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On Weakly 1-Absorbing Primary Ideals of Commutative Semirings

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Abstract
Let R be a commutative semiring with $1 \neq 0$. In this paper, we study the concept of weakly 1-absorbing primary ideal which is a generalization of 1-absorbing ideal over commutative semirings . A proper ideal I of a semiring R is called a weakly 1-absorbing primary ideal if whenever nonunit elements $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$. A number of results concerning weakly 1-absorbing primary ideals and examples of weakly 1-absorbing primary ideals are given. An ideal is called a subtractive ideal I of a semiring R is an ideal such that if $x, x+y \in I$, then $y \in I$. Subtractive ideals or k-ideals are helpful in proving in many results related to ideal theory over semirings.

Keywords:
1-absorbing primary ideal, 2-absorbing primary ideal, Prime ideal, Weakly 1-absorbing primary ideal, Weakly 2-absorbing primary ideal, Weakly prime ideal, Weakly primary, Weakly primary ideal
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1. Introduction

The algebraic structure of semirings, that are considered as a generalization of rings, plays an important role in different branches of mathematics, especially in applied sciences and computer engineering. For general references on semiring theory one may refer to [1],[4],[13] and [16].
The first formal definition of semirings was introduced by H.S Vandiver in 1934 [20] ”Note on a simple type of algebra in which cancelation law of addition does not holds”.
In this paper we need a special kind of ideals that was defined by Henriksen [14] in 1958 which is called k-ideal or subtractive ideals. A subtractive ideal I of a semiring R is an ideal such that if $x, x+y \in I$, then $y \in I$.

Since prime and primary ideals have key roles in commutative semiring theory, many authors have studied generalizations of prime and primary ideals. One of the generalization of that concept is 2-absorbing ideals.
In 2012, Darani [12] introduced the connotation of a 2-absorbing ideal of a commutative semiring. A proper ideal I of a semiring R is said to be a 2-absorbing primary ideal if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$, or $bc \in I$, or $ac \in I$.
In [8], the concept of weakly 1-absorbing primary ideal which is a generalization of 1-absorbing ideal was introduced. A proper ideal I of a ring R is called a weakly 1-absorbing primary ideal if whenever nonunit elements $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$ and studied n number of results concerning weakly 1-absorbing primary ideals and examples of weakly

1-absorbing primary ideals .
We assume throughout this paper that all semirings are commutative with unity $1 \neq 0$. We start by recalling some background material. By a proper ideal I of R, we mean an ideal I of R with $I \neq R$. Let I be a proper ideal of R. Before we state some results, let us introduce some notation and terminology. By $\sqrt{I}$, we mean the radical of R, that is, $\{a \in R \mid an \in I\}$ for some positive integer $n\}$. In particular, $\sqrt{0}$ denotes the set of all nilpotent elements of R. We define $Z_I(R) = \{r \in R \mid rs \in I$ for some $s \in R \setminus I\}$. A semiring R is called a reduced semiring if it has no non-zero nilpotent elements; i.e., $\sqrt{0} = 0$. For two ideals I and J of R, the residual division of I and J is defined to be the ideal $(I : J) = \{a \in R \mid aJ \subseteq I\}$. Let R be a commutative semiring with identity and M a unitary R-semimodule. Then $R(+)M = R \bigoplus M$(direct sum) with coordinate-wise addition and multiplication $(a,m)(b,n) = (ab, an+bm)$ is a commutative semiring with identity called the idealization of M. A semiring R is called a quasilocal semiring if R has exactly one maximal ideal. As usual we denote Z and Z_n by the semiring of integers and the semiring of integers modulo n.
In this paper, we introduce the concept of (weakly) 1-absorbing ideal of a semiring R. A proper ideal I of a semiring R is called a weakly 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$, and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$. A proper ideal I of a semiring R is called 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$, and $abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$. It is clear that a 1-absorbing primary ideal of R is a weakly 1-absorbing primary ideal of R. However, since 0 is always weakly 1-absorbing primary, a weakly 1-absorbing primary ideal of R needs not be a 1-absorbing primary ideal of R. Among many results, we show (Theorem 2.5) that if a proper ideal I of R is a weakly 1-absorbing ideal of R such that $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is 1-absorbing primary ideal of R. We show (Theorem 2.6) that if R is a reduced semiring, and I is a weakly 1-absorbing primary ideal of R, then $\sqrt{I}$ is a prime ideal of R. If I is a proper nonzero ideal of a von-Neumann regular semiring R, then we show (Theorem 2.7) that I is a weakly 1-absorbing primary ideal of R if and only if I is a 1-absorbing primary ideal of R if and only if I is a primary ideal of R. We show (Theorem 2.8) that if R is a nonquasilocal semiring, and I be a proper ideal of R such that $ann(i) = \{r \in R \mid ri = 0\}$ is not a maximal ideal of R for every element $i \in I$, then I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R. If I is a proper ideal of a reduced divided semiring R, then we show (Theorem 2.11) that I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R. If I is a weakly 1-absorbing primary of a semiring R that is not a 1-absorbing primary ideal of R, then we give (Theorem 3.4) sufficient conditions so that $I^3 = 0$ (i.e., $I \subseteq \sqrt{I}$). In Theorem 3.2, we obtain some equivalent conditions for weakly 1-absorbing primary ideals of u-semirings. In (Theorem4.1), a characterization of weakly 1-absorbing primary ideals in $R = R_1 \times R_2$, where R_1 and R_2 are commutative semirings with identity that are not semifields is given. If $R_1, R_2, ..., R_n$ are commutative semirings with identity for some $2 \leq n < \infty$, and let $R = R_1 \times ..... \times R_n$, then it is shown in (Theorem 4.2) that every proper ideal of R is a weakly 1-absorbing primary ideal of R if and only if $n = 2$ and R_1, R_2 are semifields. For a weakly 1-absorbing primary ideal of a semiring R, we show (Theorem 4.8) that $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$ for every multiplicatively closed subset S of R that is disjoint from I, and we show that the converse holds if $S \cap Z(R) = \phi$ and $S \cap Z_I(R) = \phi$.

