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Spherical Curves with Modified Orthogonal Frame

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Abstract - In [2-4,8-9], the authors have characterized the spherical curves in different spaces.In this paper, we shall characterize the spherical curves according to modified orthogonal frame in Euclidean 3-space.

Keywords - *Spherical curves, Modified orthogonal frame*

1 Introduction

In the Euclidean space E^3 a spherical unit speed curves and their characterizations are given in [8,9]. In [2-4,7], the authors have characterized the Lorentzian and Dual spherical curves in the Minkowski 3-space E_1^3. In this paper, we shall characterize the spherical curves according to modified orthogonal frame in the Euclidean 3-space.

2 Preliminaries

We first recall the classical fundamental theorem of space curves, i.e., curves in Euclidean 3-space E^3. Let $\alpha(s)$ be a curve of class C^3, where s is the arc-length parameter. Moreover we assume that its curvature $\kappa(s)$ does not vanish anywhere. Then there exists an orthonormal frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ which satisfies the Frenet-Serret equation

$$\begin{bmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix} \tag{1}$$

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where $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ are the tangent, principal normal and binormal unit vectors, respectively, and $\tau(s)$ is the torsion. Given a function $\kappa(s)$ of class C^1 and a continuous function $\tau(s)$, there exists a curve of class C^3 which admits an orthonormal frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ satisfying the Eq.(1) with given κ and τ as its curvature and torsion, respectively. Such a curve is uniquely determined by a motion of E^3.

Let α_i $(i = 1, 2, 3)$ be coordinates in E^3. Let $\alpha(s)$ an analytic curve , where s runs through some interval and $\alpha(s)$ be analytic in s. We assume that α is non-singular, i.e.,

$$\sum_{i=1}^{3} \left(\frac{d\alpha_i}{ds} \right)^2$$

is nowhere zero. Therefore we can parametrize α by its arc length s. In the rest of this paper, we only consider α in the following form:

$$\alpha = \alpha(s) = (\alpha_1, \alpha_2, \alpha_3) \ , \ s \in I$$

where $\alpha(s)$ is analytic in s and I is a non-empty open interval. We assume that the curvature $\kappa(s)$ of α is not identically zero. Now we define an orthogonal frame $\{T, N, B\}$ as follows:

$$T = \frac{d\alpha}{ds}, N = \frac{dT}{ds}, B = T \times N,$$

where $T \times N$ is the vector product of T and N. The relations between those and the classical Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ at non-zero points of κ are

$$\begin{cases} T = \mathbf{t} \\ N = \kappa \mathbf{n} \\ B = \kappa \mathbf{b}. \end{cases} \tag{2}$$

Thus $N(s_0) = B(s_0) = 0$ when $\kappa(s_0) = 0$ and squares of the length of N and B vary analytically in s. By the definition of $\{T, N, B\}$ or Eq.(2), a simple calculation shows that

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \frac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} \tag{3}$$

where a dash denotes the differentiation with respect to the arc length s and

$$\tau = \tau(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}$$

is the torsion of α. From the Frenet-Serret equation, we know that any zero point of κ^2 is a removable singularity of τ. The Eq.(3) corresponds to the Frenet-Serret equation in the classical case. Moreover, $\{T, N, B\}$satisfies:

$$\begin{cases} \langle T, T \rangle = 1, \langle N, N \rangle = \langle B, B \rangle = \kappa^2, \\ \langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0 \end{cases} \tag{4}$$

where $\langle , \rangle$ denotes the inner product of E^3. We note that the essential quantities in Eqs.(3) and (4) are $\kappa^2(s)$ and $\tau(s)$ which are analytic in s [7].

3 Spherical Curves with Modified Orthogonal Frame

Definition 3.1. *Let α be in E^3given with coordinate neighborhood (I,α). If $\alpha \subset E^3$ then α is called a spherical curve of E^3.*

Definition 3.2. *The sphere having sufficiently close common four points at $m \in \alpha$ with the curve $\alpha \subset E^3$ is called the osculating sphere or curvature sphere of the curve α at the point $m \in \alpha$.*

Now let us calculate the geometric locus of the sphere having sufficiently close common three points with curve $\alpha \subset E^3$ at the point $m \in \alpha$.

