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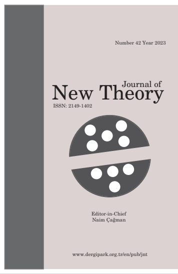

Characterizations of Unit Darboux Ruled Surface with Quaternions

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Abstract — This paper presents a quaternionic approach to generating and characterizing the ruled surface drawn by the unit Darboux vector. The study derives the Darboux frame of the surface and relates it to the Frenet frame of the base curve. Moreover, it obtains the quaternionic shape operator and its matrix representation using the normal and geodesic curvatures to provide a more detailed analysis. To illustrate the concepts discussed, the paper offers a clear example that will help readers better understand the concepts and showcases the quaternionic shape operator, Gauss curvature, mean curvature, and rotation matrix. Finally, it emphasizes the need for further research on this topic.

Keywords *Gauss curvature, mean curvature, ruled surfaces, quaternions, quaternionic shape operator*

Mathematics Subject Classification (2020) 11R52, 53A05

1. Introduction (Compulsory)

Quaternions include diverse fields such as game programming, robotics, animation, and navigation systems [1–3]. In addition to these areas, quaternions are important for the theory of curves and surfaces. Bharathi and Nagaraj have introduced the Serret-Frenet formulae for quaternionic curves in $\mathbb{R}^3$ and $\mathbb{R}^4$ [4]. By utilizing this study, numerous studies have examined quaternionic curves. One of them, the authors have proved that if the bitorsion of a quaternionic curve does not vanish, then there is no quaternionic curve in E^4. Therefore, they have expressed (1, 3) type Bertrand curves for quaternionic curves [5]. Babaarslan and Yaylı have examined constant slope surfaces with quaternions [6]. In [7], the authors have expressed the ruled surface as quaternionic and computed some properties of the ruled surface. Moreover, they have investigated the dual ruled surface using dual quaternion [8]. In light of these studies, Çalışkan have examined the quaternionic and dual quaternionic Darboux ruled surfaces [9]. In [10], the authors have examined the advantage of the dual number of Clifford algebra to make the singular ruled surfaces transform into dual singular curves. Aslan and Yaylı have defined the quaternionic shape operator by the quaternion. Their article has aimed to find a way to the invariants of the surface using Darboux frames and quaternions [11]. In [12,13], the connection between split quaternions and surfaces with the constant slope in Minkowski 3-space has been explored. It is demonstrated that these surfaces can be transformed using rotation matrices associated with quaternions and homothetic motions. A surface is said to be ruled if it is generated by moving a straight line continuously in $\mathbb{R}^3$. Thus, a ruled surface has a parameterization

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in the form $\vec{\Lambda}(s,v) = \vec{\alpha}(s) + v\vec{x}(s)$ where we call α the base curve and $\vec{x}$ the generator vector of the ruled surface [14]. Ruled surfaces are important for robot kinematics. Ryuh has suggested that ruled surfaces play an important role in robot end-effectors [15]. In another study, the ruled surfaces' differential properties drawn by the developed trihedron's generator vector have been examined [16]. In [17], the authors have introduced a new type of ruled surface defined using an orthonormal Sannia frame on a base curve. They have studied the properties of these surfaces using the first and second fundamental forms, as well as the mean and Gaussian curvatures. They have provided conditions for when these surfaces are developable and minimal and present some examples of these ruled surfaces. Eren et al. have introduced new types of ruled surfaces in Euclidean 3-space. These surfaces have been obtained using the evolution of an involute-evolute curve pair and studied with the modified orthogonal frame. They have provided some results on these surfaces' Gaussian and mean curvatures [18, 19]. Bilici has examined ruled surfaces produced by a Frenet trihedron of closed dual involute for a specific dual curve. He has specifically focused on the relations between the pitch, the angle of the pitch, and the drall of these surfaces [20]. Some ruled surfaces produced using the Frenet trihedron, Blaschke frame, and the surface family are studied in [21–24].

In Section 2, we provide some necessary background information about the problem of the paper that was mentioned in the introduction. In Section 3, we give characterizations of ruled surfaces drawn by the unit Darboux vector using quaternions. We obtain the quaternionic shape operators and their matrix representations using normal and geodesic curvatures. In the last section, we exemplify the findings.

2. Preliminaries

Let $\alpha : I \to \mathbb{R}^3$ be a unit-speed curve. Then, the three vector fields $\vec{t}(s), \vec{n}(s)$, and $\vec{b}(s)$ on the curve α are unit vector fields that are mutually orthogonal at each point. We call $\vec{t}(s), \vec{n}(s)$, and $\vec{b}(s)$ the Frenet vectors on the curve α. The Frenet formulas can be given

$$\vec{t}'(s) = \kappa(s)\vec{n}(s), \quad \vec{n}'(s) = -\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s), \quad \text{and} \quad \vec{b}'(s) = -\tau(s)\vec{n}(s)$$

where $\kappa(s)$ and $\tau(s)$ are the first and second curvature of the unit-speed curve, respectively [25]. For any unit-speed curve $\alpha : I \to \mathbb{R}^3$, the vector $\vec{W}(s)$ is called Darboux vector defined by