2. Properties of Weakly 1 -absorbing Primary Ideals

In this section, we will define some basic properties of weakly 1-absorbing primary ideals in a commutative semi-ring R.

Definition 2.1. *Let R be a commutative semiring, and I a proper ideal of R. We call I a weakly 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$ and $0 \neq abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$.*

Definition 2.2. *Let R be a commutative semiring, and I a proper ideal of R. We call I a 1-absorbing primary ideal of R if whenever nonunit elements $a,b,c \in R$ and $abc \in I$, then $ab \in I$, or $c \in \sqrt{I}$.*

It is clear that every 1-absorbing primary ideal of a semiring R is a weakly 1-absorbing primary ideal of R.
The following example shows that the converse is not true.

Example 2.3.
1. *$I = \{0\}$ is a weakly 1-absorbing primary ideal of $R = Z_6$ that is not a 1-absorbing primary of R. Indeed, $2.2.3 \in I$, but neither $2.2 \in I$ nor $3 \in \sqrt{I}$.*
2. *Let $J = \{0,6\}$ as an ideal of Z_{12}, and let $R = Z_{12}(+)J$. Then an ideal $I = \{(0,0),(0,6)\}$ is a weakly 1-absorbing primary ideal of R. Observe that $abc \in I$ for some $a,b,c \in R \mid I$ if and only if $abc = (0,0)$. However, it is not a 1-absorbing primary ideal of R. Indeed; $(2,0)(2,0)(3,0) \in I$, but neither $(2,0)(2,0) \in I$ nor $(3,0) \in \sqrt{I}$.*

We begin with the following trivial result:

Theorem 2.4. *Let be a proper ideal of a commutative semiring R. Then the following statements hold.*

1. *If I is a weakly prime ideal, then I is a weakly 1-absorbing primary ideal.*

2. *If I is a weakly primary ideal, then I is a weakly 1-absorbing primary ideal.*
3. *If I is a 1-absorbing primary ideal, then I is a weakly 1-absorbing primary ideal.*
4. *If I is a weakly 1-absorbing primary ideal, then I is a weakly 2-absorbing primary ideal.*
5. *If R/I is an semi-integral domain, then I is a weakly 1-absorbing primary ideal if and only if I is a 1-absorbing primary ideal of R.*
6. *Let R be a quasilocal semiring with maximal ideal $\sqrt{0}$. Then every proper ideal of R is a weakly 1-absorbing primary ideal of R.*

Theorem 2.5. *Let R be a semiring and I be a weakly 1-absorbing primary ideal of R. If $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is a 1-absorbing ideal primary of R.*
In particular, If I a weakly 1-absorbing primary ideal of R that is not a 1-absorbing ideal primary of R, then is not a maximal ideal of R.

Proof. Suppose that $\sqrt{I}$ is a maximal ideal of R. Then I is a semiprimary ideal of R. by [21] since I. Now, assume nonunit elements $a,b,c \in R$ and $abc \in I$. Assume ab not belong I . Since I is primary ideal, we have for some positive integer m, we have $c \in \sqrt{I}$. Hence, I is 1-absorbing primary ideal. □

Theorem 2.6. *Let R be a reduced semiring. If I is a nonzero weakly 1-absorbing primary ideal of R, then $\sqrt{I}$ is a prime ideal of R. In particular, if $\sqrt{I}i$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is a 1-absorbing primary ideal of R.*

Proof. Proof: Suppose that $0 \neq ab \in \sqrt{I}f$, for some $a,b \in R$. We may assume that a,b are nonunit. Then there exists an even positive integer $n = 2m(m \geq 1)$ such that $(ab)^n \in I$. Since $\sqrt{0} = \{0\}$, we have $(ab)^n \neq 0$. Hence, $0 \neq a^m a^m b^n \in I$. Thus, $a^m a^m = a^n \in I$ or $b^n \in \sqrt{I}$, and therefore $\sqrt{I}$ is a weakly prime ideal of R. Since R is reduced and $I \neq \{0\}$, we conclude that $\sqrt{I}$ is a prime ideal of R by [2] . The proof of the "in particular" statement : by Theorem 2, $\sqrt{I}$ is a maximal ideal of R, then I is a primary ideal of R, and hence I is a 1-absorbing ideal primary of R. □

Recall that a commutative semiring R is called a von-Neumann regular semiring if and only if for every $x \in R$, there is a $Y \in y$ such that $x^2 y = x$. It is known that a commutative semiring R is a von-Neumann regular semiring if and only if for each $x \in R$, there is an idempotent $e \in R$ and a unit $u \in R$ such that $x = eu$. We have the following result.

Theorem 2.7. *Let R be a von-Neumann regular semiring and I be a nonzero ideal of R. Then the following statements are equivalent.*

1. *I is a weakly 1-absorbing primary ideal of R.*
2. *I is a primary ideal of R.*
3. *I is a 1-absorbing ideal primary of R.*

Proof. $(1) \Rightarrow (2)$. R is a von-Neumann regular semiring, we know that R is reduced. Hence $\sqrt{I}$ is a prime ideal of R by Theorem 2.6. Since every prime ideal of a von-Neumann regular semiring is maximal, we conclude that $\sqrt{I}$ is a maximal ideal of R. Hence I is a primary ideal of R by Theorem 2.5.
$(2) \Rightarrow (3)$. Let nonunit elements $a,b,c \in R$, and $abc \in I$. Assume ab not belong I. Since I is a primary ideal, we have $c^m \in I$ for some positive integer m, so $c \in \sqrt{I}$. Thus, I is a 1-absorbing primary ideal.
$(3) \Rightarrow (1)$. Let nonunit elements $a,b,c \in R$, and $0 \neq abc \in I$. Since I is a 1-absorbing primary ideal, we have $ab \in I$, or $c \in \sqrt{I}$. Now, if a,b and $c \neq 0$, then $0 \neq abc \in I$. As a result I is a weakly 1-absorbing primary ideal. □

