})$ (henceforth, in this paper we shall denote it by ζ), and suppose $\mathbb{I}$ is an ideal on $\mathbf{Z}$. The set-valued map $(\cdot)^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ associated by the formula '$H^\star = \{a \in \mathbf{Z} : G_a \cap H \notin \mathbb{I}$ for every $G_a \in \mathbb{T}_a\}$ for every $H \subseteq \mathbf{Z}$' is designated as the local function [11] w.r.t. the ideal $\mathbb{I}$ and the topology $\mathbb{T}$,

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where $\mathbb{T}_a = \{G \in \mathbb{T} : a \in G\}$ and $\wp(\mathbf{Z})$ stands for power set of $\mathbf{Z}$. Other notations used instead of $H^\star$ are $H^\star(\mathbb{I}, \mathbb{T})$ and $H^\star(\mathbb{I})$. For the trivial ideals $\{\varnothing\}$ and $\wp(\mathbf{Z})$, values of $(\cdot)^\star$ are $H^\star(\{\varnothing\}) = \mathrm{Cl}(H)$ (closure operator) and $H^\star(\wp(\mathbf{Z})) = \varnothing$ (zero operator), respectively. An interesting ideal on $\mathbf{Z}$ is $\mathbb{I}_n$ consisting of all nowhere dense sets of ζ, and $H^\star(\mathbb{I}_n) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(H)))$ (see [11]), where 'Int' stands for interior operator. Further, for the ideals $\mathbb{I}_f = \{I \subseteq \mathbf{Z} : I \text{ is finite}\}$ and $\mathbb{I}_c = \{I \subseteq \mathbf{Z} : I \text{ is countable}\}$, $H^\star(\mathbb{I}_f) = H^\omega$ (collection of all ω-accumulation point of H) and $H^\star(\mathbb{I}_c) = H^{cd}$ (collection of all condensation point of H) (see [11]). Thus one can think the local function $(\cdot)^\star$ as a generalization of closure operator.

An important set-operator familiar to researchers as a complement of the local function $(\cdot)^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ is $\Psi : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$, and its value acting on $H \subseteq \mathbf{Z}$ is calculated by the formula $\Psi(H) = \mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\star$ [22]. Note that $(\cdot)^\star$ (resp., Ψ) is not necessarily a closure (resp., interior) operator. However, the operator $\mathrm{Cl}^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ given by the formula $\mathrm{Cl}^\star(H) = H \cup H^\star$ determines Kuratowski's closure operator [2, 11, 13, 27], and henceforth $\mathbf{Z}$ gets a new topology, named $\star$-topology [1,2,8–10,16,23], induced by $\mathrm{Cl}^\star$. Let's name this topology as $\mathbb{T}^\star$. Clearly, $\mathbb{T} \subseteq \mathbb{T}^\star$ (see [11]). The interior operator of the space $\zeta^\star = (\mathbf{Z}, \mathbb{T}^\star)$ is given by $Int^\star(H) = \mathbf{Z} \setminus \mathrm{Cl}^\star(\mathbf{Z} \setminus H)$.

Moreover, if $H \subseteq H^\star$, then H is known as $\star$-dense in itself [10], and if $H = H^\star$, then H is termed as $\star$-perfect [10].

2. $L^\star$ Operator

We are beginning this section with an example to draw interest to the fact that through idealizing a space ζ by way of a proper ideal $\mathbb{I}$ (i.e., $\mathbf{Z} \notin \mathbb{I}$), one can find an $H \subseteq \mathbf{Z}$ for which $H^\star$ intersects $\Psi(H)$ i.e., the assertion '$K^\star \cap \Psi(K) = \varnothing$ for every $K \subseteq \mathbf{Z}$' need no longer be correct. The notations $\zeta_{\mathbb{I}}$ and $\zeta_{\mathbb{I}}^\star$ will be used to recognize respectively the triplets $(\mathbf{Z}, \mathbb{T}, \mathbb{I})$ and $(\mathbf{Z}, \mathbb{T}^\star, \mathbb{I})$, ideal topological spaces, in this write-up.

Example 1. *Consider* $\mathbb{T} = \{\varnothing, \{\ell_1\}, \mathbf{Z}\}$ *and* $\mathbb{I} = \{\varnothing, \{\ell_2\}\}$ *on* $\mathbf{Z} = \{\ell_1, \ell_2, \ell_3\}$. *Then for* $H = \{\ell_1, \ell_2\}$, $H^\star = \mathbf{Z}$, $\Psi(H) = \{\ell_1\}$ *and* $H^\star \cap \Psi(H) \neq \varnothing$.

Definition 1. *We define the* $L^\star$ *operator on* $\zeta_{\mathbb{I}}$ *as a set-valued map* $L^\star : \wp(\mathbf{Z}) \to \wp(\mathbf{Z})$ *by the equation* $L^\star(H) = H^\star \cap \Psi(H)$ *for every* $H \subseteq \mathbf{Z}$.

Remark 1. *As, we know from [11] that* $H^\star(\mathbb{I}, \mathbb{T}) = H^\star(\mathbb{I}, \mathbb{T}^\star)$, *so* $L^\star$ *values of every* $H \subseteq \mathbf{Z}$ *w.r.t.* $\zeta_{\mathbb{I}}$ *and* $\zeta_{\mathbb{I}}^\star$ *are same.*

We shall now discuss the value of $L^\star(H)$ for different ideals on a topological space.

- $\mathbb{I} = \{\varnothing\}$ implies $L^\star(H) = \mathrm{Cl}(H) \cap (\mathbf{Z} \setminus \mathrm{Cl}(\mathbf{Z} \setminus H)) = \mathrm{Cl}(H) \cap \mathrm{Int}(H) = \mathrm{Int}(H)$.
- $\mathbb{I} = \wp(\mathbf{Z})$ implies $L^\star(H) = \varnothing \cap \Psi(H) = \varnothing$.
- $\mathbb{I} = \mathbb{I}_n$ implies $L^\star(H) = \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(H))) \cap \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(H))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(H)))$.

- $\mathbb{I} = \mathbb{I}_f$ implies $L^\star(H) = H^\omega \cap (\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\omega) \subseteq H^\omega$.
- $\mathbb{I} = \mathbb{I}_c$ implies $L^\star(H) = H^{cd} \cap (\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^{cd}) \subseteq H^{cd}$.

