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Design of Lag/Lead Controller for Fractional Order Systems Containing Time-Delay and Uncertainty

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Abstract – This paper presents a methodology to design a phase-lead and phase-lag controllers for a fractional order system with time-delay and parameter uncertainty. The method that was used in the study is a classic design method used by D. P. Atherton. The method has been shown to be successful by adding time-delay and parameter uncertainty to this classic design method. The controllers are obtained by the implementation of individual design steps for the phase-lead and phase-lag controller. The unit-step responses and Bode diagrams of the systems with controllers are plotted. Considering the results obtained, it is observed that the method gave successful results for a fractional order plant with time-delay and parameter uncertainty.

Keywords *- Controller design, fractional control systems, phase lead and lag controller*

1. Introduction

Controller design has an important place in the field of control engineering. The type of controller to be selected is important for achieving the desired design criteria. In practice, simple and low-degree controller structures are preferred most of the time. Among these controller structures, proportional-gain (*P*), proportional-integral (*PI*), proportional-derivative (*PD*), proportional-integral-derivative (*PID*), phase-lead and phase-lag controllers are preferred the most [1,2].

The fractional order calculations has gained great importance in control engineering with the increase in use of fractional order mathematics recently. In the literature, there are numerous scientific studies conducted on the controller design for the fractional order control systems [3-8]. The fractional order systems and fractional controller structures are

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started to be used in dynamic systems since the midst of the 20th century [9]. One of the areas of use is to perform a controller design for a fractional order system that contains time delay and parameter uncertainty. In this study, phase-lead and phase-lag controllers are used. Phase-lead controller leads to a lower overshoot in the time response of the system, but it shortens parameters such as rise time and settling time. It increases the gain crossover-frequency and bandwidth of the phase-lead controller system. This ensures acceleration of the system response. The most important feature of the phase-lag controllers is their ability of achieving decent phase-lag at high frequencies. This controller structure decreases percent overshoot through the reduced bandwidth, but extends the rise and settling times and this causes system to respond slowly [1-2,10-11].

Figure 1 shows the block diagram of the simplest control system with feedback.

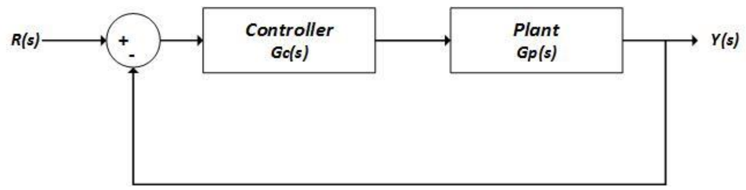

Figure 1. Simple feedback control system

In Figure 1, *Gp(s)* is the transfer function of the system to be controlled, *Gc(s)* is the transfer function of the controller, *R(s)* is the system input, and *Y(s)* is the output of the system. When the transfer functions in Figure 1 are of fractional order, then the control system is called as fractional order control system.

The controller structure used in the study is as in Equation 1.

$$G_c(s) = \frac{1+sT}{1+s\alpha T} \tag{1}$$

The α and T parameters to be determined in Equation 1 are obtained by following the design steps given in Section 3, which are different for each controller.

The plant used in the study for the application is given by Equation 2.

$$G_p(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}} e^{-Ls} \tag{2}$$

In Equation 2, a_i $(\mathrm{i}=0,..,\mathrm{n})$ and b_i $(\mathrm{i}=0,..,\mathrm{m})$ are the constant terms, α_k $(\mathrm{k}=0,..,\mathrm{n})$ and β_k $(\mathrm{k}=0,..,\mathrm{m})$ indicate the fractional orders. Here, if $a_i \in \left[\underline{a_i}, \overline{a_i}\right]$ and $b_j \in \left[\underline{b_j}, \overline{b_j}\right]$, $i = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, m$ then there is parameter uncertainty [11].

For the plant given in Equation 2, Bode diagrams are created. α and T parameters to be determined are obtained by applying the design steps to the transfer function that has the smallest phase margin in these Bode diagrams. Thus, the controller design is completed in this way. The Bode diagrams and unit-step responses of the outputs of both the uncontrolled system and system with controllers are created for each controller. Thus, the success of the method used to design controller has been seen from the results obtained.

This paper consists of five sections. The first section is the introduction, and the importance of controllers as well as the structures of phase-lead and phase-lag controllers are discussed in this section briefly. In the second section, fractional order systems with time-delay parameter uncertainty are addressed. In the third section, phase-lead and phase-lag controllers design methods are described. In the fourth section, the method presented in the third section is implemented. The results are presented in the fifth section.

