Comparison of Multiple Scales Method and Finite Difference Method for Solving Singularly Perturbed Convection Diffusion Problem

Singüler Pertürbe Özellikli Konveksiyon Difüzyon Problemleri İçin Çoklu Ölçekler Metodu ve Sonlu Fark Metodunun Karşılaştırılması

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Abstract
In this study, multiple scale method is introduced for singularly perturbed convection-diffusion equation. In this context, the mentioned problem is transformed into partial differential equation. Besides exponentially fitted difference scheme is established by the method of integral identities with using linear basis functions and interpolating quadrature rules with weight functions and remainder term in integral form. Some numerical experiments have been carried out to validate the theoretical results. The main objective of this article is to compare the multiple scale method and finite difference method for singularly perturbed convection-diffusion problems.

Keywords: Boundary Layer, Difference Scheme, Multiple Scales Method, Singular Perturbation, Uniform Convergent

Öz
Bu çalışmada singüler pertürbe özellikli konveksiyon difüzyon problemi için çoklu ölçekler metodu tanıtılmıştır. Bu bağlamda, söz konusu problem kısmi diferansiyel denklemlere dönüştürülmüştür. Ayrıca ağırlık fonksiyonu içeren ve kalan terimi integral biçiminde olan interpolasyon kuadratür kuralları ve lineer baz fonksiyonlarının kullanımı ile üstel katsayılı fark şeması kurulmuştur. Teorik sonuçları doğrulamak için bazı nümerik çalışmalara yer verilmiştir. Bu makalenin temel amacı, singüler pertürbe özellikli konveksiyon-difüzyon problemleri için çoklu ölçekler metodu ile sonlu fark metodunu karşılaştırmaktır.

Anahtar Kelimeler: *Sınır Katmanı, Fark Şeması, Çoklu Ölçekler Metodu, Singüler Pertürbasyon, Düzgün Yakınsaklık*

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1. Introduction

Singularly perturbed convection-diffusion equations arise in many scientific area. Applications of engineering, fluid mechanics, oceanography, heat transfer, bifurcation analysis, electron plasma waves, structural mechanics and chemical processes are among these (Amiraliyev and Çimen, 2010; Çakır and Amiraliyev, 2005; Nayfeh, 1973; Kevorkian and Cole, 1981; Linβ, 2010; Roos et. al., 2008).

In this paper, we concerned with the following singularly perturbed convection-diffusion equation:

$$\varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = 0, \quad 0 < x < l \tag{1}$$

with boundary conditions

$$u(0) = \kappa_0\,, \;\; u(l) = \kappa_1 \tag{2}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, κ_0, κ_1 are constants, $a(x) \geq \alpha > 0$ and $b(x)$ are sufficiently smooth functions. The solution of problems (1)-(2) has general boundary layers at the neighborhood of $x = 0$ and $x = l$ which the solution changes quickly. These problems depend on small positive parameter ε which highest derivative term is multiplied.

Singularly perturbed convection-diffusion problems have located important in literature. The existence and uniqueness of the solution of these problems are mentioned (Reddy and Chakravarthy, 2003). Solving of such kind of problems is so difficult. Due to existence of perturbation parameter, traditional methods don't give reliable results. Therefore, various numerical methods have been presented. Numerical patching method is applied with using spline functions (Sakar et. al., 2019). An initial value technique is introduced (Subburayan and Ramanujam, 2013). Multiscale finite element method is used for the elliptic and dominated form of these equations (Park and Hou, 2004). Collocation method is considered with using exponential trial functions to solve singularly perturbed reaction-convection-diffusion equations (Liu and Wen, 2019). A Wavelet-Galerkin method is developed to solve singulary perturbed convection dominated diffusion equation (El-Gamel, 2006).

Finite difference method (FDM) is one of the most suitable and effective methods for solving singularly perturbed problems. Many authors have studied this method on different meshes. A piecewise uniform Shishkin mesh is designed to estimate system of these equations (Bellew and Riordan, 2004). By using finite difference method on uniform mesh, boundary and interior layers are studied (Farrell et. al., 2004). For parabolic type, standard finite difference scheme is constructed on uniform meshes (Shishkin and Shishkina, 2019) and classical finite approximations are obtained on piecewise uniform meshes (Shishkin, 2004). For delay type of these equations, exponentially fitted difference scheme is constructed on a uniform mesh (Amiraliyev et. al., 2010) and nonlinear form is considered (Amiraliyeva et. al., 2010). Finite difference scheme is established on Shishkin mesh with integral boundary conditions (Sekar and Tamilselvan, 2019).