$$\vec{W}(s) = \tau(s)\vec{t}(s) + \kappa(s)\vec{b}(s)$$

If consider the normalization of the Darboux $\vec{C}(s) = \dfrac{1}{\|\vec{W}(s)\|}\vec{W}(s)$, we have

$$\vec{C}(s) = \sin\xi(s)\vec{t}(s) + \cos\xi(s)\vec{b}(s)$$

where $\cos\xi(s) = \frac{\kappa(s)}{\|\vec{W}(s)\|}$, $\sin\xi(s) = \frac{\tau(s)}{\|\vec{W}(s)\|}$, and $\angle(\vec{W}(s), \vec{b}(s)) = \xi(s)$. A quaternion is a unit length of four-vectors $q = d + a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$ characterized by the following properties:

$$\begin{cases} \vec{e_1}^2 = \vec{e_2}^2 = \vec{e_3}^2 = \vec{e}_1 \times \vec{e}_2 \times \vec{e}_3 = -1, \\ \vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \vec{e}_3 \times \vec{e}_1 = \vec{e}_2 \end{cases} \quad \vec{e}_1, \vec{e}_2, \vec{e}_3 \in \mathbb{R}^3$$

The quaternion product of two quaternions q_1 and q_2, which we write as $q_1 \times q_2$, takes the form

$$\begin{aligned} q_1 \times q_2 &= d_1 d_2 - (a_1 a_2 + b_1 b_2 + c_1 c_2) + (d_1 a_2 + a_1 d_2 + b_1 c_2 - c_1 b_2)\vec{e}_1 \\ &\quad + (d_1 b_2 + b_1 d_2 + b_1 a_2 - a_1 b_2)\vec{e}_2 + +(d_1 c_2 + c_1 d_2 + a_1 b_2 - b_1 a_2)\vec{e}_3 \end{aligned}$$

The complex conjugate of a quaternion q is denoted as $\bar{q} = d - a\vec{e}_1 - b\vec{e}_2 - c\vec{e}_3$. The norm of q is

$$\mathbf{N}(q) = \sqrt{d^2 + a^2 + b^2 + c^2}$$

Pure quaternion is denoted as $\bar{q}+q=0=a\vec{e}_1+b\vec{e}_2+c\vec{e}_3$. The quaternion multiplication of two pure quaternions is $q_1\times q_2=-\langle q_1,q_2\rangle+q_1\wedge q_2$. The unit quaternion can be written in the form as $q=\cos\varphi+\sin\varphi\vec{v}$ where $\vec{v}\in\mathbb{R}^3$ and $\|\vec{v}\|=1$. Let q be a unit quaternion and $\vec{w}$ be a pure quaternion. Then,

$$\vec{w}'=q\times\vec{w}\times q^{-1}$$

is rotated 2φ about the axis $\vec{v}$. We say finally that the desired rotation matrix fixing the direction v is

$$R=\begin{bmatrix} 1+\sin^2\varphi(u_1^2-u_2^2-u_3^2-1) & -\sin 2\varphi u_3+2\sin^2\varphi u_1u_2 & \sin 2\varphi u_2+2\sin^2\varphi u_1u_3 \\ \sin 2\varphi u_3+2\sin^2\varphi u_1u_2 & 1+\sin^2\varphi(u_2^2-u_1^2-u_3^2-1) & 2\sin^2\varphi u_2u_3-\sin 2\varphi u_1 \\ 2\sin^2\varphi u_1u_3-\sin 2\varphi u_2 & \sin 2\varphi u_1+2\sin^2\varphi u_2u_3 & 1+\sin^2\varphi(u_3^2-u_2^2-u_1^2-1) \end{bmatrix}$$

where R is an orthogonal matrix. For detailed information on the theory of quaternion, see the references [1,3,26].

If p is a point of M, for each tangent vector $\vec{X}$ to M at p, $S_p(\vec{X})=-\nabla_{\vec{X}}\vec{Z}$. S_p is defined as the shape operator of M at p. The shape operator is the symmetric linear map. Here, Z is the unit normal vector field. A surface M in $\mathbb{R}^3$ is flat provided its Gauss curvature is zero, and minimal provided its mean curvature is zero. Moreover, the Gauss and minimal curvatures are independent of the choice of basis. These curvatures are found in the equations

$$K=\frac{\|S(\vec{T}(u))\wedge S(\vec{T}(t))\|}{\|\vec{T}(u)\wedge\vec{T}(t)\|},\ H=\frac{\|S(\vec{T}(u))\wedge\vec{T}(t)+\vec{T}(u)\wedge S(\vec{T}(t))\|}{2\|\vec{T}(u)\wedge\vec{T}(t)\|}$$

where $\vec{T}(u)$ and $\vec{T}(t)$ are the tangent vectors of $\beta(u)$ and $\zeta(t)$, respectively [25]. Let β be a curve that is traced on a surface and Darboux frame $\{\vec{T}(u),\vec{Y}(u),\vec{Z}(u)\}$ is an orthogonal frame. The equations of motion of the Darboux frame can be written as

$$\begin{bmatrix}\vec{T}'(u)\\ \vec{Y}'(u)\\ \vec{Z}'(u)\end{bmatrix}=\|\Lambda_u\|\begin{bmatrix}0 & k_g(u) & k_n(u)\\ -k_g(u) & 0 & t_g(u)\\ -k_n(u) & -t_g(u) & 0\end{bmatrix}\begin{bmatrix}\vec{T}(u)\\ \vec{Y}(u)\\ \vec{Z}(u)\end{bmatrix} \tag{1}$$

Here, k_n, k_g, and t_g are the normal curvature, the geodesic curvature, and the geodesic torsion, respectively [14,25].