Study of the $L^\star$ operator will be therefore fascinating if we are deal with a non-trivial ideal (non-trivial means other than $\{\varnothing\}$ and $\wp(\mathbf{Z})$).

Theorem 1. *For H, $K \subseteq \mathbf{Z}$, the followings are true in $\zeta_{\mathbb{I}}$:*

(1) $L^\star(\varnothing) = \varnothing$,
(2) $L^\star(\mathbf{Z}) = \mathbf{Z}^\star$,
(3) $L^\star(\mathbf{Z}) = \mathbf{Z}$ *if and only if* $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$,
(4) $L^\star(H) = \Psi(H) \setminus \Psi(\mathbf{Z} \setminus H)$,
(5) $L^\star(H) = H^\star \setminus (\mathbf{Z} \setminus H)^\star$,
(6) $\mathbf{Z} \setminus L^\star(H) = (\mathbf{Z} \setminus H)^\star \cup (\mathbf{Z} \setminus H^\star)$,
(7) $L^\star(\mathbf{Z} \setminus H) = \mathbf{Z} \setminus (\Psi(H) \cup H^\star)$,
(8) *For* $H \subseteq K$, $L^\star(H) \subseteq L^\star(K)$,
(9) $L^\star(H) \cup L^\star(K) \subseteq L^\star(H \cup K)$,
(10) $L^\star(H \cap K) \subseteq L^\star(H) \cap L^\star(K)$,
(11) $L^\star(H) \subseteq H^\star$,
(12) $L^\star(H) \subseteq \Psi(H)$,
(13) $H \cap L^\star(H) = H^\star \cap \mathrm{Int}^\star(H)$,
(14) $H \cap L^\star(H) \subseteq \mathrm{Int}^\star(H)$,
(15) $L^\star(H) \subseteq H^\star \subseteq \mathrm{Cl}^\star(H) \subseteq \mathrm{Cl}(H)$,
(16) *For* $H \in \mathbb{T}^\star$, $H \cap H^\star \subseteq L^\star(H) \subseteq H^\star$,
(17) *For* $H \in \mathbb{T}$, $H \cap H^\star \subseteq L^\star(H) \subseteq H^\star$,
(18) *For a regular open* H *[25]*, $L^\star(H) = H \cap H^\star$,
(19) $\mathrm{Int}(L^\star(H)) = \Psi(H) \cap \mathrm{Int}(H^\star)$,
(20) $\mathrm{Int}^\star(L^\star(H)) \supseteq \Psi(H) \cap \mathrm{Int}^\star(H^\star)$,
(21) $\mathrm{Cl}(L^\star(H)) \subseteq \mathrm{Cl}(\Psi(H)) \cap H^\star$,
(22) $\mathrm{Cl}^\star(L^\star(H)) \subseteq \mathrm{Cl}^\star(\Psi(H)) \cap H^\star$,
(23) $\mathrm{Int}^\star(H^\star) \cap \Psi(H) \subseteq \mathrm{Int}^\star(L^\star(H)) \subseteq \mathrm{Cl}^\star(L^\star(H)) \subseteq H^\star \cap \mathrm{Cl}^\star(\Psi(H))$,
(24) *For a* $\star$*-perfect set* H, $L^\star(H) = H \cap \Psi(H) = \mathrm{Int}^\star(H)$,
(25) *For a* $\star$*-dense in itself set* H, $L^\star(H) \supseteq \mathrm{Int}^\star(H)$.

Proof. (1) $L^\star(\varnothing) = \varnothing^\star \cap \Psi(\varnothing) = \varnothing$.
(2) $L^\star(\mathbf{Z}) = \mathbf{Z}^\star \cap \Psi(\mathbf{Z}) = \mathbf{Z}^\star \cap \mathbf{Z} = \mathbf{Z}^\star$.
(3) Follows from the fact $\mathbf{Z}^\star = \mathbf{Z}$ if and only if $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$.
(4) $L^\star(H) = H^\star \cap \Psi(H) = (\mathbf{Z} \setminus \Psi(\mathbf{Z} \setminus H)) \cap \Psi(H) = \Psi(H) \setminus \Psi(\mathbf{Z} \setminus H)$.
(5) $L^\star(H) = H^\star \cap \Psi(H) = H^\star \cap (\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\star) = H^\star \setminus (\mathbf{Z} \setminus H)^\star$.
(6) $\mathbf{Z} \setminus L^\star(H) = \mathbf{Z} \setminus (H^\star \cap \Psi(H)) = (\mathbf{Z} \setminus H^\star) \cup (\mathbf{Z} \setminus \Psi(H)) = (\mathbf{Z} \setminus H^\star) \cup (\mathbf{Z} \setminus H)^\star$.
(7) Obvious.
(8) Obvious.
(9) Follows from 8.
(10) Follows from 8.
(11) Obvious.

(12) Obvious.
(13) $H \cap L^\star(H) = H \cap (H^\star \cap \Psi(H)) = H^\star \cap \mathrm{Int}^\star(H)$.
(14) From 10, $L^\star(H) \subseteq \Psi(H)$. Therefore, $H \cap L^\star(H) \subseteq H \cap \Psi(H) = \mathrm{Int}^\star(H)$.
(15) Obvious from the fact $H^\star \subseteq H^\star \cup H = \mathrm{Cl}^\star(H) \subseteq \mathrm{Cl}(H)$.
(16) $H \in \mathbb{T}$ implies $H \subseteq \Psi(H)$. Now $L^\star(H) = H^\star \cap \Psi(H)$ implies $H^\star \cap H \subseteq L^\star(H)$.
(17) Obvious from the fact $\mathbb{T} \subseteq \mathbb{T}^\star$.
(18) Since H is regular open, so $H = \Psi(H)$ [2,8,18]. Now, $L^\star(H) = H^\star \cap \Psi(H) = H^\star \cap H$.
(19) $\mathrm{Int}(L^\star(H)) = \mathrm{Int}(\Psi(H) \cap H^\star) = \mathrm{Int}(\Psi(H)) \cap \mathrm{Int}(H^\star) = \Psi(H) \cap \mathrm{Int}(H^\star)$.
(20) $\mathrm{Int}^\star(L^\star(H)) = \mathrm{Int}^\star(H^\star \cap \Psi(H)) = [H^\star \cap \Psi(H)] \cap \Psi[H^\star \cap \Psi(H)] = [H^\star \cap \Psi(H)] \cap [\Psi(H^\star) \cap \Psi(\Psi(H))] \supseteq [H^\star \cap \Psi(H)] \cap [\Psi(H^\star) \cap \Psi(H)] = [H^\star \cap \Psi(H^\star)] \cap \Psi(H) = \mathrm{Int}^\star(H^\star) \cap \Psi(H)$.
(21) Similar to 19.
(22) Similar to 19.
(23) Follows from 20.
(24) Trivial.
(25) Trivial.