Theorem 2.8. *Let R be a non-quasilocal semiring and I be a k-ideal of R such that $ann(i) = \{r \in R \mid ri = 0\}$ is not a maximal ideal of R for every element $i \in I$. Then I is a weakly 1-absorbing primary ideal of R if and only if I is a weakly primary ideal of R.*

Proof. If I is a weakly primary ideal of R, then I is a weakly 1-absorbing primary ideal of R by Theorem 2.4. Now, suppose that I is a weakly 1-absorbing primary k-ideal of R and suppose that $0 \neq ab \in I$ for some elements $a,b \in R$. We show that $a \in I$ or $b \in \sqrt{I}$. We may assume that a,b are nonunit elements of R. Let $ann(ab) = \{c \in R \mid cab = 0\}$. Since $ab \neq 0$, $ann(ab)$ is a proper ideal of R. Let L be a maximal ideal of R such that $ann(ab) \subseteq L$. Since R is a non-quasilocal semiring, there is a maximal ideal M of R such that $M \neq L$. Let $m \in M \setminus L$. Hence m not belong to $ann(ab)$, and $0 \neq mab \in I$. Since I is a weakly 1-absorbing primary ideal of R, we have $ma \in I$ or $b \in \sqrt{I}$. If $b \in \sqrt{I}$, then we are done. Hence assume that b not belong to $\sqrt{I}$.

Hence $ma \in I$. Since m not belong to L and L is a maximal ideal of R, we conclude that m not belong to $J(R)$. Hence there exists an $r \in R$ such that $1+rm$ is a nonunit element of R. Suppose that $1+rm$ not belong to $ann(ab)$. Hence $0 \neq (1+rm)ab \in I$. Since I is a weakly 1-absorbing primary k-ideal of R and b not belong to $\sqrt{I}$, we conclude that $(1+rm)a = a+rma \in I$. Since $rma \in I$, we have $a \in I$ and we are done. Suppose that $1+rm \in ann(ab)$. Since $ann(ab)$ is not a maximal ideal of R and $ann(ab) \subseteq L$, there is a $w \in L \setminus ann(ab)$. Hence $0 \neq wab \in I$. Since I is a weakly 1-absorbing primary k-ideal of R and b not belong to $\sqrt{I}$, we conclude that $wa \in I$. Since $1+rm \in ann(ab) \subseteq L$ and $w \in L \setminus ann(ab)$, we have $1+rm+w$ is a nonzero nonunit element of L. Hence $0 \neq (1+rm+w)ab \in I$. Since I is a weakly 1-absorbing primary k-ideal of R and b not belong $\sqrt{I}$, we conclude that $(1+rm+w)a = a+rma+wa \in I$. Since $rma, wa \in I$, we conclude that $a \in I$. □

In light of the proof of Theorem 2.8, we have the following result.

Theorem 2.9. *Let I be a weakly 1-absorbing primary k-ideal of R such that for every nonzero element $i \in I$, there exists a nonunit $w \in R$ such that $wi \neq 0$, and $w+u$ is a nonunit element of R for some unit $u \in R$. Then I is a weakly primary k-ideal of R.*

Proof. Suppose that $0 \neq ab \in I$ and b not belong to $\sqrt{I}$ for some $a, b \in R$. We may assume that a, b are nonunit elements of R. Hence there is a nonunit $w \in R$ such that $wab \neq 0$ and $w+u$ is a nonunit element of R for some unit $u \in R$. Since $0 \neq wab \in I$ and b not belong to $\sqrt{I}$ and I is a weakly 1-absorbing primary k-ideal of R, we conclude that $wa \in I$.

Since $(w+u)ab \in I$ and I is a weakly 1-absorbing primary k-ideal of R and b not belong $\sqrt{I}$, we conclude that $(w+u)a = wa+ua \in I$. Since $wa \in I$ and $wa+ua \in I$, we conclude that $ua \in I$. Since u is a unit, we have $a \in I$. □

Corollary 2.10. *Let R be a semiring and $A = R[x]$. Suppose that I is a weakly 1-absorbing primary k-ideal of A. Then I is a weakly primary k-ideal of A.*

Proof. Since $xi \neq 0$ for every nonzero $i \in I$ and $x+1$ is a nonunit element of A, we are done by Theorem 2.9. □

Recall that a semiring R is called divided if for every prime ideal P of R and for every $x \in R \setminus P$, we have $x \mid p$ for every $p \in P$. We have the following result.

Theorem 2.11. *Let R be a reduced divided semiring and I be a proper ideal of R. Then the following statements are equivalent:*

1. *I is a weakly 1-absorbing primary ideal of R.*
2. *I is a weakly primary ideal of R.*

Proof. $(1) \Rightarrow (2)$. Suppose that $0 \neq ab \in I$ for some $a, b \in R$ and b not belong to $\sqrt{I}$. We may assume that a, b are nonunit elements of R. Since $\sqrt{I}$ is a prime ideal of R by Theorem 2.6, we conclude that $a \in \sqrt{I}$. Since R is divided, we conclude that $b \mid a$. Thus $a = bc$ for some $c \in R$. Observe that c is a nonunit element of R as b not belong to $\sqrt{I}$ and $a \in \sqrt{I}$. Since $0 \neq ab = bcb \in I$ and I is weakly 1-absorbing primary, and b not belong to $\sqrt{I}$, we conclude that $bc = a \in I$. Thus I is a weakly primary ideal of R.

$(2) \Rightarrow (1)$. It is clear by Theorem 2.4. □

Recall that a semiring R is called a chained semiring if for every $x, y \in R$, we have $x \mid y$ or $y \mid x$. Every chained semiring is divided. So, if R is a reduced chained semiring, then a proper ideal I of R is a weakly 1-absorbing primary ideal if and only if it is a weakly primary ideal of R.

Theorem 2.12. *Let R be a semiDedekind domain and I be a nonzero proper ideal of R. Then I is a weakly 1-absorbing primary ideal of R if and only if $\sqrt{I}$ is a prime ideal of R.*

Proof. $(\rightarrow)$. Suppose that I is a weakly 1-absorbing primary ideal of R. Then $\sqrt{I}$ is a prime ideal of R by Theorem 2.6.