Theorem 2.1. [11] Let M be a surface with parameter u and $\beta(u)$ be a unit speed curve in M. Using the quaternion operator $Q(u)=k_n(u)+t_g(u)\vec{Z}(u)$, the shape operator can be given as

$$S(\vec{T}(u))=Q(u)\times\vec{T}(u) \tag{2}$$

The quaternion Q will be called a quaternionic shape operator.

Quaternionic shape operator can be given by the unit quaternion $p(u)=\cos 2\varphi(u)+\sin 2\varphi(u)\vec{Z}(u)$ as

$$Q(u)=\sqrt{{k_n}^2(u)+{t_g}^2(u)}\big(\cos 2\varphi(u)+\sin 2\varphi(u)\vec{Z}(u)\big)$$

Then, we can say that the vector $Q(u)\times\vec{T}(u)$ is obtained by revolving $\vec{T}(u)$ around the normal vector $\vec{Z}(u)$ of the surface through twice the angle of φ [11].

Theorem 2.2. [11] Let M be a surface, X be a local parameterization of M, N be the unit normal vector field of M, and $Q(t)$ and $Q(u)$ be the quaternionic shape operators. Then, the Gauss curvature K and mean curvature H of M are as follows:

$$K=\frac{\|(Q(u)\times\vec{T}(u))\wedge(Q(t)\times\vec{T}(t))\|}{\|\vec{T}(u)\wedge\vec{T}(t)\|} \tag{3}$$

and

$$H = \frac{\|(Q(u) \times \vec{T}(u)) \wedge \vec{T}(t) + \vec{T}(u) \wedge (Q(t) \times \vec{T}(t))\|}{2\|\vec{T}(u) \wedge \vec{T}(t)\|} \tag{4}$$

3. Main Results

In this section, the quaternionic expression of the ruled surfaces drawn by the unit Darboux vector and the striction curve on the surface are given. We obtain some interesting results, such as rotation matrices, Gauss, and mean curvatures of the surface.

Theorem 3.1. Let $\bar{\alpha}$ be a striction curve belonging to a ruled surface $\vec{\Lambda}(s,v) = \vec{\alpha}(s) + v\vec{C}(s)$. The quaternionic equations of the ruled surface and striction curve are given by

$$\vec{\Lambda}(s,v) = \vec{\alpha}(s) + vp(s) \times \vec{t}(s)$$

and

$$\bar{\alpha} = \vec{\alpha}(s) - v\frac{\langle C'(s), \vec{t}(s)\rangle}{\|C'(s)\|^2}p(s) \times \vec{t}(s)$$

Proof.
By taking into account the unit quaternion $p(s) = \frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}} - \frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}}\vec{n}(s)$ and the pure quaternion $\vec{t}(s)$, we obtain the ruled surface as follows:

$$\begin{aligned}\vec{\Lambda}(s,v) &= \vec{\alpha}(s) + vp(s) \times \vec{t}(s)\\ &= \vec{\alpha}(s) + v\left(\frac{\tau(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}} - \frac{\kappa(s)}{\sqrt{\kappa^2(s)+\tau(s)^2(s)}}\vec{n}(s)\right) \times \vec{t}(s)\\ &= \vec{\alpha}(s) + v\vec{C}(s).\end{aligned}$$

Similarly, we can obtain a striction curve using quaternion. □

Theorem 3.2. Let $\vec{\Lambda}(s,v) = \vec{\alpha}(s) + v\vec{C}(s)$ be a ruled surface. There exists a frame of the curve $\alpha(s)$ which is called Frenet frame and denoted by $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$. The relations among frames can be given by

$$\begin{bmatrix} \vec{T}(s) \\ \vec{Y}(s) \\ \vec{Z}(s) \end{bmatrix} = \begin{bmatrix} \frac{m}{\sqrt{m^2+l^2}} & 0 & \frac{l}{\sqrt{m^2+l^2}} \\ \frac{-l}{\sqrt{m^2+l^2}} & 0 & \frac{m}{\sqrt{m^2+l^2}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix} \tag{5}$$

and

$$\begin{bmatrix} \vec{T}(v) \\ \vec{Y}(v) \\ \vec{Z}(v) \end{bmatrix} = \begin{bmatrix} \frac{\tau(s)}{\|\vec{W}(s)\|} & 0 & \frac{\kappa(s)}{\|\vec{W}(s)\|} \\ -\frac{\kappa(s)}{\|\vec{W}(s)\|} & 0 & \frac{\tau(s)}{\|\vec{W}(s)\|} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix} \tag{6}$$

where $m = 1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'$ and $l = v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'$.

