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The Source of Primeness of Rings

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Abstract — In this study, we define a new concept, i.e., source of primeness of a ring R, as $P_R := \bigcap_{a \in R} S_R^a$ such that $S_R^a := \{b \in R \mid aRb = (0)\}$. We then examine some basic properties of P_R related to the ring's idempotent elements, nilpotent elements, zero divisor elements, and identity elements. Finally, we discuss the need for further research.

Keywords — *Prime ring, semiprime ring, source of primeness, source of semiprimeness*

Mathematics Subject Classification (2020) — 16N60, 13A15

1. Introduction

Our primary aim in this study is to describe the ring types that are not included in the literature. These definitions were derived in light of the original existing definitions in ring theory and can be seen as generalizations of division rings, reduced rings and domain, respectively. To define these new concepts in rings, we will first examine the structure of the subset, which we call the source of primality in rings, in consideration of articles [1,2]. Before we get to the point, let us summarize the terminology we will use throughout the study.

We will now give the basic definitions in [3–5]. An element with a right (left) multiplicative inverse in a unit ring is called a right (left) unit, and accordingly, by the unit is meant a two-sided unit. An element a of a ring R is called a right (left) zero divisor if there is a nonzero element $b \in R$ such that $ba = 0$ ($ab = 0$, respectively). A nonzero-divisor element is neither a left nor a right zero-divisor. A domain is a ring with no nonzero right or left zero-divisors. A ring with nonzero elements, which are all units, is called a division ring. An element a of a ring R is called a nilpotent element of index n if n is the least positive integer such that $a^n = 0$. A reduced ring has no nonzero nilpotent elements. An idempotent element e of R is $e = e^2$. An element of R is called central if it commutes with every element of R.

R is a prime ring if $aRb = (0)$ with $a, b \in R$ requires $a = 0$ or $b = 0$, and R is a semiprime ring if $aRa = (0)$ with $a \in R$ requires $a = 0$. Here, the ideal generated by the zero element is shown by (0). In [1] authors, defined the source of semiprimeness of the subset A in R where A is a nonempty subset of ring R as follows: $\mathcal{S}_R(A) = \{a \in R \mid aAa = (0)\}$ and $\mathcal{S}_R$ is written in place of $\mathcal{S}_R(R)$ for a ring R. As is known, every ring is isomorphic to the subdirect sum of the prime rings. Now let us introduce our primary instrument, which we have focused on throughout the article. We define the subset S_R^a of R as

$$S_R^a = \{b \in R \mid aRb = (0)\}$$

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Moreover, we call the following set as the source of primeness of R

$$P_R = \bigcap_{a \in R} S_R^a$$

This set is always different from empty because it contains $\{0\}$. If P_R consists only of $\{0\}$, then P_R is trivial, and at another end, P_R may be equal R. P_R is trivial if only if R is a prime ring. If we ignore both situations, our general interest will be situations between these two ends. A diligent reader must have intuited that P_R has always been located at the prime radical of R because $P_R \subseteq S_R$.

Thus, by examining the elements of the $R - P_R$ set more closely, we have a chance to examine the P_R subset to obtain the structural properties of R. We choose the term "the source of primeness of R" because every element in $R - P_R$ acts as a nonzero element in any prime ring. For every $b \in R - P_R$, $RRb \neq (0)$ comes from the "primeness" part.

At that, we examined the properties of P_R for the arbitrary ring R . The first important property is that while I is a right ideal, P_R is an ideal. Another important consequence is that the source of primeness of R is preserved under the ring isomorphisms.

2. Results

Definition 2.1. Let R be a ring, $\emptyset \neq A \subseteq R$ and $a \in R$. We define $S_R^a(A)$ as follows:

$$S_R^a(A) = \{b \in R \mid aAb = (0)\}$$

$P_R(A) = \bigcap_{a \in R} S_R^a(A)$ is called the source of primeness of the subset A in R. We write S_R^a instead of $S_R^a(R)$. In particular, we can similarly define the source of primeness of the semigroup R as follows:

$$P_R = \bigcap_{a \in R} S_R^a$$

Some of the basic inferences that will help understand the concept are as follows:

i. Since R be a ring, we obtain $aA0 = (0)$ for all $a \in R$. Hence, $P_R = \bigcap_{a \in R} S_R^a \neq \emptyset$.

ii. $S_R^0(A) = R$

iii. $S_A^a \subseteq S_R^a(A)$. If $b \in S_A^a$, then $b \in A$ such that $aAb = (0)$. Since $A \subseteq R$, we have $b \in R$ and $aAb = (0)$. This means that $b \in S_R^a(A)$.

If $x \in P_R(A)$, then $aAx = (0)$, for all $a \in R$. Hence, $RAx = (0)$. Therefore, $P_R(A) = \{x \in R : RAx = (0)\}$.