2. Fractional Order Systems Containing Time-Delay and Uncertainty

Fractional order calculations is a field of mathematics, and is related to non-integer derivatives and integrals [12]. The fractional order math has a history of three hundred years. The fractional order mathematics has been discussed in the correspondence between Leibniz and L'Hospital in 1695. Later, the first studies were conducted by Liouville, Riemann and Holmgren. Its applications in the field of control engineering has a history of fifty years [13,14]. The position control was realized first time in 1958 by Tustin [9]. In 1961 and 1963, Manabe [15, 16] has applied the fractional order integrator in the control systems. At first, there are limited number of applications because of the computational difficulties. In parallel with developments in computer technology, studies in recent years has increased rapidly. In the last two decades, the fractional order calculations have been rediscovered by scientists and engineers, and applied gradually in many areas [12].

Time-delay and uncertainty is encountered in real systems frequently. Therefore, the modeling of systems with parameter uncertainty and time-delay is of importance [17]. Therefore, control of systems with time delay and parameter uncertainty is also important. In addition, fractional order control system enables to get more realistic results. Expressing real systems with fractional order models instead of integer order model is known to give more realistic results [10].

There are two important definitions that are used for fractional order derivatives and fractional order integrals. These definitions are in the form of Grünwald-Letnikov and Riemann-Liouville. The Fractional Order Control System (FOCS) can be expressed as in Equation 3 [18].

$$a_n \mathrm{D}^{\alpha_n} \mathrm{y(t)} + a_{n-1} \mathrm{D}^{\alpha_{n-1}} \mathrm{y(t)} + ... + a_0 \mathrm{D}^{\alpha_0} \mathrm{y(t)} = b_m \mathrm{D}^{\beta_m} \mathrm{r(t)} + b_{m-1} \mathrm{D}^{\beta_{m-1}} \mathrm{r(t)} + ... + b_0 \mathrm{D}^{\beta_0} \mathrm{r(t)} \tag{3}$$

Laplace transformations of the Riemann-Liouville's fractional order integral and derivative are given in Equation 4 and Equation 5 [18]. In Equation 3, $\mathrm{r(t)}$ is the input signal, $\mathrm{y(t)}$ is the output signal, $D^{\gamma} \equiv {}_0D_t^{\gamma}$ is the fractional order derivative, a_k $(\mathrm{k}=0,..,\mathrm{n})$ and b_k $(\mathrm{k}=0,..,\mathrm{m})$ are the constants, α_k $(\mathrm{k}=0,..,\mathrm{n})$ and β_k $(\mathrm{k}=0,..,\mathrm{m})$ are the fractional orders.

$$L\left\{ {}_0D_t^{-\alpha} f(\mathrm{t});\mathrm{s} \right\} = s^{-\alpha} F(\mathrm{s}) \tag{4}$$

$$L\left\{ {}_0D_t^{\alpha} f(\mathrm{t});\mathrm{s} \right\} = s^{\alpha} F(\mathrm{s}) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{\alpha-k-1} f(\mathrm{t}) \right]_{t=0} \tag{5}$$

Equation 3 and Equation 4 are used in order to obtain the Laplace transformation of Equation 3. Thus, the Laplace transformation of Equation 3 can be found as in Equation 6 [18,19].

$$G(s)=\frac{b_m s^{\beta_m}+b_{m-1}s^{\beta_{m-1}}+\ldots+b_1 s^{\beta_1}+b_0 s^{\beta_0}}{a_n s^{\alpha_n}+a_{n-1}s^{\alpha_{n-1}}+\ldots+a_1 s^{\alpha_1}+a_0 s^{\alpha_0}} \tag{6}$$

The transfer function to be used in the application is obtained by adding the exponential function, which implies a time delay, to Equation 6. In Equation 6, a_i $(\mathrm{i}=0,..,\mathrm{n})$ and b_i $(\mathrm{i}=0,..,\mathrm{m})$ indicate the constant terms, α_k $(\mathrm{k}=0,..,\mathrm{n})$ and β_k $(\mathrm{k}=0,..,\mathrm{m})$ show the fractional orders. This equation has parameter uncertainty if $a_i \in \left[\underline{a_i}, \overline{a_i}\right]$ and $b_j \in \left[\underline{b_j}, \overline{b_j}\right]$ $i=0,1,2,\ldots,n$, $j=0,1,2,\ldots,m$ [11].

3. Phase-Lead and Phase-Lag Controller Design

The phase-lead and phase-lag controllers are often preferred in the design of control systems. It is mainly preferred for its small number of parameters that need to be set. They give successful results with the calculation and adjustment of two or three parameters. In addition, the simplicity of their control structure, and wide range of application areas make them preferable. Although the phase-lead and phase-lag controllers are similar to each other in terms of general structure, they are different in terms of placement of the zeros and poles [10].

Phase-lead controller adds a zero and a pole to the forward-path transfer function of the system. As a result of this, the response time of the system showed lower percent overshoot. The time parameters such as rise time and settling time become shorter. In addition, gain-phase margin of the closed-loop system increases, and the relative stability of the system improves. It increases the gain crossover-frequency and bandwidth of the phase-lead controller system. This ensures acceleration of the system response [1,10].