Theorem 3.3. *Let α be in E^3given with coordinate neighborhood (I,α). The geometric locus of the centers of the spherical curves having sufficiently close common three points with the curve α providing the modified orthogonal frame vectors $\{T,N,B\}$at the point $\alpha(s), s \in I$ is*

$$\mathbf{a}(s) = \alpha(s) + m_2(s)N(s) + m_3(s)B(s),$$

where

$$m_2 : I \to R,\ m_2(s) = \kappa^{-2}, \quad m_3(s) = \pm\kappa^{-2}\sqrt{r^2\kappa^2 - 1}.$$

Proof. Let (I,α) be a coordinate neighborhood, and $s \in I$ be parameter for the curve α. Let also $\mathbf{a}$ be the center and r be the radius of the sphere having sufficiently close common three points with α. In accordance to this, let us consider

$$\begin{array}{rcccl} f & : & I & \to & R \\ & & s & \to & f(s) = \langle \mathbf{a} - \alpha(s), \mathbf{a} - \alpha(s)\rangle - r^2. \end{array} \tag{5}$$

Since

$$f(s) = f'(s) = f''(s) = 0 \tag{6}$$

at the point $\alpha(s)$, then the sphere

$$S^2 = \left\{x \in E^3 : \langle x - \mathbf{a}, x - \mathbf{a}\rangle = r^2\right\}, (x \text{ generic point of the sphere})$$

with the curve α at this point passes sufficiently close three points. So, considering Eqs.(5) and (6) together

$$f'(s) = -2\langle T, \mathbf{a} - \alpha(s)\rangle = 0.$$

is obtained. From this, since $f''(s) = 0$, we get

$$\langle T, T\rangle + \langle a - \alpha(s), -N\rangle = 0.$$

Considering Eq.(3) with this, we have

$$\langle \mathbf{a} - \alpha(s), N\rangle = 1.$$

On the other hand, for the base $\{T,N,B\}$,

$$\mathbf{a} - \alpha(s) = m_1(s)T(s) + m_2(s)N(s) + m_3(s)B(s); \quad m_1(s), m_2(s), m_3(s) \in R \tag{7}$$

is obtained. However, by using Eq.(6), we have

$$m_1(s) = \langle \mathbf{a} - \alpha(s), T(s) \rangle \Rightarrow m_1(s) = 0, \tag{8}$$

and

$$\kappa^2 m_2 = \langle \mathbf{a} - \alpha(s), N(s) \rangle \Rightarrow m_2 = \frac{1}{\kappa^2}, \tag{9}$$

By using $f(s) = 0$, we get

$$\langle \mathbf{a} - \alpha(s), \mathbf{a} - \alpha(s) \rangle = r^2 \Rightarrow m_1^2(s) + m_2^2(s)\kappa^2(s) + m_3^2(s)\kappa^2(s) = r^2 \ . \tag{10}$$

Considering Eq.(8) in Eq.(10), we have

$$\left[m_2^2(s) + m_3^2(s)\right] \kappa^2(s) = r^2 \quad . \tag{11}$$

Considering also Eq.(9) in Eq.(11) then

$$m_3(s) = \pm\kappa^{-2}\sqrt{r^2\kappa^2 - 1} = \lambda. \tag{12}$$

Therefore, subtituting Eqs.(8),(9),(12) in Eq.(3)

$$\mathbf{a}(s) = \alpha(s) + \frac{1}{\kappa^2}N(s) \pm \kappa^{-2}\sqrt{r^2\kappa^2 - 1}B(s).$$

Here $\mathbf{a}$ and r change when the spheres change. Hence, $m_3(s) = \lambda \in R$ is a parameter. This completes the proof of the theorem. □

Corollary 3.4. *Let the curve α in E^3 be given with neighborhood coordinate (I, α). Then the centers of the spheres which pass sufficiently close common three points with α at the points $\alpha(s) \in \alpha$ are located on a straight line.*