□

Inequality of the result (9) of Theorem 1 is highlighted in next example.

Example 2. *Take* $\mathbf{Z} = \mathbb{R}$ *(set of reals) with usual topology and* $\mathbb{I} = \{\varnothing\}$. *Pick* $H = [0, 2021)$ *and* $K = [2021, 2022)$. *Then* $L^\star(H) = \mathrm{Int}(H) = (0, 2021)$, $L^\star(K) = \mathrm{Int}(K) = (2021, 2022)$ *and* $L^\star(H \cup K) = L^\star([0, 2022)) = (0, 2022)$. *Evidently,* $L^\star(H) \cup L^\star(K) \neq L^\star(H \cup K)$.

Theorem 2. *Suppose* $\mathbb{I}$ *is an ideal on* ζ *and* $H \subseteq \mathbf{Z}$. *If* $a \in L^\star(H)$, *then there exists at least one* $K_a \in \mathbb{T}_a$ *such that* $K_a \notin \mathbb{I}$ *but* $K_a \setminus H \in \mathbb{I}$.

Proof. $a \in L^\star(H)$ gives $a \in H^\star$ but $a \notin (\mathbf{Z} \setminus H)^\star$. Now, $a \notin (\mathbf{Z} \setminus H)^\star$ assures the existence of a $K_a \in \mathbb{T}_a$ such that $K_a \cap (\mathbf{Z} \setminus H) = K_a \setminus H \in \mathbb{I}$. On the other hand, $a \in H^\star$ tells that $K_a \cap H \notin \mathbb{I}$. This directs that $K_a \notin \mathbb{I}$, since $\mathbb{I}$ is an ideal. Hence, $K_a \notin \mathbb{I}$ but $K_a \setminus H \in \mathbb{I}$, as aimed. □

We talk about the validation of the converse part of Theorem 2 in next example.

Example 3. *Take* $\mathbb{T} = \{\varnothing, \{\ell_1\}, \{\ell_2\}, \mathbf{Z}\}$ *and* $\mathbb{I} = \{\varnothing, \{\ell_1\}\}$ *on* $\mathbf{Z} = \{\ell_1, \ell_2\}$. *Let* $H = \{\ell_2\}$. *Then* $H^\star = \{\ell_2\}$ *and* $\Psi(H) = \mathbf{Z}$ *and hence* $L^\star(H) = \{\ell_2\}$. *Now, pick up the point* ℓ_1 *and choose* $K_{\ell_1} = \mathbf{Z} \in \mathbb{T}_{\ell_1}$. *Evidently,* $K_{\ell_1} \notin \mathbb{I}$, $K_{\ell_1} \setminus H = \{\ell_1\} \in \mathbb{I}$ *but* $\ell_1 \notin L^\star(H)$. *Therefore, the reverse direction of Theorem 2 will usually not work.*

3. ⋆-Locally Closed Sets

Definition 2. *We call an* $H \subseteq \mathbf{Z}$ *as* ⋆*-Locally closed in* $\zeta_{\mathbb{I}}$ *if there is a* $K \subseteq \mathbf{Z}$ *such that* $H = L^\star(K)$, *and use the symbol* $L^\star(\zeta_{\mathbb{I}})$ *to mean* $\{H \subseteq \mathbf{Z} : H$ *is* ⋆*-Locally closed*$\}$.

Example 4. *Topologize* $\mathbf{Z} = \mathbb{R}$ *by considering* $\mathbb{T} = \{\varnothing, \mathbb{Q}, \mathbb{R}\}$ *and* $\mathbb{I} = \wp(\mathbb{Q})$*, where* $\mathbb{Q}$ *is the set of all rationals. Then for any* $H \subseteq \mathbf{Z}$*,*

$$H^\star = \begin{cases} \varnothing, & \text{if } H \cap (\mathbb{R} \setminus \mathbb{Q}) = \varnothing \\ \mathbb{R} \setminus \mathbb{Q}, & \text{if } H \cap (\mathbb{R} \setminus \mathbb{Q}) \neq \varnothing. \end{cases}$$

Take $L = \mathbb{R} \setminus \mathbb{Q}$*. We observe that* $L = L^\star \cap \Psi(L)$*. So,* $\mathbb{R} \setminus \mathbb{Q}$ *is a* $\star$*-Locally closed set.*

Example 5. *Consider* $\zeta_{\mathbb{I}}$ *discussed in Example 1, and take* $H = \{\ell_1\}$*,* $K = \{\ell_1, \ell_2\}$*. Since* $H = K^\star \cap \Psi(K)$*, so* H *is* $\star$*-Locally closed in* $\zeta_{\mathbb{I}}$*.*

Definition 3. *An* $L \subseteq \mathbf{Z}$ *of a space* ζ *is familiar with the name locally closed [7] (resp., semi-locally closed [26],* λ*-locally closed [20]) if we can give the form* $L = H \cap K$*, where* H *is open (resp., semi-open [14],* λ*-open [20]) and* K *is closed (resp., semi-closed, closed).*

Definition 4. *An* $L \subseteq \mathbf{Z}$ *is addressed as* $\mathbb{I}$*-locally closed [6] (resp., semi-*$\mathbb{I}$*-locally closed [12]) if we can present* L *as* $L = H \cap K$*, where* $H \in \mathbb{T}$ *and* K *is* $\star$*-perfect (resp.,* $L = H \cap L^\star$*, where* H *is semi-open). An equivalent definition of* L *to be* $\mathbb{I}$*-locally closed is* $L = H \cap L^\star$*, where* $H \in \mathbb{T}$ *(see [12]).*