Proposition 2.2. Let R be a ring and $\emptyset \neq A, B \subseteq R$. Then, $P_{R\times R}(A \times B) = P_R(A) \times P_R(B)$.

Proof. $P_{R\times R}(A \times B) = \{(x, y) \in R \times R \mid (R \times R)(A \times B)(x, y) = (0, 0)\}$. Assume that $(x, y) \in P_{R\times R}(A \times B)$. Then, $(R \times R)(A \times B)(x, y) = (0, 0)$. Namely, $RAx = (0)$, $RBy = (0)$. Hence, $x \in P_R(A)$, $y \in P_R(B)$. Thus, $(x, y) \in P_R(A) \times P_R(B)$. Similarly, the reverse side is also seen. □

Proposition 2.3. Let R be a ring. If $1_R \in R$, then $P_R \subseteq \{x \in R \mid x^2 = 0\}$.

Proof. Let $K = \{x \in R \mid x^2 = 0\}$. If $x \in P_R$, then $RRx = (0)$. Since $1_R \in R$, then $0 = x1_Rx = x^2$. Hence, $x \in K$ is satisfied. Thus, $P_R \subseteq K$. □

Proposition 2.4. Let A and B be two nonempty subsets of a ring R. Then, the following conditions hold:

i. If $A \subseteq B$, then $P_R(B) \subseteq P_R(A)$. In particular, $P_R \subseteq P_R(A)$.

ii. If A is a subring of R, then $A \cap P_R(A) \subseteq P_A$.

PROOF.

i. Let $x \in P_R(B)$. We have $x \in \bigcap_{a \in R} S_R^a(B)$ and $aBx = (0)$, for all $a \in R$. Since $A \subseteq B$, we get $aAx = (0)$ for all $a \in R$. This means that $x \in S_R^a(A)$ for all $a \in R$. Hence, we get $x \in \bigcap_{a \in R} S_R^a(A)$ and $x \in P_R(A)$. This gives up $P_R(B) \subseteq P_R(A)$. Specially, $P_R \subseteq P_R(A)$ is satisfied for $A \subseteq R$.

ii. Let $x \in A \cap P_R(A)$. Then, $x \in A$ and $x \in P_R(A)$. Hence, we get $x \in A$ and $x \in \bigcap_{a \in R} S_R^a(A)$. Using $x \in A$, $x \in S_A^a$ for all $a \in A$. This expression gives us $x \in \bigcap_{a \in R} S_A^a = P_A$. Thus, $A \cap P_R(A) \subseteq P_A$.

□

It is well known that every prime ring is a semiprime ring. Therefore, we present a relationship between the source of primeness and the source of semiprimeness as follows:

Proposition 2.5. Let R be a ring, $\emptyset \neq A \subseteq R$. Then, $P_R(A) \subseteq S_R(A)$.

PROOF. If $b \in P_R(A)$, then $b \in \bigcap_{a \in R} S_R^a(A)$. In particular, $b \in S_R^b(A)$. Therefore, $bAb = (0)$. Hence, $b \in S_R(A)$. □

Proposition 2.6. Let R be a ring, $a \in R$ and I be a nonempty subset of R. In this case, the following properties are provided.

i. $S_R^a(I)$ is a right ideal of R.

ii. If I is a right ideal of R, then $S_R^a(I)$ is a left ideal of R.

iii. If I is a right ideal of R, then $S_R^a(I)$ is an ideal of R.

PROOF.

i. Let $x, y \in S_R^a(I)$. Then, $aIx = aIy = (0)$ for $a \in R$. From here $aI(x - y) = aIx - aIy = (0)$, we obtain $x - y \in S_R^a(I)$. Besides that, we have $aI(xr) = (aIx)r = (0)$ for any $r \in R$. Thus, we get $xr \in S_R^a(I)$. Hereby, $S_R^a(I)$ is a right ideal of R.

ii. Let I is a right ideal of R and $x \in S_R^a(I)$, $r \in R$. Then, we get $aI(rx) = a(Ir)x \subseteq aIx = (0)$. Hence, we have $rx \in S_R^a(I)$ and $S_R^a(I)$ is a left ideal of R.

iii. We can easily see that if I is a right ideal of R, then $S_R^a(I)$ is an ideal of R from *i* and *ii*.

□

Theorem 2.7. Let R be a ring and I be a nonempty subset of R. If I is a right ideal of R, then $P_R(I)$ is an ideal of R.