Proof. By Theorem 3.3, we have

$$\mathbf{a}(s) = \alpha(s) + \frac{1}{\kappa^2(s)}N(s) + \lambda B(s)$$

The equation with λ parameter denotes a line which pass through the point $C(s) = \alpha(s) + \frac{1}{\kappa^2(s)}N(s)$ and is parallel to the B. □

Definition 3.5. *The line $\mathbf{a}(s) = \alpha(s) + \frac{1}{\kappa^2}N(s) + \lambda B(s)$ is the geometric locus of the centers of the spheres which have sufficiently close common three points with the curve $\alpha \subset E^3$ at the point $m \in \alpha$ is called curvature the axis at the point $m \in \alpha$ of curve $\alpha \subset E^3$. The point*

$$C(s_0) = \alpha(s_0) + \frac{1}{\kappa^2(s_0)}N(s_0)$$

on curvature the axis is called curvature the center at the point $m = \alpha(s_0)$ of curve $\alpha \subset E^3$.

Theorem 3.6. *Let curve $\alpha \subset E^3$ be given by (I, α) coordinate neighborhood. If*

$$\mathbf{a}(s) = \alpha(s) + m_2(s)N(s) + m_3(s)B(s)$$

is the center of the osculating sphere at the point $\alpha(s) \in \alpha$, then

$$m_2(s) = \kappa^{-2}(s), \ m_3(s) = \frac{(\kappa^{-2})'}{2\tau} \ \text{or} \ m_3(s) = \frac{m_2'}{2\tau}.$$

Proof. The proof of the theorem is similar to the proof of Theorem3.3. The osculating sphere with the curve α have sufficiently close common four points. Therefore, since $f''(s) = 0$ in (6) thus $f'''(s) = 0$. Then we have

$$\langle -T, N\rangle - \kappa^2 \langle \mathbf{a} - \alpha(s), T\rangle + \frac{\kappa'}{\kappa} \langle \mathbf{a} - \alpha(s), N\rangle + \tau \langle \mathbf{a} - \alpha(s), B\rangle = 0$$

Considering Eqns.(1), (8) and (9) in the last equality, we obtain

$$\langle \mathbf{a} - \alpha(s), B\rangle = -\frac{\kappa'}{\tau\kappa}.$$

or

$$m_3(s) = -\frac{\kappa'}{\tau\kappa^3} \text{ or } m_3 = \frac{(\kappa^{-2})'}{2\tau}$$

Using Eq.(4) in the last equation yields

$$m_3 = \frac{m_2'}{2\tau}.$$

□

Corollary 3.7. *Let curve α in E^3 is given by (I, α) neighbouring coordinate. If r is the radius of the osculating sphere at $\alpha(s) \in \alpha$, then*

$$r = \sqrt{[m_2^2(s) + m_3^2(s)]\,\kappa^2} = \sqrt{\frac{1}{\kappa^2} + \left[\frac{(\kappa^{-2})'}{2\tau}\right]^2}.$$

Proof. If the center of the osculating sphere at $\alpha(s)$ is $\mathbf{a}$, then by Theorem 3.3,

$$\mathbf{a} = \alpha(s) + m_2(s)N(s) + m_3(s)B(s).$$

Thus we have

$$r = \|\mathbf{a} - \alpha(s)\| = \sqrt{\kappa^2(s)m_2^2(s) + \kappa^2(s)m_3^2(s)} = \sqrt{\frac{1}{\kappa^2} + \left[\frac{(\kappa^{-2})'}{2\tau}\right]^2}.$$

□

Theorem 3.8. *Let S_0^2 be a sphere centered at 0 and also $\alpha \subset S_0^2$ be a spherical curve. In this case, since (I, α) is a neighbouring coordinate for α and $s \in I$ is arclength parameter, then*

$$-m_1(s) = \langle \alpha(s), T\rangle \;, -m_2(s) = \frac{\langle \alpha(s), N\rangle}{\kappa^2}, -m_3(s) = \frac{\langle \alpha(s), B\rangle}{\kappa^2}.$$