Remark 2. *As we know from [11],* $H^\star$ *is closed, and from [22],* $\Psi(H)$ *is open, it is derived that* $\star$*-Locally closed sets are locally closed. For reverse direction, we consider next example.*

Example 6. *Take* $\mathbb{T} = \{\varnothing, \{\ell_1\}, \{\ell_2\}, \{\ell_4\}, \{\ell_1, \ell_2\}, \{\ell_1, \ell_4\}, \{\ell_2, \ell_4\}, \{\ell_1, \ell_2, \ell_4\}, \mathbf{Z}\}$ *and* $\mathbb{I} = \{\varnothing, \{\ell_1\}, \{\ell_3\}, \{\ell_1, \ell_3\}\}$ *on* $\mathbf{Z} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$*. Different values of* $K \subseteq \mathbf{Z}$ *under the operators* Cl, Int, $(\cdot)^\star$ *and* Ψ *are considered in TABLE 1.*

TABLE 1. Values of $K \subseteq \mathbf{Z}$ under various operators

K	$\mathrm{Cl}(K)$	$\mathrm{Int}(K)$	$\mathrm{Cl}(\mathrm{Int}(K))$	$K^\star$	$\Psi(K)$	$L^\star(K)$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\{\ell_1\}$	$\varnothing$
$\{\ell_1\}$	$\{\ell_1, \ell_3\}$	$\{\ell_1\}$	$\{\ell_1, \ell_3\}$	$\varnothing$	$\{\ell_1\}$	$\varnothing$
$\{\ell_2\}$	$\{\ell_2, \ell_3\}$	$\{\ell_2\}$	$\{\ell_2, \ell_3\}$	$\{\ell_2, \ell_3\}$	$\{\ell_1, \ell_2\}$	$\{\ell_2\}$
$\{\ell_3\}$	$\{\ell_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{\ell_1\}$	$\varnothing$
$\{\ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_1, \ell_4\}$	$\{\ell_4\}$
$\{\ell_1, \ell_2\}$	$\{\ell_1, \ell_2, \ell_3\}$	$\{\ell_1, \ell_2\}$	$\{\ell_1, \ell_2, \ell_3\}$	$\{\ell_2, \ell_3\}$	$\{\ell_1, \ell_2\}$	$\{\ell_2\}$
$\{\ell_1, \ell_3\}$	$\{\ell_1, \ell_3\}$	$\{\ell_1\}$	$\{\ell_1, \ell_3\}$	$\varnothing$	$\{\ell_1\}$	$\varnothing$
$\{\ell_1, \ell_4\}$	$\{\ell_1, \ell_3, \ell_4\}$	$\{\ell_1, \ell_4\}$	$\{\ell_1, \ell_3, \ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_1, \ell_4\}$	$\{\ell_4\}$
$\{\ell_2, \ell_3\}$	$\{\ell_2, \ell_3\}$	$\{\ell_2\}$	$\{\ell_2, \ell_3\}$	$\{\ell_2, \ell_3\}$	$\{\ell_1, \ell_2\}$	$\{\ell_2\}$
$\{\ell_2, \ell_4\}$	$\{\ell_2, \ell_3, \ell_4\}$	$\{\ell_2, \ell_4\}$	$\{\ell_2, \ell_3, \ell_4\}$	$\{\ell_2, \ell_3, \ell_4\}$	$\mathbf{Z}$	$\{\ell_2, \ell_3, \ell_4\}$
$\{\ell_3, \ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_1, \ell_4\}$	$\{\ell_4\}$
$\{\ell_1, \ell_2, \ell_3\}$	$\{\ell_1, \ell_2, \ell_3\}$	$\{\ell_1, \ell_2\}$	$\{\ell_1, \ell_2, \ell_3\}$	$\{\ell_2, \ell_3\}$	$\{\ell_1, \ell_2\}$	$\{\ell_2\}$
$\{\ell_1, \ell_2, \ell_4\}$	$\mathbf{Z}$	$\{\ell_1, \ell_2, \ell_4\}$	$\mathbf{Z}$	$\{\ell_2, \ell_3, \ell_4\}$	$\mathbf{Z}$	$\{\ell_2, \ell_3, \ell_4\}$
$\{\ell_1, \ell_3, \ell_4\}$	$\{\ell_1, \ell_3, \ell_4\}$	$\{\ell_1, \ell_4\}$	$\{\ell_1, \ell_3, \ell_4\}$	$\{\ell_3, \ell_4\}$	$\{\ell_1, \ell_4\}$	ℓ_4
$\{\ell_2, \ell_3, \ell_4\}$	$\{\ell_2, \ell_3, \ell_4\}$	$\{\ell_2, \ell_4\}$	$\{\ell_2, \ell_3, \ell_4\}$	$\{\ell_2, \ell_3, \ell_4\}$	$\mathbf{Z}$	$\{\ell_2, \ell_3, \ell_4\}$
$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\{\ell_2, \ell_3, \ell_4\}$	$\mathbf{Z}$	$\{\ell_2, \ell_3, \ell_4\}$

We observe that $\{\ell_3\}$ is locally closed but not $\star$-Locally closed. Also, $\{\ell_2\}$ is $\star$-Locally closed but not $\star$-perfect whereas $\{\ell_2, \ell_3\}$ is $\star$-perfect but not $\star$-Locally closed. Further, $\{\ell_3, \ell_4\}$ is $\mathbb{I}$-locally closed but not $\star$-Locally closed; $\{\ell_2, \ell_4\}$ is semi-$\mathbb{I}$-locally closed but not $\star$-Locally closed. Here, $\star$-Locally closed sets are precisely $\varnothing$, $\{\ell_2\}$, $\{\ell_4\}$ and $\{\ell_2, \ell_3, \ell_4\}$, and these are also $\mathbb{I}$-locally closed and hence, they are semi-$\star$-locally closed (as we know from [12] that $\mathbb{I}$-locally closed implies semi-$\mathbb{I}$-locally closed). Because $\{\ell_4\}$ is $\star$-Locally closed is locally closed and hence, λ-locally closed (since locally closed implies λ-locally closed [20]), whereas $\{d\}$ in Example 2.3 of [20] λ-locally closed but not $\star$-Locally closed.

