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DISJOINT SETS IN PROJECTIVE PLANES OF SMALL ORDER

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Abstract. In this paper, results of a computer search for disjoint sets associated with maximal arcs and unitals in projective planes of order 16, and disjoint sets associated with unitals in projective planes of orders 9 and 25 are reported. It is shown that the number of pairs of disjoint unitals in planes of order 9 is exactly *four*, and new pairs and triples of disjoint degree 4 maximal arcs are shown to exist in some of the planes of order 16. New bounds on the number of 104-sets of type $(4, 8)$ and 156-sets of type $(8, 12)$ are achieved. A combinatorial method for finding new maximal arcs, new unitals, and new v-sets of type (m, n) is introduced. All disjoint sets found in this study are explicitly listed.

1. Introduction

We assume familiarity with the basic facts from finite geometries and design theory [3, 5, 10].

Let q be a prime power and π be a plane of order q^2. A *v-set* of type (m, n) in π is defined to be a set S of v points of π such that any line of π intersects with S in either m or n points.

There are *four* projective planes of order 9 [12]. Through this paper, the following abbreviations will be used for the names of these planes: $PG(2, 9)$, HALL(9), HALL(9)$^{\perp}$, and HUGHES(9). Up to isomorphism, the number of known projective planes of orders 16 and 25 is 22 and 193, respectively. The following abbreviations will be used for the names of the planes of order 16: PG(2,16), JOHN, MATH, HALL, DEMP, JOWK, SEMI2, SEMI4, DSFP, LMRH, BBH1, BBS4, and BBH2 [15], and we will follow the notations used in [13] for the known planes of order 25.

In this study, we will be interested in disjoint sets associated with $(n(q+1)-q)$-sets of type $(0, n)$ and (q^3+1)-sets of type $(1, q+1)$ in projective planes of orders

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$q^2 \in \{9, 16, 25\}$, where the former set is called a *maximal* $(n(q+1)-q, n)$-*arc* and later is called a *unital*.

The set of lines of π which have no points in common with a maximal $(n(q+1)-q, n)$-arc A determines a maximal $(\frac{q}{n}(q-n+1), \frac{q}{n})$-arc, denoted by $A^{\perp}$, in the dual plane of π. The sets of the intersections of the lines of π with A at n points form a 2-$(n(q+1)-q, n, 1)$ design $D(A)$.

In 1997, for odd prime power q, Ball et al. showed that degree n maximal arcs do not exist in $PG(2, q)$, where $1 < n < q$ [2]. Non-trivial maximal arcs do exist in some of the projective planes of even order with $n = 2^i$, $i \geq 1$ [6–9, 20, 21].

Penttila et al. classified all degree 2 maximal arcs in the known planes of order 16 [15], and it was shown that $PG(2, 16)$ contains exactly two inequivalent degree 4 maximal arcs [1]. Maximal (52, 4)-arcs have not been completely classified in the remaining of the known planes of order 16, yet.

Details of the known degree 4 maximal arcs and unitals in the known planes of order 16 are given in Table 1, where Column 1 gives the name of the planes, Column 2 shows how many degree 4 maximal arcs are known to exist in each plane, Column 3 lists the name of the maximal arcs, and the last column provides the number of known unitals in each plane.

TABLE 1. The known number of maximal (52, 4)-arcs and unitals in the known planes of order 16.

Plane	Known number of maximal (52, 4)-arcs [7–9]	maximal arcs denoted by	Known number of unitals [11, 16, 17, 19]
$PG(2,16)^*$	2	$PG(2,16)$.1 and $PG(2,16)$.2	2
BBH1*	3	bbh1.1, bbh1.2, etc.	16
BBH2	0	-	26
BBS4	0	-	13
DEMP	5	demp.1, demp.2, etc.	4
DSFP	1	dsfp.1	2
HALL	2	hall.1 and hall.2	6
JOHN	4	john.1, john.2, etc.	29
JOWK	2	jowk.1 and jowk.2	7
LMRH	2	lmrh.1 and lmrh.2	2
MATH	7	math.1, math.2, etc.	16
SEMI2*	7	semi2.1, semi2.2, etc.	21
SEMI4*	1	semi4.1	12

The specific point sets of the known degree 4 maximal arcs and unitals in the known planes of order 16 used in this study are from [8] and [17], respectively.