Proof. Since $\alpha(s) \in S_0^2$ for all $s \in I$, and r is a radius, then we have

$$\overrightarrow{O} = \alpha(s) + m_1T + m_2N + m_3B$$

and

$$\langle \alpha(s), \alpha(s)\rangle = r^2.$$

Thus, taking consecutive derivatives from the above equations with respect to s we get

$$\begin{aligned} \langle \alpha(s), T\rangle &= 0, \\ \langle \alpha(s), N\rangle &= -1, \end{aligned}$$

and

$$\langle \alpha(s), B\rangle = \frac{\kappa'}{\tau\kappa}$$

or

$$\frac{\kappa'}{\tau\kappa^3} = \frac{\langle \alpha(s), B\rangle}{\kappa^2}.$$

Thus since $\frac{\kappa'}{\tau\kappa^3} = -m_3$, we can write the last equality as

$$-m_3 = \frac{\langle \alpha(s), B\rangle}{\kappa^2}.$$

The following theorem characterize the relationship between the radii and the centers of the osculating spheres. □

Theorem 3.9. *Let $S_0^2 \subset E^3$ be a sphere centered at 0. If α is a curve on S_0^2, then the osculating sphere of the curve α at every point is S_0^2.*

Proof. Let the curve α with (I, α) neighbouring coordinate such that $s \in I$ is arclenght parameter. By Theorem 3.6,

$$\mathbf{a}(s) = \alpha(s) + m_2(s)N(s) + m_3(s)B(s).$$

By Theorem 3.8, this expression can be written as

$$\mathbf{a}(s) = \alpha(s) - \frac{\langle \alpha(s), N(s)\rangle}{\kappa^2}N(s) - \frac{\langle \alpha(s), B(s)\rangle}{\kappa^2}B(s).$$

Since $\langle \alpha(s), T(s)\rangle = 0$, we have

$$\alpha(s) = \frac{\langle \alpha(s), N(s)\rangle}{\kappa^2}N(s) + \frac{\langle \alpha(s), B(s)\rangle}{\kappa^2}B(s).$$

Thus we get

$$\mathbf{a} = \alpha(s) - \alpha(s) \Rightarrow \mathbf{a} = 0.$$

On the other hand, we can write

$$d(\alpha(s), O) = r.$$

This completes the proof of the theorem. □

Theorem 3.10. *Let the curve $\alpha \in E^3$ be given with neighbouring coordinate (I, α). The radius of the osculating sphere at the point $\alpha(s)$ for all $s \in I$ such that $m_3(s) \neq 0, \tau \neq 0$ is constant if and only if the centers of the osculating sphere are the same.*

Proof. $\Rightarrow$: By Corollary 3.4, we can write as follows

$$\kappa^2 \left(m_2^2(s) + m_3^2(s) \right) = r^2(s).$$

Since r =constant, from the derivative of this equation with respect to s, we have

$$\kappa m_2 m_2' + \kappa m_3 m_3' + \kappa' m_2^2 + \kappa' m_3^2 = 0$$

or

$$m_3' + \frac{\kappa'}{\kappa} m_3 = -\frac{\kappa'}{\kappa} \frac{m_2}{m_3} m_2 - \frac{m_2}{m_3} m_2'.$$

Inserting values $m_2 = \dfrac{1}{\kappa^2}$, $m_2' = \frac{-2\kappa'}{\kappa\tau}$and $m_3 = -\frac{\kappa'}{\tau\kappa^3}$ in right side of the last equality, we obtain

$$m_3' + \frac{\kappa'}{\kappa} m_3 = -\frac{1}{\kappa^2}\ \tau.$$

Finally, since $m_2 = \frac{1}{\kappa^2}$, we get

$$m_3' + \frac{\kappa'}{\kappa} m_3 + \tau m_2 = 0. \tag{13}$$

On the other hand for base $\{T, N, B\}$ we have

$$\mathbf{a}(s) = \alpha(s) + m_1 T + m_2(s) N(s) + m_3(s) B(s).$$

From derivative with respect to s of the last equality, we get

$$\begin{aligned} \mathbf{a}'(s) &= \left(1 + m_1' - m_2 \kappa^2\right) T + \left(m_1 + m_2 \frac{\kappa'}{\kappa} + m_2' - \tau m_3 \right) N \\ &+ \left(\frac{\kappa'}{\kappa} m_3 + m_3' + \tau m_2 \right) B. \end{aligned} \tag{14}$$

