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Oscillation Criteria for a Class of Second Order Nonlinear Differential Equations

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ABSTRACT

In this paper, we are concerned with the oscillation of a class of second order nonlinear differential equations. By using the Riccati technique some new oscillation criteria are established, therefore, we generalize and extend a number of existing oscillation criteria. An example is also given to illustrate our results.

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1. INTRODUCTION

This paper is concerned with the oscillation of the solutions of the second-order nonlinear differential equation

$$k'\left(t,x(t),x'(t)\right)+q(t)\varphi\left(f\left(x(t)\right),k\left(t,x(t),x'(t)\right)\right)=0, \tag{1.1}$$

where $q\in C\left([t_0,\infty),\mathbb{R}\right)$, $f\in C\left(\mathbb{R},\mathbb{R}\right)$ with $xf(x)\neq 0$ for $x\neq 0$, $k\in C^1\left(\mathbb{R}^3,\mathbb{R}\right)$ with $wk(u,v,w)>0$ for all $w\neq 0$, $\varphi\in C\left(\mathbb{R}^2,\mathbb{R}\right)$ with $u\varphi(u,v)>0$ and $\varphi(\lambda u,\lambda v)=\lambda\varphi(u,v)$ where $\lambda\in(0,\infty)$.

A function $x:[t_0,t_1)\to(-\infty,\infty)$, $t_1>t_0$, is called a solution of Eq. (1.1) if $x(t)$ satisfies Eq. (1.1) for all $t\in[t_0,t_1)$. In this paper we restrict our attention to these solutions, $x(t)$ of Eq. (1.1) which exists on $[t_0,\infty)$ and satisfy

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$\sup\{|x(t)|: t > t_x\} \neq 0$ for all $t_x \geq t_0$. Such a solution of Eq. (1.1) is called oscillatory if the set of its zeros is unbounded from above, otherwise, it is said to be nonoscillatory. Eq. (1.1) is called oscillatory, if its all solutions are oscillatory.

There are a great number of papers devoted to particular cases of Eq. (1.1) such as

$$x''(t)+q(t)x(t)=0,$$

$$\left(r(t)x'(t)\right)'+q(t)f(x(t))=0,$$

see for example [1]-[5] and references cited therein.

For the oscillation of

$$x''(t)+q(t)\phi\left(x(t),x'(t)\right)=0, \tag{1.2}$$

Bihari [6] proved that if $q(t)>0$ for all $t\geq t_0$ and

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)ds=\infty,$$

then Eq. (1.2) is oscillatory.

The following theorem extends the result of Bihari [6] to an equation in which q is of arbitrary sign:

Kartsatos [7] proved that Eq. (1.2) is oscillatory if

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)ds=\infty$$

and there exists a constant $c_1\in(0,\infty)$ such that

$$\Phi(m)=\int_0^m \frac{dw}{\phi(1,w)} \geq -c_1 \text{ for every } m\in\mathbb{R}. \tag{1.3}$$

Recently, Elabbasy and Elzeiny [8] extended the results of Bihari [6] and Kartsatos [7] for the equation

$$\left(r(t)\psi(x)f(x')\right)'+q(t)\varphi\left(g(x),r(t)\psi(x)f(x')\right)=0, \tag{1.4}$$

with the following criteria:

Eq. (1.4) is oscillatory if (1.3) holds and

$$\limsup_{t\to\infty}\frac{1}{t^{\beta}}\int_{t_0}^{t}(t-s)^{\beta}q(s)ds=\infty \text{ for some } \beta\geq 0. \tag{1.5}$$

Considering the type of Eq. (1.1), the aim of this paper is to obtain some new oscillation criteria which extend the results of Bihari [6], Kartsatos [7] and Elabbasy and Elzeiny [8].

2. MAIN RESULTS

In this section, we will establish sufficient conditions to Eq. (1.1) to be oscillatory. Therefore, we will use the Riccati substitution technique in our proofs. An example is also will be given to illustrate results.