Proof.
The partial derivative is taken according to s and v for the ruled surface Λ, we obtain

$$\vec{\Lambda}_s = \vec{t}(s) + v\left(\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\vec{t}(s) + \left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\vec{b}(s)\right)$$

and

$$\vec{\Lambda}_v = \vec{C}(s)$$

Arriving at this equation, we reach the tangent vectors of the parameters curve as follows:

$$\vec{T}(s)=\frac{\vec{\Lambda}_s}{\|\vec{\Lambda}_s\|}=\frac{\left(1+v(\frac{\tau(s)}{\|\vec{W}(s)\|})'\right)\vec{t}(s)+v(\frac{\kappa(s)}{\|\vec{W}(s)\|})'\vec{b}(s)}{\sqrt{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{T}(v)=\frac{\vec{\Lambda}_v}{\|\vec{\Lambda}_v\|}=\vec{C}(s)$$

For $v<\frac{\kappa(s)\sqrt{\kappa^2(s)+\tau(s)^2(s)}}{\tau(s)\kappa'(s)-\kappa(s)\tau'(s)}$, the unit normal vector of the ruled surface Λ is given as

$$\vec{Z}=\frac{\vec{\Lambda}_s\wedge\vec{\Lambda}_v}{\|\vec{\Lambda}_s\wedge\vec{\Lambda}_v\|}=-\vec{n}(s)$$

$\vec{Y}(s)$ and $\vec{Y}(v)$ depending on Frenet frame at point $\alpha(s)$ can be obtained as

$$\vec{Y}(s)=\vec{Z}(s)\wedge\vec{T}(s)=\frac{-v(\frac{\kappa(s)}{\|\vec{W}(s)\|})'\vec{t}(s)+\left(1+v(\frac{\tau(s)}{\|\vec{W}(s)\|})'\right)\vec{b}(s)}{\sqrt{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{Y}(v)=\vec{Z}(v)\wedge\vec{T}(v)=-\frac{\kappa(s)}{\|\vec{W}(s)\|}\vec{t}(s)+\frac{\tau(s)}{\|\vec{W}(s)\|}\vec{b}(s)$$

If the Darboux frame denoted by $\{\vec{T}(s),\vec{Y}(s),\vec{Z}(s)\}$ is written in matrix form, this completes the proof of the theorem. □

Theorem 3.3. Let $\vec{\Lambda}(s,v)=\vec{\alpha}(s)+v\vec{C}(s)$ be a ruled surface. The singular point of the ruled surface is given by $P(s_0,v_0)=\left(s_0,-\frac{\|W(s_0)\|\kappa(s_0)}{\kappa(s_0)\tau'(s_0)+\tau(s_0)\kappa'(s_0)}\right)$.

Proof.
The unit normal vector field of the ruled surface Λ is defined by $Z=\frac{\Lambda_s\times\Lambda_v}{\|\Lambda_s\times\Lambda_v\|}$ at those points $(s_0,v_0)\in Z$ at which $\Lambda_s\times\Lambda_v$ does not vanish. Then, Λ is a regular surface if and only if the unit normal vector field Z is everywhere well defined. The points for which $\Lambda_s\times\Lambda_v$ vanishes can be called singular points. The equation

$$\|\Lambda_s\times\Lambda_v\|(s_0,v_0)=\frac{1}{\|W(s_0)\|^2}\sqrt{(\|W(s_0)\|\kappa(s_0)+v_0(\kappa(s_0)\tau'(s_0)-\tau(s_0)\kappa'(s_0))^2}=0$$

can be written to have singular points. Hence, we can write

$$v_0=-\frac{\|W(s_0)\|\kappa(s_0)}{\kappa(s_0)\tau'(s_0)+\tau(s_0)\kappa'(s_0)}$$

Then, the singular point of the ruled surface is $P(s_0,v_0)=\left(s_0,-\frac{\|W(s_0)\|\kappa(s_0)}{\kappa(s_0)\tau'(s_0)+\tau(s_0)\kappa'(s_0)}\right)$. □

Taking into account Equation 1, we can write proposition as follows:

Proposition 3.4. The normal and the geodesic curvatures of the ruled surface Λ can be given by

$$k_n(s)=\frac{-\kappa(s)\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)+\tau(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'}{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2},\quad k_n(v)=0$$

and

$$t_g(s)=\frac{\tau(s)\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)+\kappa(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'}{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2},\quad t_g(v)=0$$

Theorem 3.5. Let Λ be a ruled surface and $Q(s)$ and $Q(v)$ be quaternionic shape operators. The shape operators $S(\vec{T}(s))$ and $S(\vec{T}(v))$ are obtained by

$$S(\vec{T}(s))=\frac{-\kappa(s)\vec{t}(s)+\tau(s)\vec{b}(s)}{\sqrt{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$S(\vec{T}(v))=\vec{0}$$