On the other hand, different perturbation techniques have been introduced by some authors. Multiple scales, asymptotic matching, streched coordinates, WKB expansions and averaging are some of them (Gupta and Kumar, 2016). This study contains implementation of the multiple scales method with second order boundary value problem.

Applications of the methods of multiple scales (MS) are addressed in many scientific fields including orbital mechanics, wave interactions, atmospheric science, hydrodynamic, statiscal mechanics, flight mechanics, model reduction and control system design (Nayfeh, 1973; Malley, 1974). This method was applied to different problems and equations in literature. Duffing equation, Van der Pol oscillator, Mathieu equation, homogenized heat equation, Klein-Gordon equation, The Earth-Moon-Spaceship problem are some of these (Nayfeh, 1973; Jager and Furu, 1966; Romanazzi et al., 2017). By using partial differential systems, nonlinear vibrations of continuous systems are considered (Boyacı and Pakdemirli, 1997). In recent times, multiple scales method has a model for various researchs. Nonlinear spring and nonlinear damper is examined to analyze vibration (Salahshoor et. al., 2016). Piezoelectric and magneto-electro-elastic structures are considered (Wu and Tsai, 2010). Quantum-optical problems are solved (Janowicz, 2003). Periodic solutions of nonlinear oscillators are obtained (Lakrad and Belhaq, 2002). Multiple scale method is improved to investigate nonlinear oscillators with fractional derivatives (Ren et. al., 2019). Multiple scale method is combined with Lindstedt-Poincare technique for linear damped vibration equation (Pakdemirli et. al., 2009).

Our goal in this work is to estimate effectiveness and robustness of Multiple Scales Method and Finite Difference Method for singularly perturbed second order boundary value problems on a uniform mesh.

The outline of this paper is organized as follows:
In section 2, multiple scales method is described for (1)-(2) problems. In section 3, the properties of the solution of (1)-(2) problems are handled. The difference scheme is constructed and error approximations are obtained. In section 4, finally some numerical experiments are presented with tables.

2. Material and Method

2.1. Multiple Scales Method For Singularly Perturbed Convection-Diffusion Problem

In this section, we consider the multiple scales method for (1)-(2) problems. For (1) equation if $a(x) > 0$, the boundary layer is at $x = 0$; and if $a(x) < 0$ the boundary layer is at $x = l$. Due to presence of the boundary layer in the problems (1)-(2), we consider two scales the outer scale at $x = x_0$ and inner or boundary layer scale at $\xi = \frac{x}{\varepsilon}$.

By using chain rule, we obtain the following derivatives:

$$\frac{d}{dx} = \frac{\partial}{\partial \xi} \times \frac{\partial}{\partial x} + \frac{\partial}{\partial x_0} \times \frac{\partial x_0}{\partial x} = \frac{1}{\varepsilon}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial x_0} \quad (3)$$

$$\frac{d^2}{dx^2} = \frac{1}{\varepsilon}\frac{\partial^2}{\partial \xi^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial \xi \partial {}_0} + \frac{\partial^2}{\partial {x_0}^2} \quad (4)$$

For two scales, we get the following multi-scale expansion

$$u = u_0(\xi, x_0) + \varepsilon u_1(\xi, x_0) + \varepsilon^2 u_2(\xi, x_0) + \cdots \quad (5)$$

Substituting (3)-(4) and (5) into (1) equation, we have:

$$\varepsilon\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \xi^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial \xi \partial x_0} + \frac{\partial^2}{\partial {x_0}^2}\right)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)$$
$$+a(x)\left(\frac{1}{\varepsilon}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial x_0}\right)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)$$
$$+b(x)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) = 0.$$

Therefore, we get:

$$\left(\frac{1}{\varepsilon}\frac{\partial^2}{\partial \xi^2}u_0 + 2\frac{\partial^2}{\partial \xi \partial x_0}u_0 + \varepsilon\frac{\partial^2}{\partial {x_0}^2}u_0 + \frac{\partial^2}{\partial \xi^2}u_1 + 2\varepsilon\frac{\partial^2}{\partial \xi \partial x_0}u_1\right.$$
$$\left.+\varepsilon^2\frac{\partial^2}{\partial {x_0}^2}u_1 + \varepsilon\frac{\partial^2}{\partial \xi^2}u_2 + 2\varepsilon^2\frac{\partial^2}{\partial \xi \partial x_0}u_2 + \varepsilon^3\frac{\partial^2}{\partial {x_0}^2}u_2 + \cdots\right)$$
$$\left(a(x)\frac{1}{\varepsilon}\frac{\partial}{\partial \xi}u_0 + a(x)\frac{\partial}{\partial x_0}u_0 + a(x)\frac{\partial}{\partial \xi}u_1\right.$$
$$\left.a(x)\varepsilon\frac{\partial}{\partial x_0}u_1 + a(x)\varepsilon\frac{\partial}{\partial \xi}u_2 + a(x)\varepsilon^2\frac{\partial}{\partial x_0}u_2 + \cdots\right)$$
$$+(b(x)u_0 + \varepsilon b(x)u_1 + \varepsilon^2 b(x)u_2 + \cdots) = 0 \quad (6)$$