Theorem 1 *Assume that* $f'\geq 0$*, (1.3) and (1.5) hold. Then Eq. (1.1) is oscillatory.*

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. We may assume that $x(t)>0$ on $[T,\infty)$ for some large $T\geq t_0$. Let $\theta(t)$ be defined by

$$\theta(t)=\frac{k(t,x,x')}{f(x)},\ t\geq T. \tag{2.1}$$

Differentiating Eq. (2.1) and using Eq. (1.1) we get

$$\theta'(t)=-q(t)\varphi(1,\theta(t))-\frac{k(t,x,x')f'(x)x'(t)}{f^2(x)}$$

which implies

$$\theta'(t) \le -q(t)\varphi(1,\theta(t))$$

or

$$q(t) \le -\frac{\theta'(t)}{\varphi(1,\theta(t))} \tag{2.2}$$

for $t \ge T$. Multiplying Eq. (2.2) by $(t-s)^{\beta}$ and then integrating, we get

$$\int_T^t (t-s)^{\beta} q(s)ds \le -\int_T^t (t-s)^{\beta} \frac{\theta'(s)}{\varphi(1,\theta(s))} ds. \tag{2.3}$$

By Bonnet's theorem [9], we see that for each $t \ge T$, there exists $\alpha_t \in [T,t]$ such that

$$\begin{aligned} -\int_T^t (t-s)^{\beta} \frac{\theta'(s)}{\phi(1,\theta(s))} ds &= -(t-T)^{\beta} \int_T^{\alpha_t} \frac{\theta'(s)}{\phi(1,\theta(s))} ds = -(t-T)^{\beta} \int_{\theta(T)}^{\theta(\alpha_t)} \frac{dw}{\phi(1,w)} \\ &= -(t-T)^{\beta} \left[\Phi(\theta(\alpha_t)) - \Phi(\theta(T))\right] \\ &\le \left[c_1 + \Phi(\theta(T))\right](t-T)^{\beta}. \end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\int_T^t (t-s)^{\beta} q(s)ds \le \left[c_1 + \Phi(\theta(T))\right](t-T)^{\beta},\ t \ge T.$$

Dividing this inequality by t^{β} and taking limit superior on both sides, we obtain

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^t (t-s)^{\beta} q(s)ds &= \limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_{t_0}^T (t-s)^{\beta} q(s)ds \\ &+ \limsup_{t\to\infty} \frac{1}{t^{\beta}} \int_T^t (t-s)^{\beta} q(s)ds < \infty, \end{aligned}$$

which contradicts (1.5).

If $x(t) < 0$ on $[T_0,\infty)$ for some large $T_0 \ge t_0$, the inequality (2.2) still holds. So the same contradiction occurs again. Hence, the proof is complete.

Corollary 1 *Let* $k(t,x(t),x'(t)) = r(t)k_1(x(t),x'(t))$ *in the Eq. (1.1) where* r *is continuous on* $[t_0,\infty)$ *with* $r > 0$ *and* $vk_1(u,v) > 0$. *Assume that the condition (1.3) holds. Furthermore, assume that*

$$\limsup_{t\to\infty} \frac{1}{R^{\beta}(t)} \int_{t_0}^t (R(t)-R(s))^{\beta} q(s)ds = \infty \text{ for some } \beta \ge 0,$$

where $R(t) = \int_{t_0}^t \frac{ds}{r(s)}$. *Then the Eq. (1.1) is oscillatory.*

Proof. In this case, if we define $\theta(t) = \dfrac{r(t)k(x(t),x'(t))}{f(x(t))}$, (2.2) holds again. Integrating (2.2) multiplied by $(R(t)-R(s))^{\beta}$ and following the same procedure as in the theorem above, we obtain

$$\int_T^t (R(t)-R(s))^{\beta} q(s)ds \le \left[c_1 + \Phi(\theta(T))\right](R(t)-R(s))^{\beta},\ t \ge T.$$

Dividing this inequality by $R^{\beta}(t)$ and taking limit superior on both sides, we get

$$\limsup_{t\to\infty}\frac{1}{R^{\beta}(t)}\int_{t_0}^{t}(R(t)-R(s))^{\beta}q(s)ds\le\limsup_{t\to\infty}[c_1+\Phi(\theta(T))]\left(1-\frac{R(T)}{R(t)}\right)^{\beta}.$$

Using $\lim_{t\to\infty}\frac{1}{R(t)}=L\in(0,\infty),$ we obtain the desired contradiction to (1.5).