The most important feature of the phase-lag controllers is their ability of achieving decent phase-lag at high frequencies. In these controller structures, the bandwidth of the system is reduced by shifting gain crossover-frequency to lower frequencies. The reduced bandwidth decreases percent overshoot, but extends the rise and settling times. And, this causes system to respond slowly [1,10].

In this section, a methodology for phase-lead and phase-lag controller design is utilized. The method used is a classic design method developed by D. P. Atherton [20]. In this paper, the method is applied to a fractional order system with time delay and parameter uncertainty.

3.1. Phase-Lead Controller Design

The phase-lead controller is designed by following the design steps below.

1. The phase margin (ϕ) of the system to be controlled is found using the Bode diagram by selecting the transfer function with the minimum phase margin.
2. A small amount of the safety angle (ε) is chosen. The safety angle is preferred between 5º and 18º approximately.
3. The phase margin of the system is obtained by Equation 7. Here, the φ angle is taken as the phase margin of the desired system.

$$\phi_m = \varphi - \phi + \varepsilon \tag{7}$$

4. The α parameter is calculated by using Equation 8.

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \tag{8}$$

5. α parameter's corresponding gain margin is obtained by using the Equation 9.

$$G_m = -20\log_{10}\sqrt{\alpha} \tag{9}$$

6. Using the Bode diagram, the frequency value (ω_m) that corresponds to the gain margin is read.
7. In the last step, the Equation 10 is used for calculating the parameter T.

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \tag{10}$$

The α and T parameters are determined by following the design steps above. The phase-lead controller is obtained by putting the α and T parameters in the Equation 11. If the time response parameters desired for the closed-loop control system were not found, then the steps are repeated with a different ε value [20-22].

$$C(s) = \frac{1+sT}{1+s\alpha T} \tag{11}$$

3.2. Phase-Lag Controller Design

The design steps for the phase-lag controller design is as follows.

1. The phase margin of the given transfer function G(s) is calculated by the Equation 12 using the transfer function with the minimum phase margin. In this equation, φ indicates the phase margin of the system, and δ indicates approximately 4º-5º of safety angle. The parameter δ is expressed as in Equation 13.

$$argG(j\omega) = -(180^\circ - \varphi) + \delta \tag{12}$$

$$\delta = \tan^{-1}10 - \tan^{-1}10\alpha \tag{13}$$

2. Using the Bode diagram of G(s), the frequency value that corresponds to gain margin, calculated in the 1st step, is read.
3. Using the Bode diagram, the frequency value that corresponds to this gain margin is found.
4. The phase margin and gain margin values found in the second and third steps are put in the Equation 14 to calculate the α and T values.

$$T = \frac{10}{\omega} \tag{14}$$

$$Gain(dB) = |G(j\omega)| = 20\log_{10}\alpha \tag{15}$$

5. This completes the phase-lag controller design. If it fails to give the desired output values, then the steps are repeated with a different δ value [20-22].

4. Implementation of the Method

Example 1. We assume a fractional order system with time-delay and parameter uncertainty as given below. The aim is to perform phase-lead and phase-lag controller design for fractional order system with time-delay and parameter uncertainty, which are frequently encountered in real systems.

$$G_p(s) = \frac{2}{a_1 s^{2.2} + a_0 s^{1.2}} e^{-0.4s} \tag{16}$$

Here, $a_1 \in [0.8, 1.2]$ and $a_0 \in [1.8, 2.2]$. There are 2 uncertain parameters in this transfer function. Here, 3 values were taken as the lower limit, upper limit, and middle value for each coefficient having parameter uncertainty. Therefore, for the $G_p(s)$ transfer function, there are $3^2 = 9$ vertex polynomials. The Bode diagrams obtained for different vertex polynomials are shown in Figure 2.