The set of lines of π meeting with a unital U at a single point determines a unital, denoted by $U^{\perp}$, in the dual plane of π. The sets of the intersections of the lines of π with U at $q+1$ points form a 2-$(q^3+1, q+1, 1)$ design $D(U)$.

In 1981, Brouwer constructed 138 non-isomorphic unital 2-$(28, 4, 1)$ designs and showed that *twelve* of them could be embedded as a unital in planes of order 9 [4], and in 1995, Penttila and Royle classified all unitals in planes of order 9 and they showed that there are exactly 18 such sets: *two* in $PG(2, 9)$, *four* in HALL(9) (so *four* in HALL$(9)^{\perp}$), and *eight* in HUGHES(9) [14].

Table 1 shows that the number of known unitals in planes of order 16 is 156, of which 38 of them were found by Stoichev and Tonchev [19], 3 of them were found by Krĉadinac and Smoljak [11] and 115 of them were found by Stoichev and Gezek [17].

The number of known unitals in the known projective planes of order 25 is 477 [18].

In this article, some results of a computer search for disjoint sets in the known planes of orders *nine*, *sixteen* and *twenty-five* are given. It is shown that disjoint sets in a projective plane of order q^2 may be useful to find a complete partitioning of the point set of the plane into disjoint sets associated with degree q maximal arcs and unitals. It is observed that new degree q maximal arcs, new unitals, new v-sets of type (m, n), and new projective planes can be found through disjoint sets as well.

The paper is organized in the following way. In Section 2, types of disjoint sets and some of possible ways of partitioning incidence matrices of projective planes are discussed. In Section 3, new pairs of disjoint degree 4 maximal arcs are shown to exist in BBH1, LMRH, and SEMI2 planes. MATH and SEMI2 planes are shown to contain new triples of disjoint degree 4 maximal arcs. In Section 4, it is shown that pairs of disjoint unitals exist in planes of order 9, and no such sets exists from the known unitals in the known planes of orders 16 and 25. In Section 5, we report the results of computer searches for 156-sets of type $(8, 12)$ associated with maximal arcs and unitals in the known planes of order 16. In Section 6, a combinatorial method for finding a complete partitioning of the point set of a projective plane into disjoint sets associated with maximal arcs and unitals, new maximal arcs, new unitals, new v-sets of type (m, n), and new projective planes is given. Point sets of all newly found disjoint sets discussed in this paper are available online at[1].

2. Types of Disjoint Sets in Projective Planes

It is well-known that some v-sets of type (m, n) might be coming from the unions of pairwise disjoint maximal arcs. In this paper, we will be interested in three different types of disjoint sets as described in Table 2:

Disjoint sets in π can be used to partition an incidence matrix of the plane in one of the following possible forms: if there exists a degree q maximal arc A_1 and

[1] euniversite.nku.edu.tr/testotomasyon/dosyalar/kullanicilar/3705/files/DisjointSets16.pdf

Table 2. Types of disjoint sets.

Type I	Disjoint pairs of maximal arcs
Type II	Disjoint pairs of unitals
Type III	A maximal arc disjoint from a unital

a unital U disjoint from A_1, then, WLOG, one may rearrange the columns (rows) of the incidence matrix according to the point sets of A_1 and U ($A_1^{\perp}$ and $U^{\perp}$) as

$$\left[\begin{array}{c||c|c} \overbrace{-\cdots\cdots\cdots-}^{(A_1\cup U)^c} & \overbrace{-\cdots-}^{A_1} & \overbrace{-\cdots-}^{U} \\ \hline q^2-2q & q & q+1 \\ \hline\hline q^2-q & \mathrm{O} & q+1 \\ \hline q^2-q & q & 1 \end{array}\right], \tag{1}$$

where O indicates a $(q^3-q^2+q)\times(q^3-q^2+q)$ zero matrix and numbers in the matrix shows the number of 1's in each row. In addition, if there exists a degree q maximal arc A_2 disjoint from $A_1\cup U$, then we may partition the incidence matrix of π as

$$\left[\begin{array}{c||c|c|c} \overbrace{-\cdots\cdots-}^{(A_1\cup A_2\cup U)^c} & \overbrace{-\cdots-}^{A_2} & \overbrace{-\cdots-}^{A_1} & \overbrace{-\cdots-}^{U} \\ \hline q^2-3q & q & q & q+1 \\ \hline\hline q^2-2q & \mathrm{O} & q & q+1 \\ \hline q^2-2q & q & \mathrm{O} & q+1 \\ \hline q^2-2q & q & q & 1 \end{array}\right]. \tag{2}$$