Theorem 2 *Assume that* $q(t)\ge 0,\ \frac{\partial\varphi}{\partial u}\ge 0,\ \frac{f(x)}{x}\ge K$ *for some* $K\ge 0$ *hold and there exists a constant* $c_2\in(0,\infty)$ *such that*

$$\Phi_1(m)=\int_0^m\frac{dw}{\phi(K,w)}\ge -c_2\ \text{for every}\ m\in\mathbb{R}. \tag{2.5}$$

Furthermore, assume that (1.5) holds. Then, Eq. (1.1) is oscillatory.

Proof. On the contrary, suppose that Eq. (1.1) has a nonoscillatory solution $x(t)$. We may assume that $x(t)>0$ on $[T,\infty)$ for some large $T\ge t_0$. Let $\psi(t)$ be defined

$$\psi(t)=\frac{k(t,x(t),x'(t))}{x(t)},\ t\ge T. \tag{2.6}$$

Differentiating (2.6) and using (1.1) we get

$$\psi'(t)=-q(t)\phi\left(\frac{f(x(t))}{x(t)},\psi(t)\right)-\frac{k(t,x(t),x'(t))x'(t)}{x^2(t)}$$

which implies

$$\psi'(t)\le -q(t)\varphi(K,\psi(t))$$

for $t\ge T.$ Rest of the proof is similar with the previous theorem, hence omitted.

Corollary 2 *Let* $k(t,x(t),x'(t))=r(t)k_1(x(t),x'(t))$ *in the Eq. (1.1) where* r *is continuous on* $[t_0,\infty)$ *with* $r>0$ *and* $vk_1(u,v)>0.$ *Assume that* $q(t)\ge 0,\ \frac{\partial\varphi}{\partial u}\ge 0$ *and the condition (2.5) hold. Furthermore, assume that*

$$\limsup_{t\to\infty}\frac{1}{R^{\beta}(t)}\int_{t_0}^{t}(R(t)-R(s))^{\beta}q(s)ds=\infty\ \text{for some}\ \beta\ge 0,$$

where $R(t)=\int_{t_0}^{t}\frac{ds}{r(s)}$. *Then the Eq. (1.1) is oscillatory.*

Remark 1 *If we replace the conditions on* φ *and* f *with* $\frac{\partial\varphi}{\partial u}\le 0,\ \frac{f(x)}{x}\le K$ *for some* $K\ge 0,$ *the theorem and the corollary above are valid as well.*

Remark 2 *If we replace the condition* $\varphi(\lambda u,\lambda v)=\lambda\varphi(u,v)$ *with* $\varphi(\lambda u,\lambda v)\le\lambda\varphi(u,v)$, *the theorem and corollary above are valid as well. Furthermore, if* $q(t)\ge 0,$ *Theorem 1 and Corollary 1 are valid with* $\phi(\lambda u,\lambda v)\le\lambda\phi(u,v)$.

Example 1 *Consider the differential equation*

$$\left((t+x(t))^2x'(t)\right)'+\left(\frac{1}{t}+2\sin t\right)x(t)\exp\left(-(t+x)^2x'(t)/x(t)\right)=0,\ t\ge 1. \tag{2.7}$$

Here, $q(t)=\frac{1}{t}+2\sin t,\ \ f(x)=x$, $k(t,x,x')=(t+x)^2x'$ *and* $\varphi(u,v)=ue^{\frac{-v}{u}}$.

Note that (1.3) is satisfied. By choosing $\beta=2,$ *we have*

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_1^t (t-s)^2 q(s)ds=\infty.$$

Thus, Theorem 1 ensures that every solution of Eq. (2.7) oscillates.

Note that, the results of Bihari [6], Kartsatos [7] and Elabbasy and Elzeiny [8] cannot be applied to Eq. (2.7).

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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