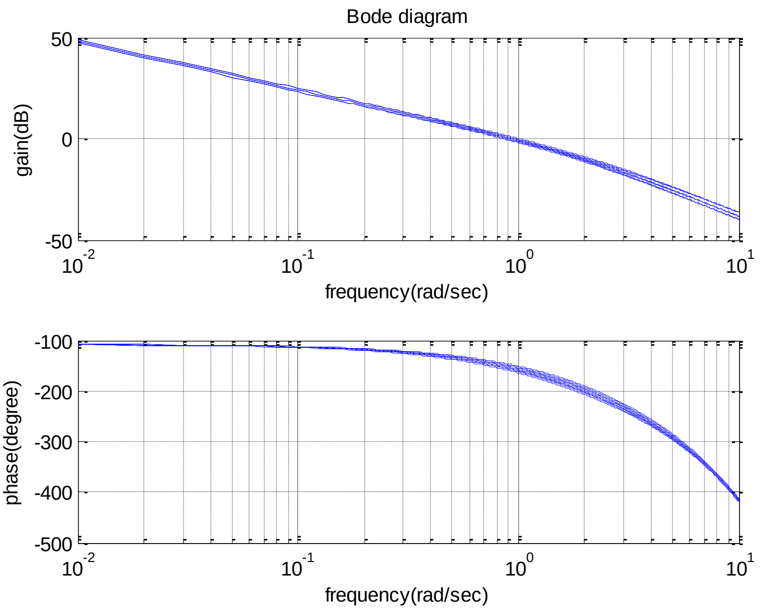

Figure 2. Bode diagrams for fractional order transfer function given in Equation 16

For this application, the transfer function with the least phase margin was chosen. The selected transfer function is given in Equation 17. The α and T parameters were calculated to be 0.23 and 1.38, respectively, by applying the design steps given in the third section to the transfer function selected. For the application, the selected phase margin (ϕ) was 40°, and the safety angle was 17.4°. The phase-lead controller was designed as in Equation 18.

$$G_{p1}(s) = \frac{2}{1.2s^{2.2} + 1.8s^{1.2}} e^{-0.4s} \tag{17}$$

$$G_{lead1}(s) = \frac{1 + 1.38s}{1 + 0.32s} \tag{18}$$

The unit-step response in Figure 3 is obtained by applying the phase-lead controller obtained to Equation 17. Figure 3 shows that percent overshoot value of the controller system decreases. In addition, it is remarkable that settling time and rise time parameters of the controller system has shortened.

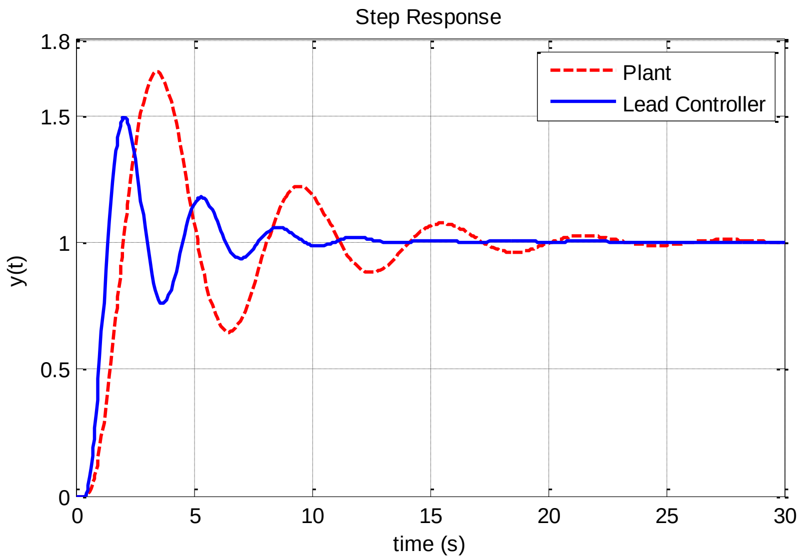

Figure 3. Step responses for Plant and Lead Controller applied system

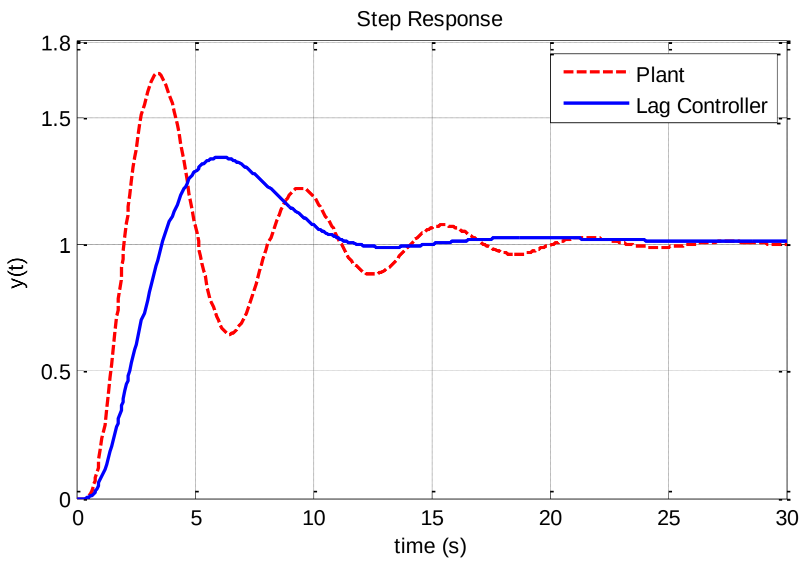

Figure 4. Step responses for Plant and Lag Controller applied system

The α and T parameters were calculated as 2.63 and 21.3, respectively, by following the design steps for the phase-lag controller design. Here, the transfer function in Equation 17 is used. For the phase-lag controller design, the phase margin (φ) of the system was chosen as 40°, and the safety angle denoted by δ parameter was chosen as 4°. The phase-lag controller is obtained as in Equation 19.

$$G_{lag1}(s) = \frac{1+21.3s}{1+56.02s} \tag{19}$$

The unit-step response in Figure 4 is obtained by applying the phase-lag controller to the system. Figure 4 shows that percent overshoot value of the controller system is about 25%. In addition, it is seen that the time parameters of the phase-lag controller system becomes longer.