Sometimes we may not have a Type III disjoint set. Instead, we could have disjoint triples of degree q maximal arcs, that is, if there exists a degree q maximal arc A_3 disjoint from $A_1\cup A_2$, but not disjoint from U, then we may partition the incidence matrix of π as

$$\left[\begin{array}{c||c|c|c} \overbrace{-\cdots\cdots-}^{(A_1\cup A_2\cup A_3)^c} & \overbrace{-\cdots-}^{A_3} & \overbrace{-\cdots-}^{A_2} & \overbrace{-\cdots-}^{A_1} \\ \hline q^2-3q+1 & q & q & q \\ \hline\hline q^2-2q+1 & O & q & q \\ \hline q^2-2q+1 & q & O & q \\ \hline q^2-2q+1 & q & q & O \end{array}\right]. \tag{3}$$

Many more forms as similar above can be derived from an incidence matrix of π, but these three will be enough for our discussion in this study.

3. Type I Disjoint Sets

Type I disjoint sets in a projective plane π are the sets coming from disjoint pairs of maximal arcs. First examples of these sets are seen in $PG(2,4)$:

Theorem 1. *It is possible to partition the points of $PG(2,4)$ into two degree 2 maximal arcs and a unital.*

Proof. $A = \{7,8,10,11,19,20\}$ is a 6-set of type $(0,2)$ in $PG(2,4)$ (the specific line set of the projective plane of order 4 that we use is available online at[2]), a degree 2 maximal arc. It can be shown that $U = \{2,3,4,6,9,15,16,17,18\}$ is a 9-set of type $(1,3)$ in $PG(2,4)$, a unital, disjoint from A. Then, the complement of $A \cup U = \{0,1,5,12,13,14\}$ is a 6-set of type $(0,2)$. □

In the known projective planes of order 16, Hamilton et al. found *thirty-seven* Type I disjoint sets (21 in $PG(2,16)$, 4 in SEMI4, 4 in SEMI2, 3 in MATH, 3 in JOWK, and 2 in BBH1) [9] and Gezek found *thirty-seven* Type I disjoint sets (33 in MATH and 4 in JOHN) [7].

The specific line sets of the known planes of order 16 used in this study are from [8].

Previously, only *four* Type I disjoint sets were known to exist in SEMI2 [9], our computations show that there are more such sets in this plane:

There are *six* isomorphic copies of *semi*2.4 disjoint from *semi*2.1. The collineation stabilizer of the unions of these sets with *semi*2.1 all have order 4, and they are equivalent. This set is denoted by *semi*2.(1, 4).1.

There are *six* isomorphic copies of semi2.5 disjoint from *semi*2.1. The collineation stabilizer of the unions of these sets with *semi*2.1 all have order 4, and they are equivalent. This set is denoted by *semi*2.(1, 5).1.

There are *thirty-six* isomorphic copies of *semi*2.3 disjoint from itself. The collineation stabilizer of the unions of *twenty-four* of these sets with *semi*2.3 have order 16, and

[2] ericmoorhouse.org/pub/planes/pg24.txt

they split into *six* inequivalent classes. These sets are denoted by $semi2.(3,3).1$, $semi2.(3,3).2$, $\cdots$, $semi2.(3,3).6$. The collineation stabilizer of the unions of the remaining *twelve* sets with $semi2.3$ have order 32, and they split into *six* inequivalent classes. These sets are denoted by $semi2.(3,3).7$, $semi2.(3,3).8$, $\cdots$, $semi2.(3,3).12$.