Proof.
By using Proposition 3.4, quaternionic shape operators are given by

$$\begin{aligned}Q(s)&=k_n(s)+t_g(s)\vec{Z}(s)\\&=\frac{-\kappa(s)\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)+\tau(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'-\left[\tau(s)\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)+\kappa(s)v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right]\vec{n}}{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}\end{aligned}$$

and

$$Q(v)=k_n(v)+t_g(v)\vec{Z}(v)=\vec{0}$$

By considering Equation 2, the shape operators

$$\begin{aligned}S(\vec{T}(s))&=Q(s)\times\vec{T}(s)=k_n(s)\vec{T}(s)+t_g(s)\vec{Y}(s)\\&=\frac{-\kappa(s)\vec{t}(s)+\tau(s)\vec{b}(s)}{\sqrt{\left(1+v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2+\left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}\end{aligned}$$

and

$$S(\vec{T}(v))=Q(v)\times\vec{T}(v)=k_n(v)\vec{T}(v)+t_g(v)\vec{Y}(v)=\vec{0}$$

are expressed. □

Corollary 3.6. The operator $Q(s)$ rotates the tangent vector $\vec{T}(s)$ in the tangent plane of the ruled surface and around the normal vector $\vec{Z}(s)$ of the surface. The rotation matrix which provides that rotation is

$$R=\begin{bmatrix}1+\sin^2\varphi(n_1^2-n_2^2-n_3^2-1) & \sin 2\varphi n_3+2\sin^2\varphi n_1n_2 & -\sin 2\varphi n_2+2\sin^2\varphi n_1n_3\\ -\sin 2\varphi n_3+2\sin^2\varphi n_1n_2 & 1+\sin^2\varphi(n_2^2-n_1^2-n_3^2-1) & 2\sin^2\varphi n_2n_3+\sin 2\varphi n_1\\ 2\sin^2\varphi n_1n_3+\sin 2\varphi n_2 & -\sin 2\varphi n_1+2\sin^2\varphi n_2n_3 & 1+\sin^2\varphi(n_3^2-n_2^2-n_1^2-1)\end{bmatrix}$$

where $Z(s)=-n(s)=(-n_1,-n_2,-n_3)$ and the cosine and sine of the angle of between $\vec{T}(s)$ and $Q(s)\times\vec{T}(s)$ are

$$\cos 2\varphi(s)=\frac{-m\kappa(s)+l\tau(s)}{\sqrt{(m^2+l^2)(\kappa^2(s)+\tau(s)^2(s))}}\quad\text{and}\quad\sin 2\varphi(s)=\frac{l\kappa(s)+m\tau(s)}{\sqrt{(m^2+l^2)(\kappa^2(s)+\tau(s)^2(s))}}$$

Theorem 3.7. The ruled surface Λ is flat. The mean curvature H of this surface is obtained by

$$H = \frac{\kappa^2(s) + \tau(s)^2(s)}{2|-m\kappa(s) + l\tau(s)|}$$

PROOF.
The Gauss curvature is a measure of the intrinsic curvature of a surface, and it is defined as quaternionic as follows:

$$K = \frac{\|(Q(s) \times \vec{t}(s)) \wedge (Q(v) \times \vec{T}(v))\|}{\|\vec{t}(s) \wedge \vec{T}(v)\|}$$

If we substitute the quaternionic shape operators and tangent vector of parameter curves, we obtain

$$\begin{aligned} K &= \frac{1}{\sqrt{\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}} \frac{\|(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)) \wedge \vec{0})\|}{\|\vec{t}(s) \wedge \vec{T}(v)\|} \\ &= 0 \end{aligned}$$

This means the surface is flat. The mean curvature is a measure of the extrinsic curvature of the ruled surface, and the curvature is calculated as quaternionic as follows:

$$H = \frac{\|(Q(s) \times \vec{t}(s)) \wedge \vec{T}(v) + \vec{t}(s) \wedge (Q(v) \times \vec{T}(v))\|}{2\|\vec{t}(s) \wedge \vec{T}(v)\|}$$

If we substitute the quaternionic shape operators $Q(v)$, we get

$$H = \frac{\|(Q(s) \times \vec{t}(s)) \wedge \vec{T}(v)\|}{2\|\vec{t}(s) \wedge \vec{T}(v)\|}$$

By taking into consideration Equations 5 and 6 and Theorem 3.5,

$$H = \frac{\sqrt{\kappa^2(s) + \tau(s)^2(s)}}{2|-m\kappa(s) + l\tau(s)|}$$

is arrived. □

Considering the above theorem, we reach the following corollary.

Corollary 3.8. If the base curve of the ruled surface Λ drawn by the unit Darboux vector is a line and planar, then the surface is minimal.

In differential geometry, the Darboux vector is a vector-valued function that measures the rate of change of the tangent vector of a curve as it moves along the curve. The ruled surface generated by the unit Darboux vector can be expressed as a function of the $\xi(s)$, which is the angle between Darboux and binormal vectors. This means some surface characterizations can be studied and analyzed as a function of $\xi(s)$.