Figure 5 shows the Bode diagrams of the open-loop transfer functions of the systems with and without a phase-lead controller. The system in the Equation 17 and the controller in the Equation 18 was used in obtaining Figure 5. As shown in the graph, the gain crossover-frequency of the system was shifted from 0.94 rad/sec to 1.5 rad/sec. Similarly, the phase crossover-frequency was increased from 1.33 rad/sec to 2.26 rad/sec. Although the gain is the same, there is approximately 13° increase in phase.

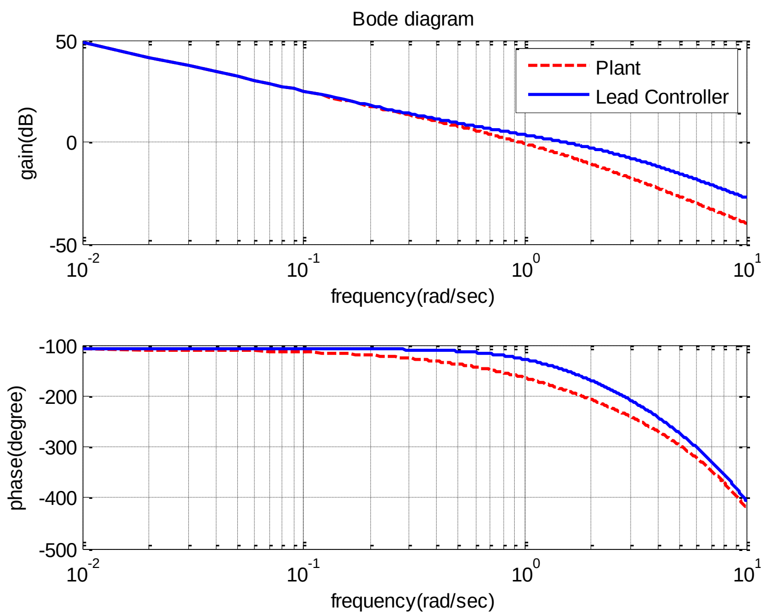

Figure 5. Bode diagrams for Plant and Lead Controller applied system

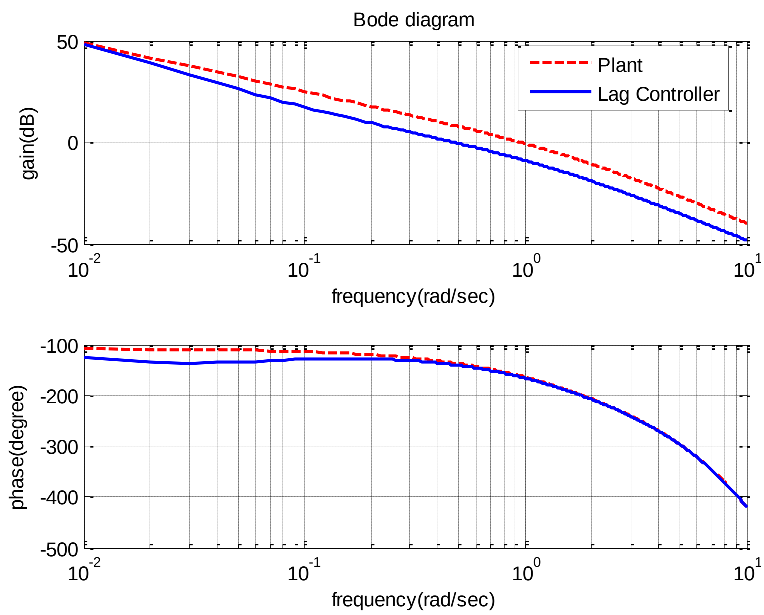

Figure 6. Bode diagrams for Plant and Lag Controller applied system

Figure 6 shows the Bode diagrams of the systems with and without a phase-lag controller. The system in the Equation 17 and the controller in the Equation 19 were used in obtaining Figure 6. As shown in Figure 6, the gain crossover-frequency of the system without controller is 0.94 rad/sec, whereas the gain crossover-frequency of the system with phase-lag controller is found to be shifted to 0.47 rad/sec. It is observed that the phase crossover-frequency remains the same. It is observed that there is an 8.5 dB of increase in gain. Approximately 4º change is observed in phase.

