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Finding Solutions to Undamped and Damped Simple Harmonic Oscillations via Kashuri Fundo Transform

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ABSTRACT

Differential equations are expressions that are frequently encountered in mathematical modeling of laws or problems in many different fields of science. It can find its place in many fields such as applied mathematics, physics, chemistry, finance, economics, engineering, etc. They make them more understandable and easier to interpret, by modeling laws or problems mathematically. Therefore, solutions of differential equations are very important. Many methods have been developed that can be used to reach solutions of differential equations. One of these methods is integral transforms. Studies have shown that the use of integral transforms in the solutions of differential equations is a very effective method to reach solutions. In this study, we are looking for a solution to damped and undamped simple harmonic oscillations modeled by linear ordinary differential equations by using Kashuri Fundo transform, which is one of the integral transforms. From the solutions, it can be concluded that the Kashuri Fundo transform is an effective method for reaching the solutions of ordinary differential equations.

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1. Introduction

Differential equations are mathematical models that are encountered in many different fields of applied sciences and include one or more functions and their derivatives [1]. Generally, laws and problems in physics, chemistry, biology, engineering, economics, finance and many other fields are expressed using differential equations. Although it is one of the subjects that applied mathematics is mostly studied, differential equations have been encountered in every field that is intertwined with mathematics in a professional sense. The principles used in the formation of many inventions that make our lives easier in daily life include differential equations [1, 2].

Simple harmonic oscillation (SHO), which is one of the advanced physics subjects, is also one of the principles modeled with differential equations. Simple harmonic oscillation (SHO) logically comes after linear oscillation and circular oscillation [3]. This oscillation has a special periodic motion. In mechanical systems, as the system moves, a restoring force is created that can do both positive and negative work. In SHO, the restoring force on the moving object is directly proportional to the displacement magnitude of the object. This indicates that the oscillation is linear. Especially in linear oscillators, the oscillation frequency is independent of amplitude. In order to physically represent linear oscillators correctly, a special damping force is defined that draws energy from the oscillator. In mechanical oscillators, this force is due to friction. Such oscillators are called damped oscillators. Damped or undamped oscillations can be driven by external forces [3–5].

Many different methods have been used in modeling damped and undamped simple harmonic oscillations, in mechanics and in reaching the solutions of differential equations that appear in many different fields. One of these methods is integral transforms [6]. Integral transforms allow the solutions of differential equations to be reached more easily and without dealing with complicated operations. There are many different integral transforms available [7–12]. The Kashuri Fundo transform, which we have used in this study, is one of these transforms [13]. When the literature is examined, many studies are encountered in which Kashuri Fundo transform is used to reach the solution of differential equations [14–22]. In this study, we sought a solution for damped and undamped simple harmonic oscillations by using Kashuri Fundo transform.

2. Kashuri Fundo Transform

2.1. Definition of Kashuri Fundo Transform

Definition 1. We consider functions in the set F defined as [13],

$$F = \left\{ f(t) | \exists M, k_1, k_2 > 0, \quad \text{such that} \quad |f(t)| \leqslant M e^{\frac{|t|}{k_i^2}}, \quad \text{if} \quad t \in (-1)^i \times [0, \infty) \right\}$$

For a function belonging to the set F, M must be finite number. k_1, k_2 may be finite or infinite.

Definition 2. Kashuri Fundo transform defined on the set F and denoted by the operator $\mathscr{K}(.)$ is defined as [13],

$$\mathscr{K}[f(t)](v) = A(v) = \frac{1}{v} \int_0^\infty e^{\frac{-t}{v^2}} f(t)\, dt, \quad t \geqslant 0, \quad -k_1 < v < k_2 \qquad (1)$$

The Kashuri Fundo transform expressed by (1) can also be expressed as [13],

$$\mathscr{K}[f(t)](v) = A(v) = v \int_0^\infty e^{-t} f(v^2 t)\, dt$$

Inverse Kashuri Fundo transform is denoted by $\mathscr{K}^{-1}[A(v)] = f(t), t \geq 0$.

Definition 3. A function $f(t)$ is said to be of exponential order $\frac{1}{k^2}$, if there exist positive constants T and M such that, $|f(t)| \leq M e^{\frac{-t}{k^2}}$, for all $t \geq T$ [13].

Theorem 1. (Sufficient Conditions for Existence of Kashuri Funfo Transform) If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $\frac{1}{k^2}$, then $\mathscr{K}[f(t)](v)$ exists for $|v| < k$ [13].

2.2. Some Properties of Kashuri Fundo Transform

Theorem 2. (Linearity Property) Let $f(t), g(t) \in F$ be functions whose Kashuri Fundo integral transforms exists and c be a constant. Then [13],

$$\mathscr{K}[(f+g)(t)](v) = \mathscr{K}[f(t)](v) + \mathscr{K}[g(t)](v)$$

$$\mathscr{K}[(cf)(t)](v) = c\mathscr{K}[f(t)](v)$$

Theorem 3. (Kashuri Fundo Transform of The Derivatives of The Function $f(t)$) Let's assume that the Kashuri Fundo transform of $f(t) \in F$ is $A(v)$. Then [13],

$$\mathscr{K}[f'(t)](v) = \frac{A(v)}{v^2} - \frac{f(0)}{v} \qquad (2)$$

$$\mathscr{K}[f''(t)](v) = \frac{A(v)}{v^4} - \frac{f(0)}{v^3} - \frac{f'(0)}{v} \qquad (3)$$

$$\mathscr{K}[f^{(n)}(t)](v) = \frac{A(v)}{v^{2n}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2(n-k)-1}} \qquad (4)$$