$(\leftarrow)$. Suppose $\sqrt{I}$ is a prime ideal of R. Since R is a semiDedekind domain, it is well known that every nonzero prime ideal of R is a maximal ideal of R. Thus $\sqrt{I}$ is a maximal ideal of R. Hence I is a primary ideal of R, and thus I is 1-absorbing primary ideal of R. □

3. Characterizations of Weakly 1-absorbing Primary Ideals in u-semirings

In this section, we will study some characterizations of weakly 1-absorbing primary ideals in u-semirings

Definition 3.1. *If an ideal of R contained in a finite union of ideals must be contained in one of those ideals, then R is said to be a u-semiring.*

Theorem 3.2. *Let R be a commutative u-semiring, and I a proper ideal of R. Then the following statements are equivalent.*

1. *I is a weakly 1-absorbing primary ideal of R.*
2. *For every nonunit elements $a,b \in R$ with ab not belong to I, $(I : ab) = (0 : ab)$, or $(I : ab) \subseteq \sqrt{I}$.*
3. *For every nonunit element $a \in R$, and every ideal I_1 of R with $I_1 \nsubseteq \sqrt{I}$. If $(I : aI_1)$ is a proper ideal of R, then $(I : aI_1) = (0 : aI_1)$, or $(I : aI_1) \subseteq (I : a)$.*
4. *For every ideals I_1, I_2 of R with $I_1 \nsubseteq \sqrt{I}$. If $(I : I_1I_2)$ is a proper ideal of R, then $(I : I_1I_2) = (0 : I_1I_2)$, or $(I : I_1I_2) \subseteq (I : I_2)$.*
5. *For every ideals I_1, I_2, I_3 of R with $0 \neq I_1I_2I_3 \subseteq II_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.*

Proof. $(1) \Rightarrow (2)$. Suppose that I is a weakly 1-absorbing primary ideal of R, ab not belong to I for some nonunit elements $a,b \in R$ and $c \in (I : ab)$. Then $abc \in I$. Since ab not belong to I, c is nonunit. If $abc = 0$, then $c \in (0 : ab)$. Assume that $0 \neq abc \in I$. Since I is weakly 1-absorbing primary, we have $c \in \sqrt{I}$. Hence we conclude that $(I : ab) \subseteq (0 : ab) \cup \sqrt{I}$. Since R is a u-semiring, we obtain that $(I : ab) = (0 : ab)$ or $(I : ab) \subseteq \sqrt{I}$.
$(2) \Rightarrow (3)$. If $aI_1 \subseteq I$, then we are done. Suppose that $aI_1 \nsubseteq I$ for some nonunit element $a \in R$ and $c \in (I : aI_1)$. It is clear that c is nonunit. Then $acI_1 \subseteq I$. Now $I_1 \subseteq (I : ac)$. If $ac \in I$, then $c \in (I : a)$. Suppose that ac not belong to I. Hence $(I : ac) = (0 : ac)$ or $(I : ac) \subseteq \sqrt{I}$ by 2. Thus $I_1 \subseteq (0 : ac)$ or $I_1 \subseteq \sqrt{I}$. Since $I_1 \nsubseteq I$ by hypothesis, we conclude $I_1 \subseteq (0 : ac)$; i.e. $c \in (0 : aI_1)$. Thus $(I : aI_1) \subseteq (0 : aI_1) \cup (I : a)$. Since R is a u-semiring, we have $(I : aI_1) = (0 : aI_1)$ or $(I : aI_1) \subseteq (I : a)$.

$(3) \Rightarrow (4)$. If $I_1 \subseteq \sqrt{I}$, then we are done. Suppose that $I_1 \nsubseteq \sqrt{I}$ and $c \in (I : I_1I_2)$. Then $I_2 \subseteq (I : cI_1)$. Since $(I : I_1I_2)$ is proper, c is nonunit. Hence $I_2 \subseteq (0 : cI_1)$ or $I_2 \subseteq (I : c)$ by 2.6. If $I_2 \subseteq (0 : cI_1)$, then $c \in (I : I_1I_2)$. If $I_2 \subseteq (I : c)$, then $c \in (I : I_2)$. So, $(I : I_1I_2) \subseteq (0 : I_1I_2) \cup (I : I_2)$ which implies that $(I : I_1I_2) = (0 : I_1I_2)$, or $(I : I_1I_2) \subseteq (I : I_2)$, as needed.

$(4) \Rightarrow (5)$. It is clear.

$(5) \Rightarrow (1)$. Let $a,b,c \in R$ be nonunit elements and $0 \neq abc \in I$. Put $I_1 = aR$, $I_2 = bR$, and $I_3 = cR$. Then 1 is now clear by 5 □

Definition 3.3. *Let I be a weakly 1-absorbing primary ideal of R and a,b,c be nonunit elements of R. We call (a,b,c) a 1-triple-zero of I if $abc = 0$, ab not belong to I, and c not belong to $\sqrt{I}$.*

Observe that if I is a weakly 1-absorbing primary ideal of R that is not 1- absorbing primary, then there exists a 1-triple-zero (a,b,c) of I for some nonunit elements $a,b,c \in R$.