Thus, equation (1) is transformed into partial differential equation (6). By considering the coefficients of each order of ε, we obtain following equations:

$$O\left({}^{1}/_{\varepsilon}\right): \frac{\partial^2 u_0}{\partial \xi^2} + a\frac{\partial u_0}{\partial \xi} = 0 \quad (7)$$

$$O(\varepsilon^0): \frac{\partial^2 u_1}{\partial \xi^2} + a\frac{\partial u_1}{\partial \xi} = -2\frac{\partial^2 u_0}{\partial \xi \partial x_0} - a\frac{\partial u_0}{\partial x_0} - bu_0 \quad (8)$$

$$O(\varepsilon^1): \frac{\partial^2 u_2}{\partial \xi^2} + a\frac{\partial u_2}{\partial \xi} = -2\frac{\partial^2 u_1}{\partial \xi \partial x_0} - \frac{\partial^2 u_0}{\partial {x_0}^2} - a\frac{\partial u_1}{\partial x_0} - bu_1 \quad (9)$$

(Gupta and Kumar, 2016). The general solution of (7) is as following:

$(D_\xi{}^2 + aD_\xi)u_0 = 0,$

$L = D_\xi{}^2 + aD_\xi = D_\xi(D_\xi + a)$

where

$L_1 = D_\xi,\ \ L_2 = D_\xi + a$

and

$a_1 = 1, b_1 = 0, c_1 = 0, a_2 = 1, b_2 = 0, c_2 = a.$

Thus, we find:

$$u_0 = e^{-\frac{c_1}{a_1}\xi} f(b_1\xi - x_0) + e^{-\frac{c_2}{a_2}\xi} g(b_2\xi - x_0),$$

$$= f(-x_0) + e^{-a\xi} g(-x_0),$$

$$= A(x_0) + B(x_0)e^{-a\xi}$$

where A and B are solution of following problems:

$aB' - bB = 0$

$aA' - bA = 0.$

Rewriting u_0 in (8), we get following equalities:

$$\frac{\partial u_0}{\partial x_0} = A'(x_0) + B'(x_0) \quad (10)$$

$$\frac{\partial^2 u_0}{\partial \xi \partial x_0} = B'^{(x_0)(-ae^{-a\xi})} = -aB'^{(x_0)(e^{-a\xi})}. \quad (11)$$

Substituting (10) and (11) in the right side of (8), we can write

$$-2\frac{\partial^2 u_0}{\partial \xi \partial x_0} - a\frac{\partial u_0}{\partial x_0} - bu_0 = (aB' - bB)e^{-a\xi} - (aA' + bA) \quad (12)$$

The solution of (12) is as follow:

$$u_{1p} = w\xi e^{-a\xi} + (Z\xi + X) \quad (13)$$

where $w = \frac{-(aB' - bB)}{a}$ and $Z = \frac{-(aA' - b\ )}{a}$.

So, we have

$$u_1 = u_{1h} + u_{1p} = A(x_0) + B(x_0)e^{-a\xi} - \frac{(aB' - bB)}{a}\xi e^{-a\xi} - \frac{(aA' - bA)}{a}\xi.$$

Similarly, we obtain

$$u_2 = A(x_0) + B(x_0)e^{-a\xi} + (B'' + (a - 2\xi)B' - bB)\xi e^{-a\xi} + \left(\frac{b^2}{a}B - aB''\right)\xi^2 e^{-a\xi}$$

$$-\left(aA'' + \frac{b^2}{a}\right)\xi^2 + \frac{(aA'' + (a^2 - 2a)A' + (a+1)bA)}{a}\xi + 2\frac{(bB' - aB'')}{a}\xi e^{-a}\ (1 + \xi^2).$$

2.2. Finite Difference Method

2.2.1. Continuous Problem

We give the some properties of the solution of (1)-(2) problems, which is needed in the analysis of the numerical method.