PROOF. Let I be a right ideal of R and $x, y \in P_R(I)$. Therefore, $x, y \in \bigcap_{a \in R} S_R^a(I)$ and $x, y \in S_R^a(I)$ for all $a \in R$. Then, $aIx = aIy = (0)$ for all $a \in R$. Since $S_R^a(I)$ is an ideal of R for all $a \in R$ from Proposition 2.6, we write $x - y, xr, rx \in S_R^a(I)$ for all $a, r \in R$. Consequently, we get $x - y, xr, rx \in \bigcap_{a \in R} S_R^a(I) = P_R(I)$. For this reason, $P_R(I)$ is an ideal of R. □

In the following theorem, if the ring R is prime, its relation to the set the source of primeness examined.

Theorem 2.8. Let R be a ring. Thus the followings are provided.

i. If R is a prime ring, then $P_R = \{0\}$.

ii. The source of primeness P_R is contained by every prime ideal of the R.

Proof.

i. Let R be a prime ring and $x \in P_R$. From definition of the set P_R, we have $RRx = (0)$. Since R is prime ring, we obtained $x = 0$. Namely, $P_R = \{0\}$.

ii. Let P be a prime ideal in R. If $x \in P_R$, then $RRx = (0) \subseteq P$. Since P is prime, we get $x \in P$. Hence, we get $P_R \subseteq P$. This gives us that every prime ideal of R includes P_R.

□

We know that the properties of idempotent, nilpotent, and zero-divisor elements in a ring are also related to the primality of that ring. Some of the relationships between the source of primeness and these special elements are as follows:

Proposition 2.9. Let R be a ring. Then, the following holds.

i. If R is a Boolean ring, then $P_R = \{0\}$.

ii. If $0 \neq a \in P_R$, then a is a zero divisor element of R.

iii. If R has identity element, then $P_R = \{0\}$.

Proof.

i. Let R be a Boolean ring and $a \in P_R$. Since $RRa = (0)$, then $aaa = 0$. Moreover, a is an idempotent element, thus $a = 0$. Then, $P_R = \{0\}$.

ii. If $0 \neq a \in P_R$, then $RRa = (0)$. First case is if $RR = (0)$, then $aa = 0$. Thus, a is a zero divisor element. The other case is $RR \neq (0)$. In this case, a is a right zero divisor element since $RRa = (0)$. Besides, it is either $Ra = 0$ or $Ra \neq (0)$. If $Ra = 0$, then a is a zero divisor element. Let $Ra \neq (0)$. Since $RRa = (0)$, $aRa = (0)$. This means that a is a left zero divisor element. Thus, a is a zero divisor element.

iii. Let R has identity element 1_R and $a \in P_R$. Then, we have $RRa = (0)$. In particular, we write $1_R 1_R a = 0$. Then, we obtain $P_R = \{0\}$.

□

From Proposition 2.9, it is easy to see that the following corollary.

Corollary 2.10. For any ring R the following is always true.

i. There is no idempotent element other than zero in P_R.

ii. Every element in P_R is nilpotent.

Proof.

i. Let $a \in P_R$ be an idempotent element. Since $RRa = (0)$, then $aaa = 0$. Moreover, a is an idempotent element, thus $a = 0$.

ii. If $0 \neq a \in P_R$. Since $RRa = (0)$, then we get $aaa = 0$. Thus, a is a nilpotent element.

□

Theorem 2.11. Let R and T be two rings and $f : R \to T$ a ring homomorphism. Therefore, $f(P_R) \subseteq P_{f(R)}$. If f is injective, then $f(P_R) = P_{f(R)}$.

Proof. Let $x \in f(P_R)$. In this case, there is $a \in P_R$ such that $x = f(a)$. Namely, $RRa = (0)$. Since

$$(0) = f(RRa) = f(R)f(R)f(a)$$

we get $f(a) \in P_{f(R)}$ and so $x \in P_{f(R)}$. Hence, $f(P_R) \subseteq P_{f(R)}$.

Let $y \in P_{f(R)}$. From the set definition, $f(R)f(R)y = (0)$ for $y \in f(R)$. Since $y \in f(R)$, we have $y = f(r)$ for $r \in R$. Hence, $f(R)f(R)f(r) = (0)$ and since f is a homomorphism $f(RRr) = (0)$ is satisfied. Using f is injective, we obtain $RRr = (0)$ for $r \in R$. So, $r \in P_R$. From here, $y = f(r) \in f(P_R)$ is obtained. Thus, $P_{f(R)} \subseteq f(P_R)$. □

3. Conclusion

We first defined the concept of the source of primeness of an associative ring. We have shown that when R is a prime ring, the source of primeness set consists only of zero element, which is very useful for examining the work done on the prime ring for the source of primeness of R. Furthermore, we adapt some well-known results in prime ring to the source of primeness of R. The properties of the set source of primeness of a ring R can be investigated in the sense of the articles in [6–8].

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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