Theorem 3.4. *Let I be a weakly 1-absorbing primary k-ideal of R, and (a,b,c) be a 1-triple-zero of I. Then*

1. *$abI = 0$.*
2. *If a,b not belong to $(I : c)$, then $bcI = acI = aI^2 = bI^2 = cI^2 = 0$.*
3. *If a,b not belong to $(I : c)$, then $I^3 = 0$.*

Proof.
1. Suppose that $abI \neq 0$. Then $abx \neq 0$ for some nonunit $x \in I$. Hence $0 \neq ab(c+x) \in I$. Since ab not belong to I, $(c+x)$ is nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R and ab not belong to I, we conclude that $(c+x) \in \sqrt{I}$. Since $x \in I$, we have $c \in \sqrt{I}$, a contradiction. Thus $abI = 0$.
2. Suppose that $bcI \neq 0$. Then $bcy \neq 0$ for some nonunit element $y \in I$. Hence $0 \neq bcy = b(a+y)c \in I$. Since b not belong to $(I : c)$, we conclude that $a+y$ is a nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R and $ab \in I$ and $by \in I$, we conclude that $b(a+y)$ not belong to I, and hence $c \in \sqrt{I}$, a contradiction. Thus $bcI = 0$. We show that $acI = 0$. Suppose that $acI \neq 0$. Then $acy \neq 0$ for some nonunit element $y \in I$. Hence $0 \neq acy = a(b+y)c \in I$. Since a not belong to $(I : c)$, we conclude that $b+y$ is a nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R and ab not belong to I and $ay \in I$, we conclude that $a(b+y)$ not belong to I, and hence $c \in \sqrt{I}$, a contradiction. Thus $bcI = 0$. We show that $acI = 0$. Suppose that $acI \neq 0$. Then $acy \neq 0$ for some nonunit element $y \in I$. Hence $0 \neq acy = a(b+y)c \in I$. Since a not belong to $(I : c)$, we conclude that $b+y$ is a nonunit element of R. Since I is a

weakly 1-absorbing primary k-ideal of R and ab not belong to I and $ay \in I$, we conclude that $a(b+y)$ not belong to I, and hence $c \in \sqrt{I}$, a contradiction.

Thus $acI = 0$. Now we prove that $aI^2 = 0$. Suppose that $axy \neq 0$ for some $x, y \in I$. Since $abI = 0$ by (1) and $acI = 0$ by (2), $0 \neq axy = a(b+x)(c+y) \in I$.

Since ab not belong to I, we conclude that $c+y$ is a nonunit element of R. Since a not belong to $(I : c)$, we conclude that $b+x$ is a nonunit element of R. Since I is a weakly 1-absorbing Primary k-ideal of R, we have $a(b+x) \in I$ or $(c+y) \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $aI^2 = 0$. We show $bI^2 = 0$. Suppose that $bxy \neq 0$ for some $x, y \in I$. Since $abI = 0$ by (1) and $bcI = 0$ by (2), $bxy = b(a+x)(c+y) \in I$. Since ab not belong to I, we conclude that $c+y$ is a nonunit element of R. Since b not belong to $(I : c)$, we conclude that $a+x$is a nonunit element of R. Since I is a weakly 1-absorbing primary k-ideal of R, we have $b(a+x) \in I$ or $(c+y) \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction.Thus $bI^2 = 0$. We show $cI^2 = 0$.

Suppose that $cxy \neq 0$ for some $x, y \in I$. Since $acI = bcI = 0$ by (2), $0 \neq cxy = (a+x)(b+y)c \in I$. Since a, b not belong to $(I : c)$, we conclude that $a+x$ and $b+y$ are nonunit elements of R. Since I is a weakly 1-absorbing primary k-ideal of R, we have $(a+x)(b+y) \in I$ or $c \in \sqrt{I}$. Since $x, y \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $cI^2 = 0$.

3. Assume that $xyz \neq 0$ for some $x, y, z \in I$. Then $0 \neq xyz = (a+x)(b+y)(c+z) \in I$ by (1) and (2). Since ab not belong to I, we conclude $c+z$ is a nonunit element of R. Since a, b not belong to $(I : c)$, we conclude that $a+x$ and $b+y$ are nonunit elements of R. Since I is a weakly 1-absorbing primary k-ideal of R, we have $(a+x)(b+y) \in I$ or $c+z \in \sqrt{I}$. Since $x, y, z \in I$, we conclude that $ab \in I$ or $c \in \sqrt{I}$, a contradiction. Thus $I^3 = 0$.

□

Theorem 3.5. *1. Let I be a weakly 1-absorbing primary k-ideal of a reduced semiring R. Suppose that I is not a 1-absorbing ideal primary ideal of R and (a, b, c) is a 1-triple-zero of I such that a, b not belong to $(I : c)$. Then $I = 0$.*

2. Let I be a nonzero weakly 1-absorbing primary k-ideal of a reduced semiring R. Suppose that I is not a 1-absorbing ideal primary ideal of R and (a, b, c) is a 1-triple-zero of I. Then $ac \in I$ or $bc \in I$.

Proof. 1. Since a, b not belong to $(I : c)$, then $I^3 = 0$ by Theorem 3.4. Since R is reduced, we conclude that $I = 0$.

2. Suppose that neither $ac \in I$ nor $bc = 0$. Then $I = 0$ by (1), a contradiction, since I is a nonzero ideal of R by hypothesis. Hence if (a, b, c) is a 1-triple-zero of I, then $ac \in I$ or $bc \in I$.

□

Theorem 3.6. *Let I be a weakly 1-absorbing primary ideal of R. If I is not a weakly primary ideal of R, then there exist an irreducible element $x \in R$ and a nonunit element $y \in R$ such that $xy \in I$, but neither $x \in I$ nor $y \in \sqrt{I}$. Furthermore, if $ab \in I$ for some nonunit elements $a, b \in R$ such that neither $a \in I$ nor $b \in \sqrt{I}$, then a is an irreducible element of R.*

Proof. Suppose that I is not a weakly primary ideal of R. Then there exist nonunit elements $x, y \in R$ such that $0 \neq xy \in I$ with x not belong to I, y not belong to $\sqrt{I}$. Suppose that x is not an irreducible element of R. Then $x = cd$ for some nonunit elements $c, d \in R$. Since $0 \neq xy = cdy \in I$ and I is weakly 1-absorbing primary and y not belong to $\sqrt{I}$, we conclude that $cd = x \in I$, a contradiction. Hence x is an irreducible element of R. □

In general, the intersection of a family of weakly 1-absorbing primary ideals need not be a weakly 1-absorbing primary ideal.

Example 3.7. *consider the semiring $R = Z_6$. Then $I = (2)$ and $J = (3)$ are clearly weakly 1-absorbing primary ideals of Z_6 but $I \cap J = 0$ is not a weakly 1-absorbing primary ideal of R.*

However, we have the following result.