There are *forty-four* isomorphic copies of $semi2.4$ disjoint from itself. The collineation stabilizer of the unions of *twenty-four* of these sets with $semi2.4$ have order 16, and they split into *six* inequivalent classes. These sets are denoted by $semi2.(4,4).1$, $semi2.(4,4).2$, $\cdots$, $semi2.(4,4).6$. The collineation stabilizer of the unions of *twelve* of the remaining sets with $semi2.4$ have order 32, and they split into *six* inequivalent classes. These sets are denoted by $semi2.(4,4).7$, $semi2.(4,4).8$, $\cdots$, $semi2.(4,4).12$. The collineation stabilizer of the unions of the remaining *eight* sets with $semi2.4$ all have order 8, and they are equivalent. This set is denoted by $semi2.(4,4).13$.

There are *sixteen* isomorphic copies of $semi2.5$ disjoint from $semi2.4$. The collineation stabilizer of the unions of these sets with $semi2.4$ all have order 8, and they split into *four* inequivalent classes. These sets are denoted by $semi2.(4,5).1$, $semi2.(4,5).2$, $\cdots$, $semi2.(4,5).4$.

There are *twenty* isomorphic copies of $semi2.5$ disjoint from itself. The collineation stabilizer of the unions of *eight* of these sets with $semi2.5$ have order 8, and they split into *two* inequivalent classes. These sets are denoted by $semi2.(5,5).1$ and $semi2.(5,5).2$. The collineation stabilizer of the unions of the *eight* of the remaining sets with $semi2.5$ have order 4, and they are equivalent. This set is denoted by $semi2.(4,5).3$. The collineation stabilizer of the unions of the remaining *four* sets with $semi2.5$ have order 16, and they split into *two* inequivalent classes. These sets are denoted by $semi2.(5,5).4$ and $semi2.(5,5).5$.

There are *thirty-six* isomorphic copies of $semi2.6$ disjoint from itself. The collineation stabilizer of the unions of these sets with $semi2.6$ all have order 16, and they split into *six* inequivalent classes. These sets are denoted by $semi2.(6,6).1$, $semi2.(6,6).2$, $\cdots$, $semi2.(6,6).6$.

There are *thirty-six* isomorphic copies of $semi2.7$ disjoint from itself. The collineation stabilizer of the unions of these sets with $semi2.7$ all have order 16, and they split into *six* inequivalent classes. These sets are denoted by $semi2.(7,7).1$, $semi2.(7,7).2$, $\cdots$, $semi2.(7,7).6$.

Previously, no Type I disjoint set were known to exist in LMRH. However, our computations show that this plane also contains such sets:

There are *thirty-six* isomorphic copies of $lmrh.2$ disjoint from $lmrh.1$. The collineation stabilizer of the unions of these sets with $lmrh.1$ all have order 8, and they split into *three* inequivalent classes. These sets are denoted by $lmrh.(1,2).1$, $lmrh.(1,2).2$, and $lmrh.(1,2).3$.

There are *twenty-four* isomorphic copies of $lmrh.2$ disjoint from itself. The collineation stabilizer of the unions of *sixteen* of these sets with $lmrh.2$ have order 8, and they split into *two* inequivalent classes. These sets are denoted by $lmrh.(2,2).1$ and $lmrh.(2,2).2$. The collineation stabilizer of the unions of the remaining *eight* sets with $lmrh.2$ have order 16, and they split into *two* inequivalent classes. These sets are denoted by $lmrh.(2,2).3$ and $lmrh.(2,2).4$.

Previously, only *two* Type I disjoint sets were known to exist in BBH1 [9], our computations show that there are more such sets in this plane as well:

There are *four* isomorphic copies of $bbh1.3$ disjoint from itself. The collineation stabilizer of the unions of *two* of these sets with $bbh1.3$ have order 8, and they split into *two* inequivalent classes. These sets are denoted by $bbh1.(3,3).1$, and $bbh1.(3,3).2$. The collineation stabilizer of the unions of the remaining *two* sets with $bbh1.3$ have order 4, and they are equivalent. This set is denoted by $bbh1.(3,3).3$.

All known Type I disjoint sets in the known projective planes of order 16 can be summarized as in Table 3, where Column 1 presents the name of the planes, Column 2 and 3 shows the group orders of the Type I disjoint sets (with their quantities) found in [9], and [7], respectively. Column 4 provides the group orders of the Type I newly discovered disjoint sets (with their quantities). The last column (row) shows the total number of such sets in each plane (study). An entry a^b in Table 3 implies that there are b inequivalent Type I disjoint sets with collineation stabilizer of order a.