Following diagram will provide a transparent idea regarding different local versions of sets just discussed above:

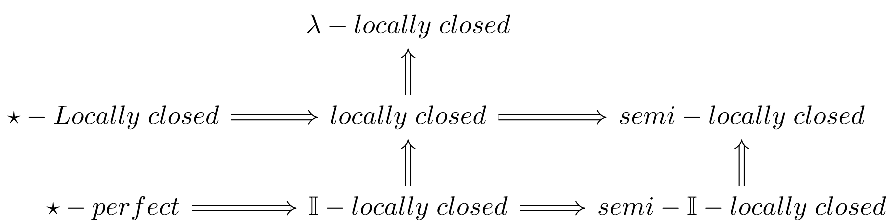

FIGURE 1. Implication Diagram

Theorem 3. *If H be $\star$-dense in itself and $\star$-Locally closed in $\zeta_{\mathbb{I}}$, then H is $\mathbb{I}$-locally closed.*

Proof. Straightforward. □

Corollary 1. *If H be $\star$-dense in itself and $\star$-Locally closed in $\zeta_{\mathbb{I}}$, then H is semi-$\mathbb{I}$-locally closed.*

Theorem 4. *An $L \subseteq \mathbf{Z}$ is $\star$-Locally closed in $\zeta_{\mathbb{I}}$ if and only if $L = H^\star \setminus (\mathbf{Z} \setminus H)^\star$ for some $H \subseteq \mathbf{Z}$.*

Proof. Immediate from Theorem 1(5). □

Theorem 5. *An $L \subseteq \mathbf{Z}$ is $\star$-Locally closed in $\zeta_{\mathbb{I}}$ if and only if $L = \Psi(H) \setminus \Psi(\mathbf{Z} \setminus H)$ for some $H \subseteq \mathbf{Z}$.*

Proof. Immediate from Theorem 1(4). □

Theorem 6. *An $L \subseteq \mathbf{Z}$ is $\star$-Locally closed in $\zeta_{\mathbb{I}}$ if and only if $\mathbf{Z} \setminus L = (\mathbf{Z} \setminus H)^\star \cup (\mathbf{Z} \setminus H^\star)$ for some $H \subseteq \mathbf{Z}$.*

Proof. Obvious from Theorem 1(6). □

It is known that in ζ, open as well as closed sets are locally closed whereas in $\zeta_{\mathbb{I}}$, this occurrence need not longer be true in case of $\star$-Locally closedness. For this purpose, consider the next example.

Example 7. *Think about Example 3, and pick $\{\ell_1\}$, a clopen set. Since no $H \subseteq \mathbf{Z}$ satisfies $\{\ell_1\} = H^\star \cap \Psi(H)$, $\{\ell_1\}$ is not $\star$-Locally closed in $\zeta_{\mathbb{I}}$.*

Theorem 7. *If $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$, then every regular open set is $\star$-Locally closed in $\zeta_{\mathbb{I}}$.*

Proof. Pick a regular open set H. So $H = \Psi(H)$. Now, $\mathbb{I} \cap \mathbb{T} = \{\varnothing\}$ yields $H \subseteq H^\star$. Evidently, $H^\star \cap \Psi(H) = H$. This allows that $H \in L^\star(\zeta_{\mathbb{I}})$. □

Example 8. *Following facts are observed in a $\zeta_{\mathbb{I}}$:*

- *In Example 3, $\{\ell_2\}$ is $\star$-Locally closed but its complement $\{\ell_1\}$ is not.*
- *In Example 3, $\{\ell_2\}$ is $\star$-Locally closed but its super set $\{\ell_2, \ell_3\}$ is not.*
- *In Example 6, $\{\ell_2, \ell_3, \ell_4\}$ is $\star$-Locally closed but its subset $\{\ell_2, \ell_3\}$ is not.*
- *In Example 6, for the subset $\{\ell_2, \ell_3, \ell_4\}$, $L^\star(\{\ell_2, \ell_3, \ell_4\})$ is not open.*
- *In Example 6, for the subset $\{\ell_4\}$, $L^\star(\{\ell_4\})$ is not closed.*
- *In Example 6, $\{\ell_2\}$ and $\{\ell_4\}$ are $\star$-closed but their union $\{\ell_2, \ell_4\}$ is not.*

Remark 3. *From above example, we say that the compilation $L^\star(\zeta_{\mathbb{I}})$ usually does not form a topology, boolean algebra, generalized topology [15], ideal, filter [4] and grill [5, 17].*

4. Homeomorphisms

Though this entire section, an ideal $\mathbb{I}$ is considered as proper, ϑ as $(\mathbf{W}, \mathbb{O})$ and $\vartheta_{\curlyvee(\mathbb{I})}$ as $(\mathbf{W}, \mathbb{O}, \curlyvee(\mathbb{I}))$.

Lemma 1. *[24] If an ideal $\mathbb{I}$ on $\mathbf{Z}$ be proper and $\curlyvee : \mathbf{Z} \to \mathbf{W}$ bijective, then the ideal $\curlyvee(\mathbb{I}) = \{\curlyvee(I) : I \in \mathbb{I}\}$ is proper on $\mathbf{W}$.*

Below, we now disclose that 'bijectivity' of $\curlyvee$ in Lemma 1 is sufficient to carry a (proper) ideal to a (proper) ideal.

Lemma 2. *Suppose $\curlyvee : \mathbf{Z} \to \mathbf{W}$ is a map, and $\mathbb{I}$ an ideal on $\mathbf{Z}$. Then $\curlyvee(\mathbb{I})$ defined in Lemma 1 is an ideal on $\mathbf{W}$. Moreover, injectivity of $\curlyvee$ preserves 'properness' of $\mathbb{I}$.*