Since $1 + m_1' - m_2 \kappa^2$ and $m_1 + m_2 \frac{\kappa'}{\kappa} + m_2' - \tau m_3$ for values $m_1 = 0$ and $m_2 = \dfrac{1}{\kappa^2}$ are zero, we can write $\mathbf{a}'(s)$ as follows

$$\mathbf{a}'(s) = \left(\frac{\kappa'}{\kappa} m_3 + m_3' + \tau m_2 \right) B. \tag{15}$$

So by (13) we find

$$\mathbf{a}'(s) = 0.$$

Thus we have $\mathbf{a}(s)$ =constant for all $s \in I$

$\Leftarrow$:Conversely, let $a(s)$ be constant for all $s \in I$. Considering the equation

$$\langle a(s) - \alpha(s), a(s) - \alpha(s) \rangle = r^2(s),$$

taking derivative of this equation with respect to s, and if necessary calculations are made, we find

$$r(s) r'(s) = 0.$$

Here, either $r(s) = 0$ or $r'(s) = 0$. If $r(s) = 0$, then by Corollary 3.7, we have

$$\kappa^2 \left[m_2^2(s) + m_3^2(s)\right] = 0, \quad \kappa \neq 0$$

or

$$m_2^2(s) = -m_3^2(s) = 0.$$

But this contradicts the theorem. Therefore $r'(s) = 0$. Thus $r(s)$ is constant for all $s \in I$. □

Theorem 3.11. *Let α be a curve in E^3 with (I, α) neighbouring coordinate and $m_3(s) \neq 0, \tau \neq 0$ for all $s \in I$. Then, α is a spherical curve if and only if the centers of the osculating spheres at the point $\alpha(s)$ for all $s \in I$ are located at the same point.*

Proof. Let α be a curve on S_b^2 which have the radius r and centered at any point b. By Theorem 3.8, the proof is clear. Conversely, let the centers of the osculating curve be the point b in $\alpha(s) \in \alpha$ for all $s \in I$. Then by Theorem 3.10 all osculating spheres have the same radius r. Therefore

$$d(\alpha(s), b) = r$$

for all $s \in I$. This completes the proof of the theorem. □

Theorem 3.12. *Let α be curve in E^3 be given with (I, α) neighbouring coordinate. If $m_3(s) \neq 0, \tau \neq 0$ such that s is a arclenght parameter, then α is a spherical curve if and only if*

$$\left[\left(\frac{1}{\kappa^2}\right)' \frac{1}{2\tau}\right]' + \frac{\tau}{\kappa^2} - \frac{\kappa'}{\tau\kappa^3} = 0.$$

Proof. Let α be a spherical curve. By Theorem 3.11, for all $s \in I$, the center $\mathbf{a}(s)$ of the osculating spheres are constant. Additionally, the Eq.(13) yields

$$m_3' + \frac{\kappa'}{\kappa} m_3 + \tau m_2 = 0$$

or

$$\left[\left(\frac{1}{\kappa^2}\right)' \frac{1}{2\tau}\right]' - \frac{\kappa'}{\tau\kappa^3} + \frac{\tau}{\kappa^2} = 0$$

Conversely, let $m_3' + \frac{\kappa'}{\kappa} m_3 + \frac{\tau}{\kappa^2} = 0$. By Theorem 3.9 and $\mathbf{a}'(s) = 0$. Therefore $\mathbf{a}(s)$=constant. Thus, by Theorem 3.11, the curve α is a spherical curve.. □

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