TABLE 3. The number of known Type I disjoint sets in the known planes of order 16.

Plane	Hamilton et al. [9]	Gezek [7]	Gezek (2022)	Total
$PG(2,16)^*$	$2^{16}, 4^3, 8^2$			21
BBH1*	$8^1, 16^1$		$4^1, 8^2$	5
JOHN		16^4		4
JOWK	$8^2, 16^1$			3
LMRH			$8^5, 16^2$	7
MATH	$4^1, 8^2$	$4^5, 8^{14}, 16^8, 32^6$		36
SEMI2*	$8^2, 16^2$		$4^3, 8^7, 16^{26}, 32^{12}$	52
SEMI4*	$16^2, 32^2$			4
Total	37	37	58	132

Previously, it was reported that $PG(2,16)$ contains *one* disjoint triples of degree 4 maximal arc having collineation stabilizer of order 2 [9]. A computer program was written to find disjoint triples and disjoint quadruples of degree 4 maximal arcs in the known planes of order 16. In addition to the one found in $PG(2,16)$, our results show that SEMI2 and MATH planes also contain disjoint triples of degree 4 maximal arcs: there is an isomorphic copy of $semi2.4$ disjoint from the union of $semi2.1$ and $semi2.(1,4).1$, having collineation stabilizer of order 8 (this set is denoted by $semi2.(1,4,4).1$). There is an isomorphic copy of $semi2.5$ disjoint from the union of semi2.1 and $semi2.(1,5).1$, having collineation stabilizer of order 8 (this set is denoted by $semi2.(1,5,5).1$). There is an isomorphic copy of math.4 disjoint from the union of math.1 and $math.(1,4).1$, having collineation stabilizer of order 8 (this set is denoted by $math.(1,4,4).1$). There is an isomorphic copy of math.5 disjoint from the union of $math.1$ and $math.(1,5).1$, having collineation stabilizer of order 8 (this set is denoted by $math.(1,5,5).1$). Our program found no disjoint quadruples of degree 4 maximal arcs in the known planes of order 16.

4. Type II Disjoint Sets

Type II disjoint sets in a projective plane π are the sets coming from disjoint pairs of unitals.

In HALL(9), there are (up to isomorphism) four unitals. The unital having group order 24 has *six* isomorphic copies disjoint from itself. Up to isomorphism, there are two such pairs of disjoint unitals with collineation stabilizer of order 16. The unions of these sets with the unital having group order 24 in HALL(9) provide 56-sets of type $(5,8)$ (or, 35-sets of type $(2,5)$).

$PG(2,9)$ and HUGHES(9) planes do not contain any Type II disjoint sets, and our computations show that no Type II disjoint sets exists from the known unitals in the known planes of orders 16 and 25.

5. Type III Disjoint Sets

Type III disjoint sets in a projective plane π are the sets coming from a maximal arc A and a unital U in π such that A and U are disjoint. The smallest plane where this type of set exists is $PG(2,4)$ (see Section 3).

Previously, no Type III disjoint sets were known to exist in the known projective planes of order 16. Our computations show that such sets exist in these planes. We provide details of the Type III disjoint sets found by our algorithm in Table 4, where the first column presents the maximal arcs, and the last column gives for which unital there exists such a set. An entry $\mathbf{j}(k)$ in row i in Table 4 implies that there are k inequivalent Type III disjoint sets coming from maximal arc i and unital $\mathbf{j}$.

Group orders of the Type III disjoint sets found in this study can be summarized as in Table 5, where Column 1 presents the name of the planes, Column 2 shows the group orders of the Type III disjoint sets (with their quantities). The last column (row) shows the total number of such sets in each plane (all planes).