Proposition 3.8. *Let $\{I_i : i \in \wedge\}$ be a collection of weakly 1-absorbing primary ideals of R such that $Q = \sqrt{I_i} = \sqrt{I_j}$ for every distinct $i, j \in \wedge$. Then $I = \cap_{i \in \wedge} I_i$ is a weakly 1-absorbing primary ideal of R.*

Proof. Suppose that $0 \neq abc \in I = \cap_{i \in \wedge} I_i$ for nonunit elements $a, b, c \in R$ and ab not belong to I. Then for some $k \in \wedge$, $0 \neq abc \in I_k$ and ab not belong to I_k. It implies that $c \in \sqrt{I_k} = Q = \sqrt{I}$. □

Proposition 3.9. *Let I be a weakly 1-absorbing primary ideal of R and c be a nonunit element of $R \setminus I$. Then $(I : c)$ is a weakly primary ideal of R.*

Proof. Suppose that $0 \neq ab \in (I : c)$ for some nonunit $c \in R \setminus I$ and assume that a not belong to $(I : c)$. Hence b is a nonunit element of R. If a is unit, then $b \in (I : c) \subseteq \sqrt{(I : c)}$, and we are done. So assume that a is a nonunit element of R. Since $0 \neq abc = acb \in I$ and ac not belong to I and I is a weakly 1-absorbing primary ideal of R, we conclude that $b \in \sqrt{I} \subseteq \sqrt{(I : c)}$. Thus, $(I : c)$ is a weakly primary ideal of R. □

4. Characterization for Weakly 1-absorbing Primary Ideal of $R = R_1 \times R_2$

The next theorem gives a characterization for weakly 1-absorbing primary ideals of $R = R_1 \times R_2$ where R_1 and R_2 are commutative semirings with identity that are not semifields

Theorem 4.1. *Let R_1 and R_2 be commutative semirings with identity that are not semifields, and let $R = R_1 \times R_2$ and I be a a nonzero proper ideal of R. Then the following statements are equivalent.*

1. *I is a weakly 1-absorbing primary ideal of R.*
2. *$I = I_1 \times R_2$ for some primary ideal I_1 of R_1 or $I = R_1 \times I_2$ for some primary ideal I_2 of R_2.*
3. *I is a 1-absorbing primary ideal of R.*
4. *I is a primary ideal of R_1.*

Proof. $(1) \Rightarrow (2)$. Suppose that I is a weakly 1-absorbing primary ideal of R. Then I is of the form $I_1 \times I_2$ for some ideals I_1 and I_2 of R_1 and R_2, respectively. Assume that both I_1 and I_2 are proper. Since I is a nonzero ideal of R, we conclude that $I_1 \neq 0$ or $I_2 \neq 0$. We may assume that $I_1 \neq 0$. Let $0 \neq c \in I_1$ Then $0 \neq (1,0)(1,0)(c,1) = (c,0) \in I_1 \times I_2$. It implies that $(1,0)(1,0) \in I_1 \times I_2$ or $(c,1) \in \sqrt{(I_1 \times I_2)} = \sqrt{I_1} \times \sqrt{I_2}$, that is $I_1 = R_1$ or $I_2 = R_2$, a contradiction. Thus either I_1 or I_2 is a proper ideal. Without loss of generality, assume that $I = I_1 \times R_2$ for some proper ideal I_1 of R_1. We show that I_1 is a primary ideal of R_1. Let $ab \in I_1$ for some $a, b \in R_1$. We can assume that a and b are nonunit elements of R_1. Since R_2 is not a semifield, there exists a nonunit nonzero element $x \in R_2$. Then $0 \neq (a,1)(1,x)(b,1)\ I_1 \times R_2$ which implies that either $(a,1)(1,x) \in I_1 \times R_2$ or $(b,1) in \sqrt{I_1 \times R_2} = \sqrt{I}_1 \times R_2$; i.e., $a \in I_1$ or $b \in \sqrt{I_1}$.
$(2) \Rightarrow (3)$. Since I is a primary ideal of R, I is a 1-absorbing primary ideal of R by [[9], Theorem (1)].
$(3) \Rightarrow (4)$ Since I a 1-absorbing primary ideal of R and R is not a quasilocal semring, we conclude that I is a primary ideal of R by [9, Theorem(3)].
$(4) \Rightarrow (1)$ Let nonunit elements $a, b, c \in R$, and $0 \neq abc \in I$. Assume ab not belong to I. Since I is primary ideal, we have $c^m \in I$ for some positive integer m, so $c \in \sqrt{I}$. So I is a weakly 1-absorbing primary ideal. □

Theorem 4.2. *Let $R_1, \ldots, R_n$ be commutative semirings with $1 \neq 0$ for some $2 \leq n < \infty$, and let $R = R_1 \times \ldots\ldots \times R_n$. Then the following statements are equivalent.*

1. *Every proper ideal of R is a weakly 1-absorbing primary ideal of R.*
2. *$n = 2$ and R_1, R_2 are semifields.*

Proof. $(1) \Rightarrow (2)$. Suppose that every proper ideal of R is a weakly 1-absorbing primary ideal. Without loss of generality, we may assume that $n = 3$. Then $I = R_1 \times \{0\} \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $a \in R_1$, we have$(0,0,0) \neq (1,0,1)(1,0,1)(a,1,0) = (a,0,0) \in I$, but neither $(1,0,1)(1,0,1) \in I$ nor $(a,1,0) \in \sqrt{I}$, a contradiction. Thus $n = 2$. Assume that R_1 is not a semifield. Then there exists a nonzero proper ideal A of R_1. Hence $I = A \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $a \in A$, we have $(0,0) \neq (1,0)(1,0)(a,1) = (a,0) \in I$, but neither$(1,0)(1,0) \in I$ nor $(a,1) \in \sqrt{I}$, a contradiction. And, assume that R_2 is not a semifield. Then there exists a nonzero proper ideal B of R_2. Hence $I = B \times \{0\}$ is a weakly 1-absorbing primary ideal of R. However, for a nonzero $b \in B$, we have $(0,0) \neq (1,0)(1,0)(b,1) = (a,0) \in I$, but neither $(1,0)(1,0) \in I$ nor $(a,1) \in \sqrt{I}$, a contradiction. Hence $n = 2$ and R_1, R_2 are semifields.