Figure 7 and 8 show the unit-step responses of the systems controlled by phase-lead and phase-lag controllers respectively. Figure 7 and 8 also show that the controllers designed for a transfer function having minimum phase margin control other transfer functions as well.

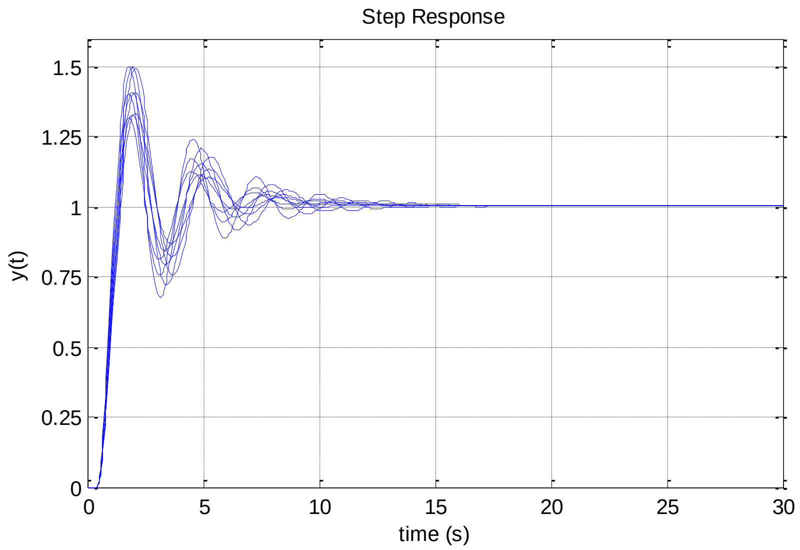

Figure 7. Step responses for Lead Controller applied system

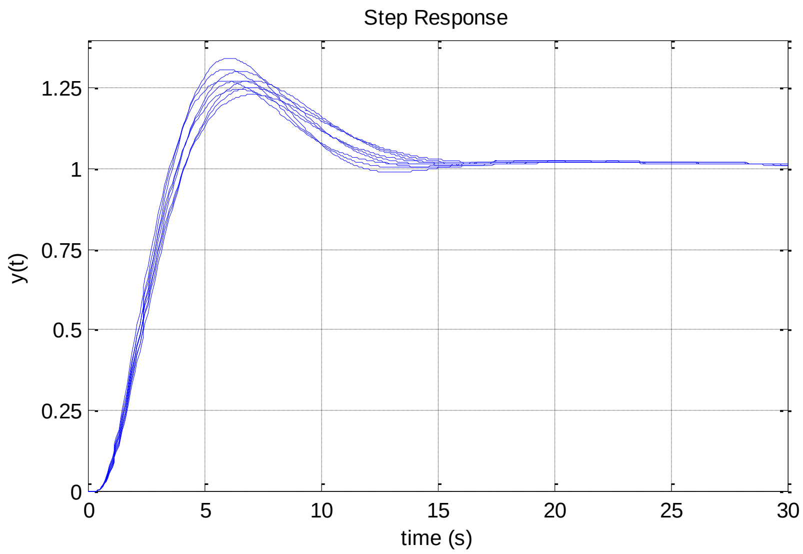

Figure 8. Step responses for Lag Controller applied system

Example 2. For this example, a transfer function as in Equation 20 was chosen. Although the number of uncertain parameters was 2 in the transfer function given in Equation 16, there are 3 coefficients with parameter uncertainty in the transfer function in this example.

$$G_p(s)=\frac{b_0}{a_1 s^{2.1}+a_0 s^{1.1}}e^{-0.5s} \tag{20}$$

These coefficients are given as $b_0 \in [1.3, 1.7]$, $a_1 \in [1, 1.4]$ and $a_0 \in [1.6, 2]$. Minimum, maximum and middle-point values are taken for each coefficient to obtain 27 transfer functions in total. The Bode diagrams obtained for different transfer functions are shown in Figure 9.