Lemma 1. The solution of (1)-(2) problems holds following estimates:

$$|u(x)| \le |\kappa_0| + |\kappa_1|,\ \ 0 \le x \le l \quad (14)$$

$$|u'(x)| \le \frac{C}{\varepsilon}e^{\frac{-\alpha x}{\varepsilon}} + \alpha^{-1}\|b\|_{C[0,l]}(\alpha^{-1}|\kappa_0| + |\kappa_1|) \quad (15)$$

where C is the arbitrary parameter.

Proof. For the proof of the lemma, we consider the following barrier function:
$\psi(x) = |\kappa_0| + |\kappa_1| \pm u(x)$.

For this function, $L\psi(x) \geq 0$, $\psi(0) \geq 0$ and $\psi(l) \geq 0$. According to maximum principle, we obtain $\psi(x) \geq 0$. So inequality (14) is true. From (15), we can write

$$u'(x) = u'(0)exp\left(-\frac{1}{\varepsilon}\int_0^x a(\eta)d\eta\right) - \frac{1}{\varepsilon}\int_0^x [b(s)u(s)]\exp\left(-\frac{1}{\varepsilon}\int_s^x a(\eta)d\eta\right)ds.$$

Thus, we have

$$|u'(x)| \leq u'(0)e^{\frac{-\alpha x}{\varepsilon}} + \alpha^{-1}\max_{[0,l]}|b(s)u(s)|\left(1 - e^{\frac{-\alpha x}{\varepsilon}}\right)$$

$$\leq u'(0)e^{\frac{-\alpha}{\varepsilon}} + \alpha^{-1}\{\|b\|_{C[0,l]}(|\kappa_0| + |\kappa_1|)\}.$$

By using equality $g'(x) = g(\alpha_0, \alpha_1) - \int_s^x K_0(x,\xi)g''(\xi)d\xi$, we obtain

$$|u'(0)| \leq \frac{C}{\varepsilon}$$

and

$$|u'(l)| \leq \frac{C}{\varepsilon}.$$

This is proof of the (15). Therefore, lemma is proved.

2.2.2 Construction of The Difference Scheme

A difference scheme is established for (1)-(2) problems on uniform mesh.

$$\varpi_N = \left\{x_i = ih, \quad i = 1,2,\ldots,N-1;\ h = \frac{l}{N}\right\}, \quad \overline{\omega_h} = \omega_h \cup \{0, l\}$$

is a uniform mesh to a set of discrete points. x_i points are called the node points.

To construct the difference scheme, we use following integral identity:

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} L u\varphi_i dx = h^{-1}\int_{x_{i-1}}^{x_{i+1}} (-\varepsilon^2 u'' + a(x)u'(x) + b(x)u(x))\varphi_i dx = 0$$

where φ_i basis function

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \dfrac{e^{-\frac{a_i}{\varepsilon}(x - x_{i-1})} - 1}{e^{-\frac{a_i}{\varepsilon}h} - 1}, & x \in (x_{i-1}, x_i), \\ \varphi_i^{(2)}(x) = \dfrac{1 - e^{-\frac{a_i}{\varepsilon}(x_{i+1} - x)}}{1 - e^{-\frac{a_i}{\varepsilon}h}}, & x \in (x_i, x_{i+1}), \\ 0, & x \notin (x_i, x_{i+1}) \end{cases}$$

is the solution of the following problems:

$$\varepsilon\varphi_i^{(1)''} + a_i\varphi_i^{(1)'} = 0, \ \ \varphi_i^{(1)}(x_i) = 1, \varphi_j^{(1)}(x_{i-1}) = 0$$

and

$$\varepsilon\varphi_i^{(2)''} + a_i\varphi_i^{(2)'} = 0, \ \ \varphi_i^{(2)}(x_i) = 1, \varphi_j^{(2)}(x_{i+1}) = 0.$$

Using interpolating quadrature rules in (Amiraliyev and Mamedov, 1995), we obtain the following difference scheme

$$l_u :\equiv -\varepsilon\theta_i u_{\bar{x}x,i} + a_i u_{\bar{x},i} + b_i u_i + R_i = 0, \quad i = 1,2,\dots,N-1 \tag{16}$$

$$u(0) = A, \quad u(l) = B \tag{17}$$

where

$$\theta_i = \frac{a_i h}{2\varepsilon}\left[h^{-1}\left(\int_{x_i}^{x_{i+1}} \varphi_i{}^{(2)}(x)dx - \int_{x_{i-1}}^{x_i} \varphi_i{}^{(1)}(x)dx\right)\right] + 1$$

$$R_i = R_i{}^{(1)} + R_i{}^{(2)} + R_i{}^{(3)} \tag{18}$$

$$R_i{}^{(1)} = h^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]u'(x)\varphi_i(x)dx$$

$$R_i{}^{(2)} = h^{-1}\int_{x_{i-1}}^{x_{i+1}} [b(x) - b(x_i)]u(x)\varphi_i(x)dx$$

$$R_i{}^{(3)} = h^{-1}b_i\left[\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_{x_{i-1}}^{x_{i+1}} u'(\xi)K_0(x,\xi)d\xi\right].$$