Proof. Firstly, $\varnothing \in \mathbb{I}$ (since an ideal) implies $\curlyvee(\varnothing) \in \curlyvee(\mathbb{I})$. But $\curlyvee(\varnothing) = \varnothing$. So, $\varnothing \in \curlyvee(\mathbb{I})$. Secondly, pick E_1, $E_2 \in \curlyvee(\mathbb{I})$. Then, by the definition of $\curlyvee(\mathbb{I})$, choose I_1, $I_2 \in \mathbb{I}$ such that $E_1 = \curlyvee(I_1)$ and $E_2 = \curlyvee(I_2)$. Now, $E_1 \cup E_2 = \curlyvee(I_1) \cup \curlyvee(I_2) = \curlyvee(I_1 \cup I_2) = \curlyvee(I_3)$, where $I_3 = I_1 \cup I_2 \in \mathbb{I}$ (since $\mathbb{I}$ is ideal). This permits that $E_1 \cup E_2 \in \curlyvee(\mathbb{I})$. Lastly, take $F_1 \subseteq F_2$ and $F_2 \in \curlyvee(\mathbb{I})$. So, there is an $I \in \mathbb{I}$ such that $F_2 = \curlyvee(I)$. Now, $F_1 \subseteq \curlyvee(I) = \{\curlyvee(u) : u \in I\}$ knocks us to construct an $I_0 \subseteq \mathbf{Z}$ as: 'Pick those $u \in I$ whose images under $\curlyvee$ goes to F_1, and keep such u in I_0'. Thus, $I_0 = \{u \in I : \curlyvee(u) \in F_1\}$. Clearly, $\curlyvee(I_0) = F_1$ and $I_0 \subseteq I$. Because $\mathbb{I}$ is an ideal, $I \in \mathbb{I}$ implies $I_0 \in \mathbb{I}$. This again implies $\curlyvee(I_0) \in \curlyvee(\mathbb{I})$ i.e., $F_1 \in \curlyvee(\mathbb{I})$. Thus, we

finally present that $\curlyvee(\mathbb{I})$ is an ideal on $\mathbf{W}$.
For second part, suppose $\mathbb{I}$ is proper and $\curlyvee$ injective. Claim: $\curlyvee(\mathbb{I})$ is proper i.e., $\mathbf{W} \notin \curlyvee(\mathbb{I})$. If not, there exists $I \in \mathbb{I}$ such that $\curlyvee(I) = \mathbf{W}$. Now, $I \subseteq \mathbf{Z}$ implies $\mathbf{W} = \curlyvee(I) \subseteq \curlyvee(\mathbf{Z}) \subseteq \mathbf{W}$ whence $\curlyvee(I) = \curlyvee(\mathbf{Z})$. This yields $\curlyvee^{-1}(\curlyvee(I)) = \curlyvee^{-1}(\curlyvee(\mathbf{Z}))$ implies $I = \mathbf{Z}$ (since $\curlyvee$ is injective). So, $\mathbf{Z} \in \mathbb{I}$, a contradiction. □

As consequences of the Lemma 1 we have following:

Theorem 8. *Let $\curlyvee : \zeta_{\mathbb{I}} \to \vartheta$ is a homeomorphism. Then, for every $H \subseteq \mathbf{Z}$, we have*

(1) $\curlyvee[H^\star(\mathbb{I})] = [\curlyvee(H)]^\star(\curlyvee(\mathbb{I}))$,
(2) $\curlyvee[\Psi(H)(\mathbb{I})] = \Psi[\curlyvee(H)](\curlyvee(\mathbb{I}))$.

Proof. (1) Assume $v \notin [\curlyvee(H)]^\star(\curlyvee(\mathbb{I}))$. Pick an $E \in \mathbb{O}$ such that $v \in E$ and $E \cap \curlyvee(H) \in \curlyvee(\mathbb{I})$. Draw an $I \in \mathbb{I}$ such that $\curlyvee(I) = E \cap \curlyvee(H)$. Because $\curlyvee$ is injective, $\curlyvee^{-1}(E) \cap H = \curlyvee^{-1}(E) \cap \curlyvee^{-1}(\curlyvee(H)) = \curlyvee^{-1}(E \cap \curlyvee(H)) = \curlyvee^{-1}(\curlyvee(I)) = I \in \mathbb{I}$, where $\curlyvee^{-1}(E) \in \mathbb{T}_{\curlyvee^{-1}(v)}$ (by continuity of $\curlyvee$). This tells that $\curlyvee^{-1}(v) \notin H^\star(\mathbb{I})$, and we have $v \notin \curlyvee[H^\star(\mathbb{I})]$. So, $\curlyvee[H^\star(\mathbb{I})] \subseteq [\curlyvee(H)]^\star(\curlyvee(\mathbb{I}))$. Reversely, pick $u \in \mathbf{W}$ such that $u \notin \curlyvee[H^\star(\mathbb{I})]$. Then, $\curlyvee^{-1}(u) \notin H^\star(\mathbb{I})$. There is $G \in \mathbb{T}_{\curlyvee^{-1}(u)}$ such that $G \cap H \in \mathbb{I}$. So, $\curlyvee(G) \cap \curlyvee(H) = \curlyvee(G \cap H) \in \curlyvee(\mathbb{I})$, where $\curlyvee(G) \in \mathbb{O}_u$. This highlights that $u \notin [\curlyvee(H)]^\star(\curlyvee(\mathbb{I}))$. Therefore, $[\curlyvee(H)]^\star(\curlyvee(\mathbb{I})) \subseteq \curlyvee[H^\star(\mathbb{I})]$. Hence, the result.

(2) $\curlyvee[\Psi(H)(\mathbb{I})] = \curlyvee[\mathbf{Z} \setminus (\mathbf{Z} \setminus H)^\star(\mathbb{I})] = \mathbf{W} \setminus \curlyvee[(\mathbf{Z} \setminus H)^\star(\mathbb{I})] = \mathbf{W} \setminus [\curlyvee(\mathbf{Z} \setminus H)]^\star(\curlyvee(\mathbb{I}))$ (by first part) $= \mathbf{W} \setminus [\mathbf{W} \setminus \curlyvee(H)]^\star(\curlyvee(\mathbb{I})) = \Psi[\curlyvee(H)](\curlyvee(\mathbb{I}))$. □

Theorem 9. *For a homeomorphism $\curlyvee : \zeta_{\mathbb{I}} \to \vartheta_{\curlyvee(\mathbb{I})}$, followings are well fulfilled:*

(1) *if H be $\star$-perfect in $\zeta_{\mathbb{I}}$, then $\curlyvee(H)$ is $\star$-perfect in $\vartheta_{\curlyvee(\mathbb{I})}$,*
(2) *if H be $\mathbb{I}$-locally closed in $\zeta_{\mathbb{I}}$, then $\curlyvee(H)$ is $\curlyvee(\mathbb{I})$-locally closed in $\vartheta_{\curlyvee(\mathbb{I})}$,*
(3) *if H be semi-$\mathbb{I}$-locally closed in $\zeta_{\mathbb{I}}$, then $\curlyvee(H)$ is semi-$\curlyvee(\mathbb{I})$-locally closed in $\vartheta_{\curlyvee(\mathbb{I})}$.*

Proof. First two results are straightforward from Theorem 8 (1), and third one follows from Theorem 8 (1) and the fact that 'E is semi-open implies $\curlyvee(E)$ is semi-open'. □

For more homeomorphic image regarding $(\cdot)^\star$ and Ψ operators interested readers can see [19].