Theorem 3.9. Let $\vec{\Lambda}(s,v) = \vec{\alpha}(s) + v\vec{C}(s)$ be a ruled surface. There exists a frame of the curve $\alpha(s)$ called Frenet frame and denoted by $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$. The relations among frames in terms of $\xi(s)$ are given by

$$\begin{bmatrix} \vec{T}(s) \\ \vec{Y}(s) \\ \vec{Z}(s) \end{bmatrix} = \begin{bmatrix} \frac{1+v\xi'(s)\cos\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} & 0 & \frac{-v\xi'(s)\sin\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} \\ \frac{v\xi'(s)\sin\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} & 0 & \frac{1+v\xi'(s)\cos\xi(s)}{\sqrt{1+v\xi'(s)(2\cos\xi(s)+v\xi'(s))}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix}$$

and

$$\begin{bmatrix} \vec{T}(v) \\ \vec{Y}(v) \\ \vec{Z}(v) \end{bmatrix} = \begin{bmatrix} \sin\xi(s) & 0 & \cos\xi(s) \\ -\cos\xi(s) & 0 & \sin\xi(s) \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix}$$

PROOF.
If the partial derivative is taken according to s and v using the angle $\xi(s)$, we have

$$\vec{\Lambda}_s = \vec{t}(s) + v\left(\xi'(s)\cos\xi(s)\vec{t}(s) - \xi'(s)\sin\xi(s)\vec{b}(s)\right)$$

and

$$\vec{\Lambda}_v = \vec{C}(s)$$

Arriving at this equation, we reach to

$$\vec{T}(s) = \frac{\vec{\Lambda}_s}{\|\vec{\Lambda}_s\|} = \frac{\left(1 + v\xi'(s)\cos\xi(s)\right)\vec{t}(s) - v(\xi'(s)\sin\xi(s)\vec{b}(s))}{\sqrt{(1 + v\xi'(s)\cos\xi(s))^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{T}(v) = \frac{\vec{\Lambda}_v}{\|\vec{\Lambda}_v\|} = \sin\xi(s)\vec{t}(s) + \cos\xi(s)\vec{b}(s)$$

For $v < \frac{\cos\xi(s)}{\xi'(s)}$, the unit normal vector of the ruled surface Λ is given as

$$\vec{Z} = \frac{\vec{\Lambda}_s \wedge \vec{\Lambda}_v}{\|\vec{\Lambda}_s \wedge \vec{\Lambda}_v\|} = -\vec{n}(s)$$

$\vec{Y}(s)$ and $\vec{Y}(v)$ depending on Frenet frame at point $\alpha(s)$ can be obtained

$$\vec{Y}(s) = \vec{Z}(s) \wedge \vec{T}(s) = \frac{v\xi'(s)\sin\xi(s)\vec{t}(s) + \left(1 + v\xi'(s)\cos\xi(s)\right)\vec{b}(s)}{\sqrt{\left(1 + v\left(\frac{\tau(s)}{\|\vec{W}(s)\|}\right)'\right)^2 + \left(v\left(\frac{\kappa(s)}{\|\vec{W}(s)\|}\right)'\right)^2}}$$

and

$$\vec{Y}(v) = \vec{Z}(v) \wedge \vec{T}(v) = -\sin\xi(s)\vec{t}(s) + \cos\xi(s)\vec{b}(s)$$

This completes the proof of the theorem. □

Theorem 3.10. Let Λ be a ruled surface and $\xi(s)$ be the angle between the Darboux vector and the binormal vector. The singular point belonging to the ruled surface is given by $P(s_0, v_0) = \left(s_0, \frac{\cos\xi(s_0)}{\xi'(s_0)}\right)$.

PROOF.
To determine the normal vector for a singular point, the denominator of the normal vector must be zero. As a result, when performing the necessary operations, the singular point becomes $P(s_0, v_0) = \left(s_0, \frac{\cos\xi(s_0)}{\xi'(s_0)}\right)$. □

Taking into account Equation 1, we have the following result.

Corollary 3.11. The normal and the geodesic curvatures of the ruled surface in terms of angle $\xi(s)$ can be expressed as follows:

$$k_n(s) = \frac{-(\cos\xi(s) + v\xi'(s))\|\vec{W}(s)\|}{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))}, \ k_n(v) = 0$$

and

$$t_g(s) = \frac{\sin\xi(s)\|\vec{W}(s)\|}{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))}, \ t_g(v) = 0$$

Using the angle $\xi(s)$, quaternionic shape operators are given by

$$Q(s) = \frac{-(\cos\xi(s) + v\xi'(s))\|\vec{W}(s)\|}{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))} + \frac{\sin\xi(s)\|\vec{W}(s)\|}{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))}\vec{n}(s)$$

and

$$Q(v) = k_n(v) + t_g(v)\vec{Z}(v) = 0$$

Theorem 3.12. Let Λ be a ruled surface and $\xi(s)$ be the angle between the Darboux vector and the binormal vector. Using the quaternionic operators, the shape operators $S(\vec{T}(s))$ and $S(\vec{T}(v))$ are obtained by

$$S(\vec{T}(s)) = Q(s) \times \vec{T}(s) = \frac{-(\cos\xi(s)\|\vec{W}(s)\| + v\xi'(s)\|\vec{W}(s)\|(\cos 2\xi(s) + v\xi'(s)\cos\xi(s) + 1))\vec{t}(s)}{(1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s)))^{3/2}}$$

$$+ \frac{(\sin\xi(s)\|\vec{W}(s)\| + v\xi'(s)\|\vec{W}(s)\|(\sin 2\xi(s) + v\xi'(s)\sin\xi(s)))\vec{b}(s)}{(1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s)))^{3/2}}$$