Table 3 shows that the number of disjoint pairs of maximal $(52,4)$-arcs is at least 132. Union of disjoint pairs of maximal $(52,4)$-arcs is a 104-set of type $(4,8)$. Some of the disjoint pairs of the degree 4 maximal arcs given in Table 3 are disjoint from some of the isomorphic copies of unitals: *two* in SEMI2 and *two* in MATH. The complement of the union of these disjoint sets is also a 104-set of type $(4,8)$: the sets in SEMI2 have collineation stabilizer of orders 4 and 8, and the sets in MATH have collineation stabilizer of orders 4 and 8. None of these sets are equivalent to any Type I disjoint sets given in Table 3. We have

TABLE 4. Type III disjoint sets in the known planes of order 16.

Maximal $(52, 4)$-arc	**Unital No.**(Quantity)
$bbh1.2$	$\mathbf{1}(2), \mathbf{16}(2)$
$john.1$	$\mathbf{26}(1), \mathbf{29}(1)$
$john.2$	$\mathbf{2}(1), \mathbf{26}(1)$
$john.3$	$\mathbf{2}(1), \mathbf{26}(1)$
$john.4$	$\mathbf{26}(1), \mathbf{29}(1)$
$jowk.1$	$\mathbf{7}(1)$
$jowk.2$	$\mathbf{7}(5)$
$lmrh.1$	$\mathbf{1}(1), \mathbf{2}(2)$
$lmrh.2$	$\mathbf{1}(3), \mathbf{2}(8)$
$math.1$	$\mathbf{4}(2), \mathbf{8}(2)$
$math.2$	$\mathbf{2}(1), \mathbf{5}(2), \mathbf{6}(1), \mathbf{10}(1), \mathbf{11}(1)$
$math.3$	$\mathbf{5}(2), \mathbf{7}(1), \mathbf{11}(1), \mathbf{12}(2), \mathbf{15}(1), \mathbf{16}(1)$
$math.4$	$\mathbf{3}(1), \mathbf{5}(2), \mathbf{10}(2), \mathbf{11}(1), \mathbf{13}(2), \mathbf{14}(2)$
$math.5$	$\mathbf{3}(1), \mathbf{10}(1)$
$semi2.1$	$\mathbf{4}(2)$
$semi2.2$	$\mathbf{11}(2)$
$semi2.3$	$\mathbf{1}(1), \mathbf{2}(2), \mathbf{9}(2), \mathbf{10}(2), \mathbf{14}(1) - \mathbf{21}(1)$
$semi2.4$	$\mathbf{2}(2), \mathbf{9}(4), \mathbf{10}(2), \mathbf{11}(2), \mathbf{14}(1), \mathbf{15}(1), \mathbf{16}(2), \mathbf{17}(1) - \mathbf{19}(1)$
$semi2.5$	$\mathbf{11}(1), \mathbf{16}(1)$
$semi2.6$	$\mathbf{5}(1), \mathbf{6}(4), \mathbf{12}(2)$
$semi2.7$	$\mathbf{5}(4), \mathbf{6}(1), \mathbf{13}(2)$
$semi4.1$	$\mathbf{8}(2), \mathbf{10}(2), \mathbf{12}(2)$

TABLE 5. The number of known Type III disjoint sets in the known planes of order 16.

Plane	Gezek (2022)	Total
BBH1*	$8^2, 16^2$	4
JOHN	16^8	8
JOWK	$4^2, 8^2, 16^2$	6
LMRH	$8^{10}, 16^4$	14
MATH	$4^7, 8^{17}, 16^6$	30
SEMI2*	$4^5, 8^{34}, 16^6$	51
SEMI4*	$8^3, 16^2$	5
Total		118

Theorem 2. *The number of* 104*-sets of type* $(4, 8)$ *in planes of order* 16 *is at least* 136*, of which all except four are coming from the unions of pairs of disjoint maximal* $(52, 4)$*-arcs.*

The complement of a Type III disjoint set is a 156-set of type $(8,12)$. Table 5 shows that the number of 156-sets of type $(8,12)$ in planes of order 16 is at least 118. As previously it was mentioned, there exist *five* triples of disjoint maximal $(52,4)$-arcs, their unions provide *five* 156-sets of type $(8,12)$. None of these sets are equivalent to any of the complement of Type III disjoint sets given in Table 5. We have

Theorem 3. *The number of* 156*-sets of type* $(8,12)$ *in planes of order* 16 *is at least* 123*, of which five of them are coming from the unions of triples of disjoint maximal* $(52,4)$*-arcs.*

6. Conclusion

The main purpose of the study presented in this paper is to answer the following question: for any prime power q, is it possible to partition the point set of a projective plane of order q^2 into q pairwise disjoint degree q maximal arcs and a unital? For $q=2$, the answer is yes (see Section 3).