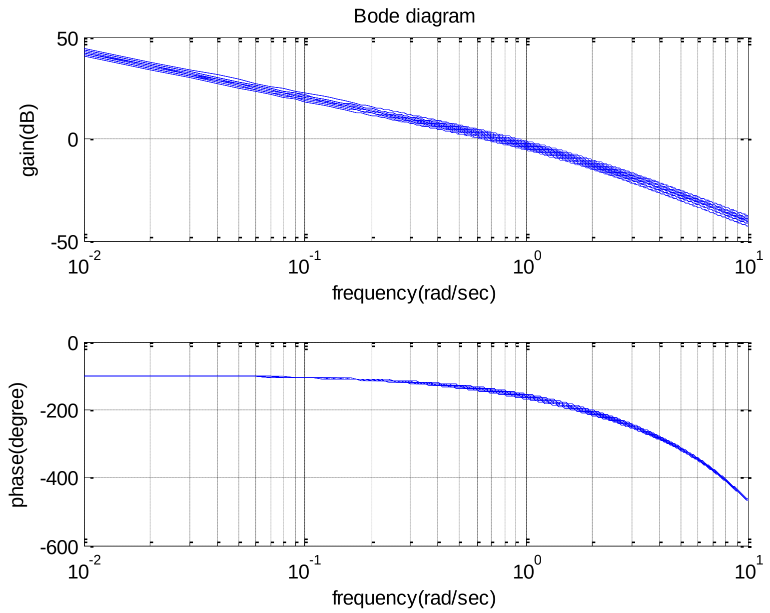

Figure 9. Bode diagrams for fractional order transfer function given in Equation 20

For the phase-lead controller design, the transfer function with the least phase margin was chosen. The selected transfer function is given in Equation 21. The α and T parameters were calculated as 0.244 and 1.5, respectively, by applying the design steps to the transfer function in Equation 21. The phase-lead controller was designed as in Equation 22.

$$G_{p2}(s)=\frac{1.7}{1.4s^{2.1}+1.6s^{1.1}}e^{-0.5s} \tag{21}$$

$$G_{lead\,2}(s)=\frac{1+1.5s}{1+0.366s} \tag{22}$$

The unit-step response in Figure 10 is obtained by applying designed controller to Equation 21. Figure 10 shows that a graph very close to the unit-step response in the Example 1 was obtained.

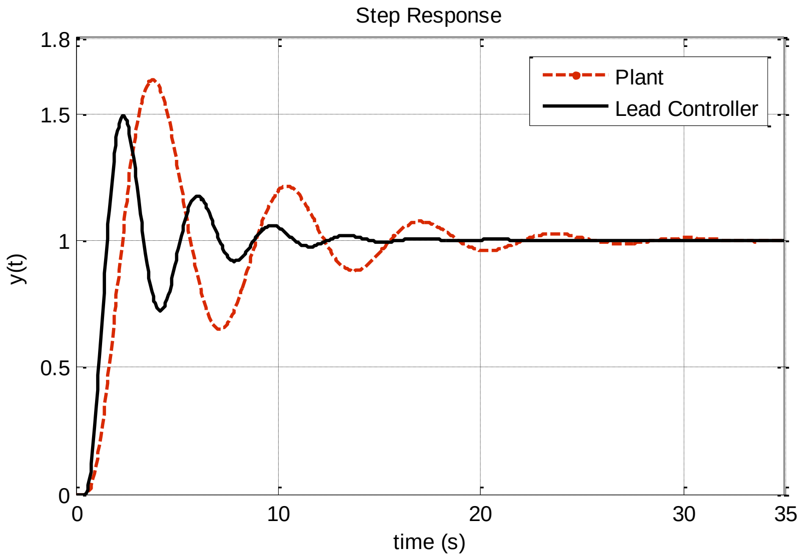

Figure 10. Step responses for Plant and Lead Controller applied system

The α and T parameters were calculated as 2.16 and 20.55, respectively, by following the design steps for the phase-lag controller design. Here, the design was made by using the transfer function in Equation 21. The phase-lag controller is obtained as in Equation 23.

$$G_{lag2}(s) = \frac{1+20.55s}{1+44.388s} \tag{23}$$

The unit-step response in Figure 11 is obtained by applying the phase-lag controller to the system. Figure 11 shows that percent overshoot value of the controller system is about 30%.

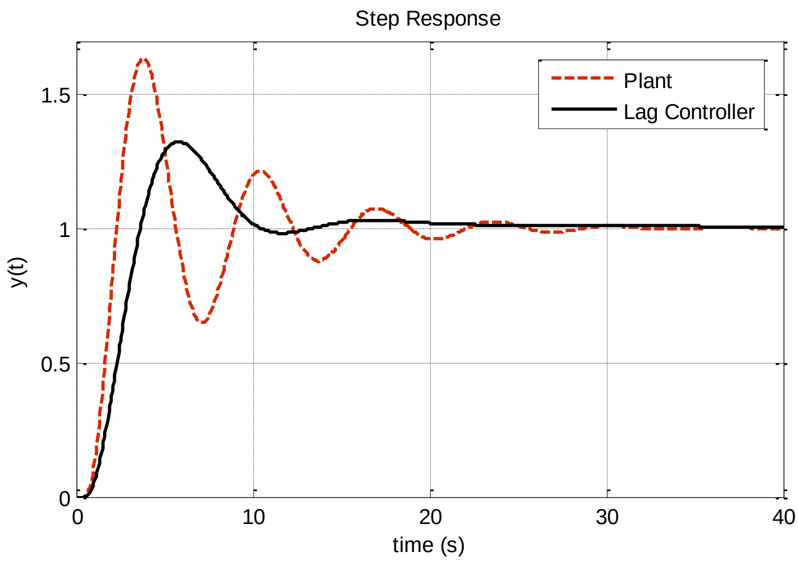

Figure 11. Step responses for Plant and Lag Controller applied system

Figure 12 and 13 show the unit-step responses of the systems controlled by phase-lead and phase-lag controllers respectively.