Theorem 10. *For a homeomorphism $\curlyvee : \zeta_{\mathbb{I}} \to \vartheta_{\curlyvee(\mathbb{I})}$ and for $H \subseteq \mathbf{Z}$, we have*

(1) $\curlyvee[L^\star(H)(\mathbb{I})] = L^\star[\curlyvee(H)](\curlyvee(\mathbb{I}))$,
(2) $H \in L^\star(\zeta_{\mathbb{I}})$ *implies* $\curlyvee(H) \in L^\star(\vartheta_{\curlyvee(\mathbb{I})})$.

Proof. First one is derived from Theorem 8, and second one is a consequence of first part. □

Lemma 3. *[24] If an ideal $\mathbb{J}$ on $\mathbf{W}$ be proper and $\Upsilon : \mathbf{Z} \to \mathbf{W}$ surjective, then the ideal $\Upsilon^{-1}(\mathbb{J}) := \{\Upsilon^{-1}(J) : J \in \mathbb{J}\}$ is proper on $\mathbf{Z}$.*

Below, by presenting a sophisticated counterexample, we will show the Lemma 3 is wrong.

Example 9. *Consider the map $\Upsilon : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ as $x \mapsto |x|$. Here, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all integers and the set of all positive integers, respectively, and $|\cdot|$ is the modulus function. Note that Υ is surjective. Consider the subset O of all odd positive integers, and take $\mathbb{J} = \wp(O)$. Then, $\mathbb{J}$ is a proper ideal on $\mathbb{N} \cup \{0\}$. Now, $\{1\} \in \mathbb{J}$ implies $\Upsilon^{-1}(\{1\}) = \{-1, +1\} \in \Upsilon^{-1}(\mathbb{J})$. Though $\{-1\} \subseteq \{-1, +1\}$, $\{-1\} \notin \Upsilon^{-1}(\mathbb{J})$. Thus, $\Upsilon^{-1}(\mathbb{J})$ is not an ideal on $\mathbb{Z}$.*

A modification of Lemma 3 is presented below:

Lemma 4. *Let $\Upsilon : \mathbf{Z} \to \mathbf{W}$ be a map, and $\mathbb{J}$ an ideal on $\mathbf{W}$. Then*

$$\Upsilon^{\leftarrow}(\mathbb{J}) := \{E \subseteq \mathbf{Z} : E \subseteq \Upsilon^{-1}(J),\ J \in \mathbb{J}\}$$

is an ideal on $\mathbf{Z}$. In addition, surjectivity of Υ preserves 'properness' of $\mathbb{J}$.

Proof. Firstly, $\varnothing \subseteq \Upsilon^{-1}(\varnothing)$, where $\varnothing \in \mathbb{J}$ (since an ideal) implies $\varnothing \in \Upsilon^{\leftarrow}(\mathbb{J})$. Secondly, take $E_1 \subseteq E_2$ and $E_2 \in \Upsilon^{\leftarrow}(\mathbb{J})$. There is a $J \in \mathbb{J}$ such that $E_2 \subseteq \Upsilon^{-1}(J)$, and so, $E_1 \subseteq \Upsilon^{-1}(J)$ implies that $E_1 \in \Upsilon^{\leftarrow}(\mathbb{J})$. Thirdly, consider E_1, $E_2 \in \Upsilon^{\leftarrow}(\mathbb{J})$. Then, pick J_1, $J_2 \in \mathbb{J}$ such that $E_1 \subseteq \Upsilon^{-1}(J_1)$ and $E_2 \subseteq \Upsilon^{-1}(J_2)$. Now, $E_1 \cup E_2 \subseteq \Upsilon^{-1}(J_1) \cup \Upsilon^{-1}(J_2) = \Upsilon^{-1}(J_1 \cup J_2)$, where $J_1 \cup J_2 \in \mathbb{J}$ (since $\mathbb{J}$ is an ideal). Therefore, $E_1 \cup E_2 \in \Upsilon^{\leftarrow}(\mathbb{J})$. Thus, we demonstrate that $\Upsilon^{\leftarrow}(\mathbb{J})$ is an ideal on $\mathbf{Z}$.
For second part, consider Υ is surjective and $\mathbb{J}$ proper. Claim: $\Upsilon^{\leftarrow}(\mathbb{J})$ is proper. If not so, $\mathbf{Z} \in \Upsilon^{\leftarrow}(\mathbb{J})$. Choose $J \in \mathbb{J}$ such that $\mathbf{Z} \subseteq \Upsilon^{-1}(J)$. Because Υ is surjective, $\mathbf{W} = \Upsilon(\mathbf{Z}) \subseteq \Upsilon(\Upsilon^{-1}(J)) = J \subseteq \mathbf{W}$ implies $\mathbf{W} = J \in \mathbb{J}$, a contradiction. □

We demonstrate another modification of Lemma 3 in next corollary:

Corollary 2. *If Υ be bijective, then $\Upsilon^{-1}(\mathbb{J})$ of Lemma 3 coincides with $\Upsilon^{\leftarrow}(\mathbb{J})$, and hence, becomes an ideal.*

Proof. It is transparent from the fact 'for each $J \in \mathbb{J}$, $\Upsilon^{-1}(J) \subseteq \Upsilon^{-1}(J)$' that $\Upsilon^{-1}(\mathbb{J}) \subseteq \Upsilon^{\leftarrow}(\mathbb{J})$. For backward part, let's pick an $E \in \Upsilon^{\leftarrow}(\mathbb{J})$. Then, $E \subseteq \Upsilon^{-1}(J)$ for some $J \in \mathbb{J}$. Because Υ is surjective, $\Upsilon(E) \subseteq \Upsilon(\Upsilon^{-1}(J)) = J$ implies $\Upsilon(E) \in \mathbb{J}$. Because Υ is injective, $E = \Upsilon^{-1}(\Upsilon(E)) \in \Upsilon^{-1}(\mathbb{J})$. Thus, $\Upsilon^{\leftarrow}(\mathbb{J}) \subseteq \Upsilon^{-1}(\mathbb{J})$, as aimed. □