$(2) \Rightarrow (1)$. Suppose that $n = 2$ and R_1, R_2 are semifields. Then R has exactly three proper ideals, i.e., $\{(0,0)\}, \{0\} \times R_2$ and $R_1 \times \{0\}$ are the only proper ideals of R. Hence it is clear that each proper ideal of R is a weakly 1-absorbing primary ideal of R. □

Since every semiring that is a product of a finite number of fields is a von-Neumann regular semiring, in light of Theorem 4 and Theorem 14 we have the following result.

Corollary 4.3. *Let $R_1, \ldots, R_n$ be commutative semirings with $1 \neq 0$ for some $2 \leq n < \infty$, and let $R = R_1 \times \ldots.. \times R_n$. Then the following statements are equivalent.*

1. Every proper ideal of R is a weakly 1-absorbing primary ideal of R.
2. Every proper ideal of R is a weakly primary ideal of R.
3. $n = 2$ and R_1, R_2 are semifields, and hence $R = R_1 \times R_2$ is a von-Neumann regular semiring.

Theorem 4.4. *Let R_1 and R_2 be commutative semirings and $f : R_1 \to R_2$ be a semiring homomorphism with $f(1) = 1$. Then the following statements hold:*

1. *Suppose that f is a monomorphism and $f(a)$ is a nonunit element of R_2 for every nonunit element $a \in R_1$ and J is a weakly 1-absorbing primary ideal of R_2. Then $f^{(-1)}(J)$ is a weakly 1-absorbing primary ideal of R_1.*
2. *If f is an epimorphism and I is a weakly 1-absorbing primary ideal of R_1 such that $Ker(f) \subseteq I$, then $f(I)$ is a weakly 1-absorbing primary ideal of R_2.*

Proof. . (1) Let $0 \neq abc \in f^{(-1)}(J)$ for some nonunit elements $a,b,c \in R$. Since $Ker(f) = 0$, we have $0 \neq f(abc) = f(a)f(b)f(c) \in J$, where $f(a), f(b), f(c)$ are nonunit elements of R_2 by hypothesis. Hence $f(a)f(b) \in J$ or $f(c) \in \sqrt{J}$. Hence $ab \in f^{(-1)}(J)$ or $c \in \sqrt{(f(-1)(J))} = f^{(-1)}(\sqrt{J})$. Thus $f^{(-1)}(J)$ is a weakly 1-absorbing primary ideal of R_1.

Let $0 \neq xyz \in f(I)$ for some nonunit elements $x,y,z \in R$. Since f is onto, there exists nonunit elements $a,b,c \in I$ such that $x = f(a), y = f(b), z = f(c)$. Then $f(abc) = f(a)f(b)f(c) = xyz \in f(I)$. Since $Ker(f) \subseteq I$, we have $0 \neq abc \in I$. It follows $ab \in I$ or $c \in \sqrt{I}$. Thus $xy \in f(I)$ or $z \in f(\sqrt{I})$. Since f is onto and $Ker(f) \subseteq I$, we have $f(\sqrt{I}) = \sqrt{(f(I))}$. Thus we are done. □

Example 4.5. *Let$A = K[x,y]$, where K is a semifield, $M = (x,y)A$, and $B = A_M$. Note that B is a quasilocal semiring with maximal ideal M_M. Then $I = xM_M = (x^2,xy)B$ is a 1-absorbing primary ideal of B and $\sqrt{I} = xB$. However $xy \in I$, but neither $x \in I$ nor $y \in \sqrt{I}$. Thus I is not a primary ideal of B. Let $f : B \times B \to B$ such that $f(x,y) = x$. Then f is a semiring homomorphism from $B \times B$ onto B such that $f(1,1) = 1$. However, $(1,0)$ is a nonunit element of $B \times B$ and $f(1,0) = 1$ is a unit of B. Thus f does not satisfy the hypothesis of 4.4. Now $f^{(-1)}(I) = I \times B$ is not a weakly 1-absorbing ideal of $B \times B$ by 4.1.*

Theorem 4.6. *Let I be a proper ideal of R. Then the following statements hold.*

1. *If J is a proper ideal of a semiring R with $J \subseteq I$ and I is a weakly 1-absorbing primary ideal of R, then I/J is a weakly 1-absorbing primary ideal of R/J.*
2. *If J is a proper ideal of a semiring R with $J \subseteq I$ such that $U(R/J) = \{a+J \mid a \in U(R)\}$. If J is a 1-absorbing primary ideal of R and I/J is a weakly 1-absorbing primary ideal of R/J, then I is a 1-absorbing primary ideal of R.*
3. *If $\{0\}$ is a 1-absorbing primary ideal of R and I is a weakly 1-absorbing primary ideal of R, then I is a 1-absorbing primary ideal of R.*
4. *If J is a proper ideal of a ring R with $J \subseteq I$ such that $U(R/J) = \{a+J \mid a \in U(R)\}$. If J is a weakly 1-absorbing primary ideal of R and I/J is a weakly 1-absorbing primary ideal of R/J, then I is a weakly 1-absorbing primary ideal of R.*

Proof.

1. Consider the natural epimorphism $\pi : R \to R/J$. Then $\pi(I) = I/J$. So we are done by Theorem 1.
2. Suppose that $abc \in I$ for some nonunit elements $a,b,c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \sqrt{J} \subseteq \sqrt{I}$ as J is a 1-absorbing primary ideal of R. Now assume that abc not belong to J. Then $J \neq (a+J)(b+J)(c+J) \in I/J$, where $a+J, b+J, c+J$ are nonunit elements of R/J by hypothesis. Thus $(a+J)(b+J) \in I/J$ or $(c+J) \in \sqrt{(I/J)}$. Hence $ab \in I$ or $c \in \sqrt{I}$.
3. The proof follows from (2).
4. Suppose that $0 \neq abc \in I$ for some nonunit elements $a,b,c \in R$. If $abc \in J$, then $ab \in J \subseteq I$ or $c \in \sqrt{J} \subseteq \sqrt{I}$ as J is a weakly 1-absorbing primary ideal of R. Now assume that abc not belong to J. Then $J \neq (a+J)(b+J)(c+J) \in I/J$, where $a+J, b+J, c+J$ are nonunit elements of R/J by hypothesis. Thus $(a+J)(b+J) \in I/J$ or $(c+J) \in \sqrt{(I/J)}$. Hence $ab \in I$ or $c \in \sqrt{I}$.