and

$$S(\vec{T}(v)) = Q(v) \times \vec{T}(v) = \vec{0}$$

Proof.
The proof of the theorem is similar to the proof of Theorem 3.3. □

Corollary 3.13. According to $\xi(s)$, the angle between Darboux and binormal vectors, the cosine and sine of the angle of between $\vec{T}(s)$ and $Q(s) \times \vec{T}(s)$ are as follows:

$$\cos 2\varphi(s) = \frac{-\cos\xi(s) - v\xi'(s)}{\sqrt{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))}}$$

and

$$\sin 2\varphi(s) = \frac{\sin\xi(s)}{\sqrt{1 + v\xi'(s)(2\cos\xi(s) + v\xi'(s))}}$$

Theorem 3.14. The ruled surface Λ is flat. The mean curvature H of this surface is obtained by

$$H = \frac{\|\vec{W}(s)\|}{2|1 + v\xi'(s)|}$$

Proof.
The proof of the theorem is similar to the proof of Theorem 3.7. □

Example 3.15. The various position of the generating unit Darboux vector is obtained from the ruled surface. Such a surface has a parameterization,

$$\vec{\Lambda}(s, v) = \left(\sqrt{1 + s^2}, \ln(s + \sqrt{1 + s^2}), s + v\right)$$

If we choose the quaternion as $p(s) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2\sqrt{1+s^2}}(1, s, 0)$, then we can write the surface with quaternions as follows:

$$\vec{\Lambda}(s, v) = \left(\sqrt{1 + s^2}, \ln(s + \sqrt{1 + s^2}), s\right) + \frac{1}{\sqrt{2 + 2s^2}} v p(s) \times (s, 1, \sqrt{1 + s^2})$$

This ruled surface is provided in Figure 1.

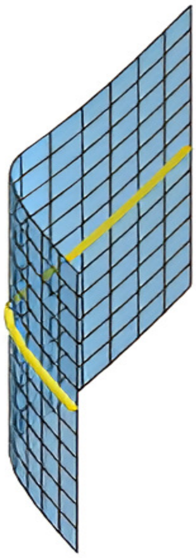

Figure 1. The ruled surface drawn by unit Darboux vector and the base curve of the surface

The quaternionic shape operator, denoted as $Q(s)$, is a mathematical construct that can be used to analyze the shape of the surface. It is calculated as

$$Q(s) = k_n(s) + t_g(s)\vec{Z} = \frac{1}{2\sqrt{2}(1+s^2)} - \frac{1}{2\sqrt{2}(1+s^2)^{3/2}}(1, -s, 0)$$

The shape operator, denoted as $S(\vec{T}(s))$, is obtained by taking the quaternionic product of the quaternionic shape operator and the tangent vector $\vec{T}(s)$. The tangent vector is a vector that is tangent to the surface at a particular point and points in the direction of the surface at that point. The shape operator is then calculated as

$$S(\vec{T}(s)) = Q(s) \times \vec{T}(s) = \frac{1}{2\sqrt{2}(1+s^2)^{3/2}}(s, 1, 0)$$

The shape operator for the parameter v, denoted as $S(\vec{T}(v))$, is equal to $\vec{0}$. Hence, by Equations 3 and 4, it is easy to express Gauss and mean curvatures as

$$K = 0 \quad \text{and} \quad H = \frac{\sqrt{2}}{4(1+s^2)^{5/2}}$$

This means that the surface is developable and is not a minimal surface. The operator $Q(s)$ rotates the tangent vector $\vec{T}(s)$ in the tangent plane of the ruled surface and around the normal vector $\vec{Z}(s)$ of the ruled surface. In this case, the rotation matrix for the unit quaternion $q = \cos\varphi + \sin\varphi\vec{Z}(s)$ is given as

$$R_1 = \frac{1}{\sqrt{2}(1+s^2)} \begin{bmatrix} \sqrt{2} - s^2(\sqrt{2}-1) & -s(\sqrt{2}-1) & s\sqrt{1+s^2} \\ -s(\sqrt{2}-1) & 1 & 1 \\ -s\sqrt{1+s^2} & -1 & \frac{1}{1+s^2} \end{bmatrix}$$

where $\vec{Z}(s) = \left(\frac{1}{\sqrt{1+s^2}}, -\frac{s}{\sqrt{1+s^2}}, 0\right)$.

4. Conclusion

Quaternions are an essential topic in animation, robot kinematics, and rotational motion in 3-dimensional space. Ruled surfaces have a vital role in technology (especially robot end-effectors). Moreover, it is known that Gauss and mean curvatures and the shape operator are the invariants in the surface of theory. These invariants are quaternionically calculated for the unit Darboux ruled surface.

In this study, we combine some points on two critical subjects. Besides, we provide some theorems related to the invariants and then show how to find a rotation matrix. Based on the quaternionic shape operator and the rotation matrix, we derive different situations of the invariants and rotations: one from the curvatures of the base curve and the other one by the angle $\xi(s)$ between $\vec{W}(s)$ and $\vec{b}(s)$. Thus, we observe what happens when we express the relation form of the frame equations using ξ instead of the curvatures. Furthermore, we obtain the shape operators by revolving tangent vectors of parameter curves around the surface's normal vector through twice the angle of φ and then get rotation matrices.