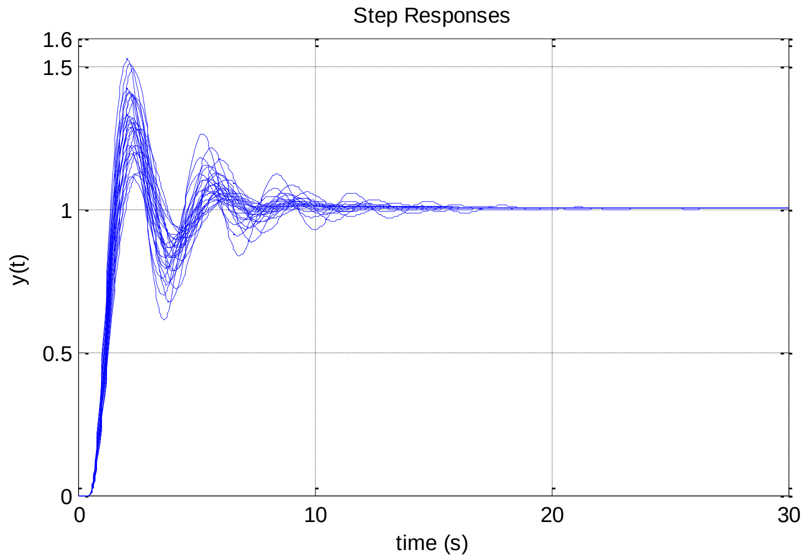

Figure 12. Step responses for Lead Controller applied system

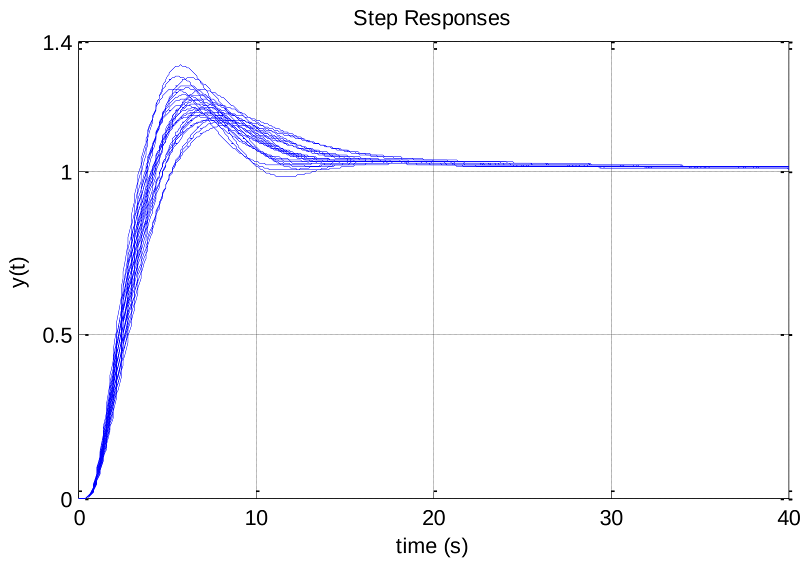

Figure 13. Step responses for Lag Controller applied system

The Bode diagrams used for the application of the method and the figures showing the unit-step responses were obtained using FOTF Matlab Toolbox [22,23].

The first version of this paper was presented in the symposium EEB2016 [24].

5. Conclusions

In this study, phase-lead and phase-lag controller design was carried out for the fractional order systems that have time delay and parameter uncertainty. The method that was used in the study is a design method used by D. P. Atherton [20]. This methodology has been applied to fractional order systems with time delay and parameter uncertainty, and successful results have been obtained.

The two parameters of the phase-lead and phase-lag controller, α and T were determined by applying the design steps for the fractional order controller systems that have time delay and parameter uncertainty. In the unit-step response of a system controlled with a phase-lead controller, the percent overshoot value was found to be decreased as well as shortened time parameters and improved response time of the system. In addition, gain crossover-frequency and bandwidth were observed to be increased as seen in the Bode diagrams of the system controlled by a phase-lead controller. This ensures acceleration of the system response. It increased the gain crossover-frequency and bandwidth of the phase-lag controller system. The reduced bandwidth has extended the rise and settling times. It was observed that this causes slower system response.

As one of the most fundamental issues of control systems, this controller design study was successfully applied in a fractional order controller system with time-delay and parameter uncertainty.

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