□

Proposition 4.7. *1. Let R_1 and R_2 be commutative semirings and $f : R_1 \to R_2$ be a ring homomorphism with $f(1) = 1$ such that R_2 is not a quasilocal semiring, then $f(a)$ is a nonunit element of R_2 for every nonunit element $a \in R_1$ and J is a 1-absorbing primary ideal of R_2. Then $f^{(-1)}(J)$ is a 1-absorbing primary ideal of R_1.*

2. *Let I and J be proper ideals of a semiring R with $I \subseteq J$. If J is a 1-absorbing primary ideal of R, then J/I is a 1-absorbing primary ideal of R/I. Furthermore, assume that if R/I is a quasilocal semiring, then $U(R/I) = a+I \mid a \in U(R)$. If J/I is a 1-absorbing primary ideal of R/I, then J is a 1-absorbing primary ideal of R.*

3. *Let R be a semiring and $A = R[x]$. Then a proper ideal I of R is a 1-absorbing primary ideal of R if and only if $(I[x]+xA)/xA$ is a 1-absorbing primary ideal of A/xA, since R is semiring-isomorphic to A/xA.*

Theorem 4.8. *Let S be a multiplicatively closed subset of R, and I a proper ideal of R. Then the following statements hold.*

1. *If I is a weakly 1-absorbing primary ideal of R such that $I \cap S = \phi$, then $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$.*

2. *If $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$ such that $S \cap Z(R) = \phi$ and $S \cap Z_I(R) = \phi$, then I is a weakly 1-absorbing primary ideal of R.*

Proof. 1. Suppose that $0 \neq \frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} \in S^{(-1)}I$ for some nonunit $a,b,c \in R\setminus S$, $s_1,s_2,s_3 \in S$ and $\frac{a}{s_1}\frac{b}{s_2}$ not belong to $S^{(-1)}I$. Then $0 \neq uabc \in I$ for some $u \in S$. Since I is weakly 1-absorbing primary and uab not belong to I, we conclude $c \in \sqrt{I}$. Thus $\frac{c}{s_3} \in S^{(-1)}\sqrt{I} = \sqrt{(S^{(-1)}I)}$. Thus $S^{(-1)}I$ is a weakly 1-absorbing primary ideal of $S^{(-1)}R$.

2. Suppose that $0 \neq abc \in I$ for some nonunit elements $a,b,c \in R$. Hence $0 \neq \frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{(-1)}I$ as $S \cap Z(R) = \phi$. Since $S^{(-1)}I$ is weakly 1-absorbing primary, we have either $\frac{a}{1}\frac{b}{1} \in S^{(-1)}I$, or $\frac{c}{1} \in \sqrt{S^{(-1)}I} = S^{-1}\sqrt{I}$. If $\frac{a}{1}\frac{b}{1} \in S^{(-1)}I$, then $uab \in I$ for some $u \in S$. Since $S \cap Z_I(R) = \phi$, we conclude that $ab \in I$. If $\frac{c}{1} \in S^{-1}\sqrt{I}$, then $(tc)^n \in I$ for some positive integer $n \geq 1$ and $t \in S$. Since t^n not belong to $Z_I(R)$, we have $c^n \in I$, i.e., $c \in \sqrt{I}$. Thus I is a weakly 1-absorbing primary ideal of R.

□

Definition 4.9. *Let I be a weakly 1-absorbing primary ideal of R and $I_1I_2I_3 \subseteq I$ for some proper ideals I_1,I_2,I_3 of R. If (a,b,c) is not 1-triple zero of I for every $a \in I_1$, $b \in I_2, c \in I_3$, then we call I a free 1-triple zero with respect to $I_1I_2I_3$.*

Theorem 4.10. *Let I be a weakly 1-absorbing primary ideal of R and J be a proper ideal of R with $abJ \subseteq I$ for some $a,b \in R$. If (a,b,j) is not a 1-triple zero of I for all $j \in J$and ab not belong to I, then $J \subseteq \sqrt{I}$.*

Proof. Suppose that $J \nsubseteq \sqrt{I}$. Then there exists $c \in J \setminus \sqrt{I}$. Then $abc \in abJ \subseteq I$. If $abc \neq 0$, then it contradicts our assumption that ab not belong to I and c not belong to $\sqrt{I}$. Thus $abc = 0$. Since (a,b,c) is not a 1-triple zero of I and ab not belong to I, we conclude $c \in \sqrt{I}$, a contradiction. Thus $J \subseteq \sqrt{I}$.

□

Theorem 4.11. *Let I be a weakly 1-absorbing primary ideal of R and $0 \neq I_1I_2I_3 \subseteq I$ for some proper ideals I_1,I_2,I_3 of R. If I is free 1-triple zero with respect to $I_1I_2I_3$, then $I_1I_2 \subseteq I$ or $I_3 \subseteq \sqrt{I}$.*

Proof. Suppose that I is free 1-triple zero with respect to $I_1I_2I_3$, and $0 \neq I_1I_2I_3 \subseteq I$. Assume that $I_1I_2 \nsubseteq I$. Then there exist $a \in I_1, b \in I_2$ such that ab not belong to I. Since I is a free 1-triple zero with respect to $I_1I_2I_3$, we conclude that (a,b,c) is not a 1-triple zero of I for all $c \in I_3$. Thus $I_3 \subseteq \sqrt{I}$ by Theorem 4.10. □

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