Theorem 4. (First Translation Theorem) Let's assume that the Kashuri Fundo transform of $f(t) \in F$ is $A(v)$. Then [13],

$$\mathscr{K}[e^{at} f(t)] = \left(\frac{1}{\sqrt{1-av^2}}\right) A\left[\frac{v}{\sqrt{1-av^2}}\right] \qquad (5)$$

2.3. Kashuri Fundo Transform of Some Special Functions

Table 1: Kashuri Fundo Transform of Some Special Functions [13, 23]

$f(t)$	$\mathscr{K}[f(t)] = A(v)$
1	v
t	v^3
t^n	$n!v^{2n+1}$
e^{at}	$\frac{v}{1-av^2}$
$sin(at)$	$\frac{av^3}{1+a^2v^4}$
$cos(at)$	$\frac{v}{1+a^2v^4}$
$sinh(at)$	$\frac{av^3}{1-a^2v^4}$
$cosh(at)$	$\frac{v}{1-a^2v^4}$
t^α	$\Gamma(\alpha+1)v^{2\alpha+1}$
$\sum_{k=0}^{n} a_k t^k$	$\sum_{k=0}^{n} k!a_k v^{2k+1}$

3. Application of Kashuri Fundo Transform to The Equation of Simple Harmonic Oscillation

3.1. Undamped Simple Harmonic Oscillation

The undamped simple harmonic oscillation is mathematically modeled as [3–5]

$$y''(x) + \omega_0^2 y(x) = 0 \tag{6}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ and is called the natural angular frequency. m is a inertial mass of the object and k is a spring constant.

Having applied bilaterally the Kashuri Fundo transform to the differential equation of undamped simple harmonic oscillation, we acquire

$$\mathscr{K}[y''(x)] + \omega_0^2 \mathscr{K}[y(x)] = 0 \tag{7}$$

Rearranging the equation (7) using the equation (3), we get

$$A(v) = \frac{v^4}{1+\omega_0^2 v^4}\left(\frac{f(0)+v^2 f'(0)}{v^3}\right) \tag{8}$$

If we assume that $f(0) = c_1$ and $f'(0) = c_2$ here (c_1, c_2 are constants), and arranging this equation to apply the inverse Kashuri Fundo transform, we get

$$A(v) = \frac{c_1 v}{1+\omega_0^2 v^4} + \frac{c_2 v^3}{1+\omega_0^2 v^4} \tag{9}$$

Having applied bilaterally the inverse Kashuri Fundo transform to the equation (9), using table 1, we acquire

$$y(x) = c_1 \cos(\omega_0 x) + \frac{c_2}{\omega_0}\sin(\omega_0 x) \tag{10}$$

which is a sinusoidal function.

3.2. Damped Simple Harmonic Oscillation

The damped simple harmonic oscillation is mathematically modeled as [3–5]

$$y''(x) + \frac{b}{m}y'(x) + \omega_0^2 y(x) = 0 \tag{11}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is called the natural angular frequency and b is called the damping coefficient. m is a inertial mass of the object and k is a spring constant.

Having applied bilaterally the Kashuri Fundo transform to the differential equation of damped simple harmonic oscillation with initial conditions $y(0) = 1$ and $y'(0) = 0$, we acquire

$$\mathscr{K}[y''(x)] + \frac{b}{m}\mathscr{K}[y'(x)] + \omega_0^2 \mathscr{K}[y(x)] = 0 \tag{12}$$

Rearranging the equation (12) using the equations (2), (3) and initial conditions, we get

$$A(v) = \frac{v + \frac{b}{m}v^3}{1 + \frac{b}{m}v^2 + \omega_0^2 v^4} \tag{13}$$

We need to get the equation (13) into the form that can be applied the inverse Kashuri Fundo transform. If we rearrange the equation (13) for this, we obtain

$$A(v) = \frac{v\left(1+\frac{b}{2m}v^2\right)}{\left(1+\frac{b}{2m}v^2\right)^2 + \omega_1^2 v^4} + \frac{b}{2m\omega_1}\frac{\omega_1 v^3}{\left(1+\frac{b}{2m}v^2\right)^2 + \omega_1^2 v^4} \tag{14}$$

where $\omega_1 = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$.

Having applied bilaterally the inverse Kashuri Fundo transform to the equation (14), using the equation (5), we acquire

$$y(x) = e^{-(\frac{b}{2m})x}cos(\omega_1 x) + \frac{b}{2m\omega_1}e^{-(\frac{b}{2m})x}sin(\omega_1 x) \tag{15}$$

which is also a sinusoidal function.

4. Conclusion

At the beginning of the study, we mentioned the importance of differential equations for science. The more clear and uncomplicated the steps taken while reaching the solutions of these equations, which are so important for science, the sooner and more clearly the result can be revealed. In this study, we have based on damped and undamped simple harmonic oscillations modeled by ordinary differential equations. We examined the Kashuri Fundo transform through these models. As a result of the applied steps, it was concluded that the Kashuri Fundo transform is a simple and effective method for reaching the solutions of ordinary differential equations.

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