In further research, it would be valuable to replicate similar approaches in different spaces, such as Galileo or Lorentz spaces. These alternative spaces could potentially yield different results and provide a deeper understanding of the results herein being studied.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

References

[1] A. J. Hanson, Visualing Quaternions, *The Morgan Kaufmann Series in Interactive 3D Technology*, Elsevier, San Francisco, 2006.

[2] J. B. Kuipers, Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality, Princeton University Press, New Jersey, 2002.

[3] K. Shoemake, *Animating Rotation with Quaternion Curves*, Siggraph Computer Graphics 19 (1985) 245–254.

[4] K. Bharathi, M. Nagaraj, *Quaternion Valued Function of a Real Variable Serret-Frenet Formulae*, Indian Journal of Pure and Applied Mathematics 18 (1987) 507–511.

[5] O. Keçilioğlu, K. İlarslan, *Quaternionic Bertrand Curves in Euclidean 4-Space*, Bulletin of Mathematical Analysis and Applications 5 (3) (2013) 27–38.

[6] M. Babaarslan, Y. Yaylı, *A New Approach to Constant Slope Surfaces with Quaternions*, ISRN Geometry 2012 (2012) Article ID 126358 8 pages.

[7] S. Şenyurt, A. Çalışkan, *The Quaternionic Expression of Ruled Surfaces*, Filomat 32 (16) (2018) 403–411.

[8] A. Çalışkan, S. Şenyurt, *The Dual Spatial Quaternionic Expression of Ruled Surfaces*, Thermal Science 23 (1) (2019) 403–411.

[9] A. Çalışkan, *Quaternionic and Dual Quaternionic Darboux Ruled Surfaces*, Turkish Journal of Mathematics and Computer Science 13 (1) (2021) 106–114.

[10] Y. Li, Z. Wang, T. Zhao, *Geometric Algebra of Singular Ruled Surfaces*, Advances in Applied Clifford Algebras 31 (2) (2021) 1–19.

[11] S. Aslan, Y. Yaylı, *Quaternionic Shape Operator*, Advances in Applied Clifford Algebras 27 (2017) 2921–2931.

[12] M. Babaarslan, Y. Yaylı, *Split Quaternions and Spacelike Constant Slope Surfaces in Minkowski 3-Space*, International Journal of Geometry 2 (1) (2013) 23–33.

[13] M. Babaarslan, Y. Yaylı, *Split Quaternions and Timelike Constant Slope Surfaces in Minkowski 3-Space*, International Journal of Geometry 8 (1) (2019) 57–71.

[14] M. P. Do Carmo, Differential Geometry of Curves and Surfaces, 2nd Edition, Dover Publications, New York, 2016.

[15] B. S. Ryuh, *Robot Trajectory Planning Using the Curvature Theory of Ruled Surfaces*, Doctoral Dissertation Purdue University (1989) West Lafayette.

[16] F. Güler, E. Kasap, *A Path Planning Method for Robot End Effector Motion Using the Curvature Theory of the Ruled Surfaces*, International Journal of Geometric Methods in Modern Physics 15 (3) (2018) 1850048 15 pages.

[17] S. Şenyurt, K. Eren, *On Ruled Surfaces with a Sannia Frame in Euclidean 3-space*, Kyungpook Mathematical Journal 62 (3) (2022) 509–531.

[18] A. Kelleci, K. Eren, *On Evolution of Some Associated Type Ruled Surfaces*, Mathematical Sciences and Applications E-Notes 8 (2) (2020) 178–186.

[19] K. Eren, H. H. Kosal, *Evolution of Space Curves and the Special Ruled Surfaces with Modified Orthogonal Frame* AIMS Mathematics 5 (3) (2020) 2027–2040.

[20] M. Bilici, *On the Invariants of Ruled Surfaces Generated by the Dual Involute Frenet Trihedron*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 66 (2) (2017) 62–70.

[21] S. Palavar, M. Bilici, *Dual Ruled Surface Constructed by the Pole Curve of the Involute Curve*, International Journal of Open Problems in Computer Science and Mathematics 15 (1) (2022) 39–53.

[22] S. Palavar, M. Bilici, *New-Type Tangent Indicatrix of Involute and Ruled Surface According to Blaschke Frame in Dual Space*, Maejo International Journal of Science And Technology 16 (3) (2022) 199–207.

[23] E. Bayram, M. Bilici, *Surface Family with a Common Involute Asymptotic Curve*, International Journal of Geometric Methods in Modern Physics 13 (5) (2016) 1650062 9 pages.

[24] M. Bilici, E. Bayram *Surface Construction with a Common Involute Line of Curvature*, International Journal of Open Problems in Computer Science and Mathematics 14 (2) (2021) 20–31.

[25] B. O'Neill, Elementary Differential Geometry, 2nd Edition, Elsevier, London, 2006.

[26] J. Vince, Quaternions for Computer Graphics, Springer-Verlag, London, 2011.