As an application of Corollary 2, we have following important result:

Theorem 11. *For a homeomorphism $\Upsilon : \zeta_{\Upsilon^{-1}(\mathbb{J})} \to \vartheta_{\mathbb{J}}$, and for $K \subseteq \mathbf{W}$, we have*
(1) $\Upsilon^{-1}[K^{\star}(\mathbb{J})] = [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$,
(2) $\Upsilon^{-1}[\Psi(K)(\mathbb{J})] = \Psi[\Upsilon^{-1}(K)](\Upsilon^{-1}(\mathbb{J}))$.

Proof. (1) Assume $u \notin [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$. Select an $E \in \mathbb{T}_u$ for which $E \cap \Upsilon^{-1}(K) \in \Upsilon^{-1}(\mathbb{J})$. Draw a $J \in \mathbb{J}$ such that $E \cap \Upsilon^{-1}(K) = \Upsilon^{-1}(J)$. Because Υ is bijective, $\Upsilon(E) \cap K = \Upsilon(E) \cap \Upsilon(\Upsilon^{-1}(K)) = \Upsilon(E \cap \Upsilon^{-1}(K)) = \Upsilon(\Upsilon^{-1}(J)) = J \in \mathbb{J}$, where continuity of Υ^{-1} implies $\Upsilon(E) \in \mathbb{O}_{\Upsilon(u)}$. This states that $\Upsilon(u) \notin K^{\star}(\mathbb{J})$, and this again implies $u \notin \Upsilon^{-1}[K^{\star}(\mathbb{J})]$. Therefore, $\Upsilon^{-1}[K^{\star}(\mathbb{J})] \subseteq [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$. For reverse part, pick $v \notin \Upsilon^{-1}[K^{\star}(\mathbb{J})]$. Then, $\Upsilon(v) \notin K^{\star}(\mathbb{J})$. Choose $F \in \mathbb{O}_{\Upsilon(v)}$ such that $F \cap K \in \mathbb{J}$. Continuity of Υ assures $\Upsilon^{-1}(F) \in \mathbb{T}_v$, and $\Upsilon^{-1}(F) \cap \Upsilon^{-1}(K) = \Upsilon^{-1}(F \cap K) \in \Upsilon^{-1}(\mathbb{J})$. This indicates $v \notin [\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$, and consequently $[\Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J})) \subseteq \Upsilon^{-1}[K^{\star}(\mathbb{J})]$.

(2) $\Upsilon^{-1}[\Psi(K)(\mathbb{J})] = \Upsilon^{-1}[\mathbf{W} \setminus (\mathbf{W} \setminus K)^{\star}(\mathbb{J})] = \mathbf{Z} \setminus \Upsilon^{-1}[(\mathbf{W} \setminus K)^{\star}(\mathbb{J})] = \mathbf{Z} \setminus [\Upsilon^{-1}(\mathbf{W} \setminus K)]^{\star}(\Upsilon^{-1}(\mathbb{J}))$ (by first part) $= \mathbf{Z} \setminus [\mathbf{Z} \setminus \Upsilon^{-1}(K)]^{\star}(\Upsilon^{-1}(\mathbb{J})) = \Psi[\Upsilon^{-1}(K)](\Upsilon^{-1}(\mathbb{J}))$.

□

Theorem 12. *For a homeomorphism $\Upsilon : \zeta_{\Upsilon^{-1}(\mathbb{J})} \to \vartheta_{\mathbb{J}}$, followings are well fulfilled:*

1. *if K be $\star$-perfect in $\vartheta_{\mathbb{J}}$, then $\Upsilon^{-1}(K)$ is $\star$-perfect in $\zeta_{\Upsilon^{-1}(\mathbb{J})}$,*
2. *if K be $\mathbb{J}$-locally closed in $\vartheta_{\mathbb{J}}$, then $\Upsilon^{-1}(K)$ is $\Upsilon^{-1}(\mathbb{J})$-locally closed in $\zeta_{\Upsilon^{-1}(\mathbb{J})}$,*
3. *if K be semi-$\mathbb{J}$-locally closed in $\vartheta_{\mathbb{J}}$, then $\Upsilon^{-1}(K)$ is semi-$\Upsilon^{-1}(\mathbb{J})$-locally closed in $\zeta_{\Upsilon^{-1}(\mathbb{J})}$.*

Proof. First two results are straightforward from Theorem 11 (1), and third one follows from Theorem 11 (1) and the fact that 'F is semi-open implies $\Upsilon^{-1}(F)$ is semi-open'. □

Theorem 13. *For a homeomorphism $\Upsilon : \zeta_{\Upsilon^{-1}(\mathbb{J})} \to \vartheta_{\mathbb{J}}$, and for $K \subseteq \mathbf{W}$, we have*

1. $\Upsilon^{-1}[L^{\star}(K)(\mathbb{J})] = L^{\star}[\Upsilon^{-1}(K)](\Upsilon^{-1}(\mathbb{J}))$,
2. $K \in L^{\star}(\vartheta_{\mathbb{J}})$ *implies* $\Upsilon^{-1}(K) \in L^{\star}(\zeta_{\Upsilon^{-1}(\mathbb{J})})$.

Proof. First one is derived from Theorem 11, and second one is a consequence of first part. □

5. Conclusion

Kuratowski's local function '$(\cdot)^{\star}$' is a generalized operator of the classic closure operator 'Cl', and 'Ψ' operator is a generalized operator of the classic interior operator 'Int'. On the other side, one can think Bourbaki's locally closed sets are applications of the operators 'Cl' and 'Int'. Replacing these classic operators by the updated generalized operator '$(\cdot)^{\star}$' and 'Ψ', we derived a new version of locally closed set, and named $\star$-Locally closed. Example 6 and FIGURE 1 show that our $\star$-Locally closed version is a stronger form of locally closed set.

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