The first open case where no such partitioning is known to exist is the case for $q=4$. If a partitioning of the point set of a plane of order 16 (as a union of *four* pairwise disjoint degree 4 maximal arcs and *one* unital) is possible, we get it from either (i) finding appropriate new maximal arcs of degree 4, or (ii) finding appropriate new unitals, or (iii) finding appropriate Type III disjoint sets, or (iv) finding an appropriate partitioning of the complement of the disjoint sets presented in this paper.

An appropriate degree 4 maximal arc in (i) in the matrix form (1) means a maximal arc A disjoint from $A_1 \cup U$ such that we have the matrix form (2), which may lead to a complete partitioning of the incidence matrix of the plane (if there exists an isomorphic copy of A disjoint from $A \cup A_1 \cup U$) and a (possible) new maximal arc (the complement of $A_1 \cup U$ may be a union of triples of disjoint maximal arcs such that maximal arcs in the union are isomorphic to A), or a (possible) new 104-set of type $(4,8)$. An appropriate degree 4 maximal arc in (i) in the matrix form (2) means a maximal arc A disjoint from $A_1 \cup A_2 \cup U$ such that we have

$$\left[\begin{array}{c||c|c|c|c} \overbrace{-\cdots-}^{A'} & \overbrace{-\cdots-}^{A} & \overbrace{-\cdots-}^{A_2} & \overbrace{-\cdots-}^{A_1} & \overbrace{-\cdots-}^{U} \\ \hline \mathrm{O} & 4 & 4 & 4 & 5 \\ \hline\hline 4 & \mathrm{O} & 4 & 4 & 5 \\ \hline 4 & 4 & \mathrm{O} & 4 & 5 \\ \hline 4 & 4 & 4 & \mathrm{O} & 5 \\ \hline 4 & 4 & 4 & 4 & 1 \end{array}\right], \tag{4}$$

which gives a complete partitioning of the incidence matrix of the plane as well as a possible new degree 4 maximal arc A' (A' may be isomorphic to A). An appropriate degree 4 maximal arc in (i) in the matrix form (3) means a maximal arc A disjoint from $A_1 \cup A_2 \cup A_3$ such that

$$\left[\begin{array}{c||c|c|c|c} \overbrace{-\cdots-}^{U'} & \overbrace{-\cdots-}^{A} & \overbrace{-\cdots-}^{A_3} & \overbrace{-\cdots-}^{A_2} & \overbrace{-\cdots-}^{A_1} \\ \hline 1 & 4 & 4 & 4 & 4 \\ \hline\hline 5 & O & 4 & 4 & 4 \\ \hline 5 & 4 & O & 4 & 4 \\ \hline 5 & 4 & 4 & O & 4 \\ \hline 5 & 4 & 4 & 4 & O \end{array}\right], \tag{5}$$

which gives a complete partitioning of the incidence matrix of the plane as well as a new unital U' (this is a new set because U' is disjoint from $A_1 \cup A_2 \cup A_3$ and none of the known unitals have isomorphic copies disjoint from $A_1 \cup A_2 \cup A_3$). Similar arguments can be made for (ii)-(iv) in the matrix forms (1)-(3).

Discussions in this study make us believe that the following is true in general:

Conjecture 1. *The points of $PG(2, q^2)$ can be partitioned into q degree q maximal arcs and a unital.*

We conclude that disjoint sets in a projective plane π may be useful to find a complete partitioning of the point set of the plane into disjoint sets associated with degree q maximal arcs and unitals, new degree q maximal arcs, new unitals, and new v-sets of type (a, b). New projective planes can be found through disjoint sets by studying submatrices in the matrix forms (1)-(3) (a possible future research project). Disjoint sets also dramatically lessen the number of computations for finding new maximal arcs (i.e., from $\binom{273}{52}$ to $\binom{104}{52}$ for the computations in the planes of order 16) and unitals (i.e., from $\binom{273}{65}$ to $\binom{117}{65}$ for the computations in the planes of order 16).

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