Thus, we can write difference problem for approximate solution of y

$$l_y :\equiv -\varepsilon\theta_i y_{\bar{x}x,i} + a_i y_{\bar{x},i} + b_i y_i = 0, \quad i = 1,2,\dots,N-1 \tag{19}$$

$$y(0) = A, \quad y(N) = B. \tag{20}$$

2.2.3 Error Analysis

To investigate the uniform convergence of this method, let u_i be the solution of the problems (1)-(2) and y_i be the solution of the problem (19)-(20). Error function $z_i = y_i - u_i$, $i = 0,1,2,\dots,N$ is the solution of following discrete problem

$lz_i = R_i, \quad 1 \le i \le N-1,$

$z_0 = z_N = 0$

where R_i is given by (18).

Lemma 2. For $a(x), b(x) \in C^1[0,l]$, the following estimate is satisfy:

$$h\sum_{i=1}^{N-1} |R_i| \le Ch.$$

Proof. First, R_i is written the following form:

$$R_i = h^{-1}\int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)]u'(x)\varphi_i(x)dx + h^{-1}\int_{x_{i-1}}^{x_{i+1}} [b(x) - b(x_i)]u(x)\varphi_i(x)dx$$

$$+h^{-1}b_i\left[\int_{x_{i-1}}^{x_{i+1}} dx\varphi_i(x)\int_{x_{i-1}}^{x_{i+1}} u'(\xi)K_0(x,\xi)d\xi\right].$$

By considering $|u(x)| \le C_0$ and $|\varphi_i(x)| \le 1$, we obtain

$$|R_i| \le Ch\left(1 + h^{-1}\int_{x_{i-1}}^{x_{i+1}} |u'(x)|dx\right).$$

Theorem 1. Under the conditions of Lemma 2, the solution of (19)-(20) is uniform convergent to the solution of (1)-(2) with respect to ε on $C(\omega_h)$ and its convergence rate is $O(h)$. Thus, we can write

$\|y - u\|_{C(\omega_h)} \le Ch.$

Proof. The proof of the theorem is by similar manner as in (Amiraliyev and Duru, 2002).

3. Numerical Results

In this section, we present two numerical examples to compare the both methods. For numerical algorithm, we can write difference problem (19)-(20) in explicit form

$$\varepsilon\theta_i\left(\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}\right)+a_i\left(\frac{y_{i+1}-y_{i-1}}{2h}\right)+b_iy_i=0.$$

We edit this equation following form

$$A_iy_{i-1}-C_iy_i+B_iy_{i+1}=0, i=1,\dots,N-1$$

where

$$A_i=\varepsilon\theta_ih^{-2}-a_i2h^{-1}$$

$$B_i=\varepsilon\theta_ih^{-2}+a_i2h^{-1}$$

$$C_i=2\varepsilon\theta_ih^{-2}-b_i.$$

Then, we apply the elimination method to following examples. The elimination method is defined by

$$y_i=y_{i+1}\alpha_{i+1}+\beta_{i+1}, i=N-1,\dots,0$$

where

$$\alpha_{i+1}=\frac{B_i}{C_i-\alpha_iA_i}, \alpha_1=0, i=1,\dots,N-1$$

$$\beta_{i+1}=\frac{F_i+A_i\beta_i}{C_i-\alpha_iA_i}, \beta_1=0, i=1,\dots,N-1$$ (Samarskii, 2001).

Example 1. We consider the following problem

$$\varepsilon u''(x)+(1+\varepsilon)u'(x)+u(x)=0,\ \ x\epsilon[0,1],$$
$$u(0)=0,\qquad u(1)=1.$$

The exact solution of the problem is $u(x)=\frac{\left(e^{-\frac{x}{\varepsilon}}-e^{-x}\right)}{\left(e^{-\frac{1}{\varepsilon}}-e^{-1}\right)}$. The exact solution of the problem is compared with the solution which is obtained from multiple scales method in Figure 1.

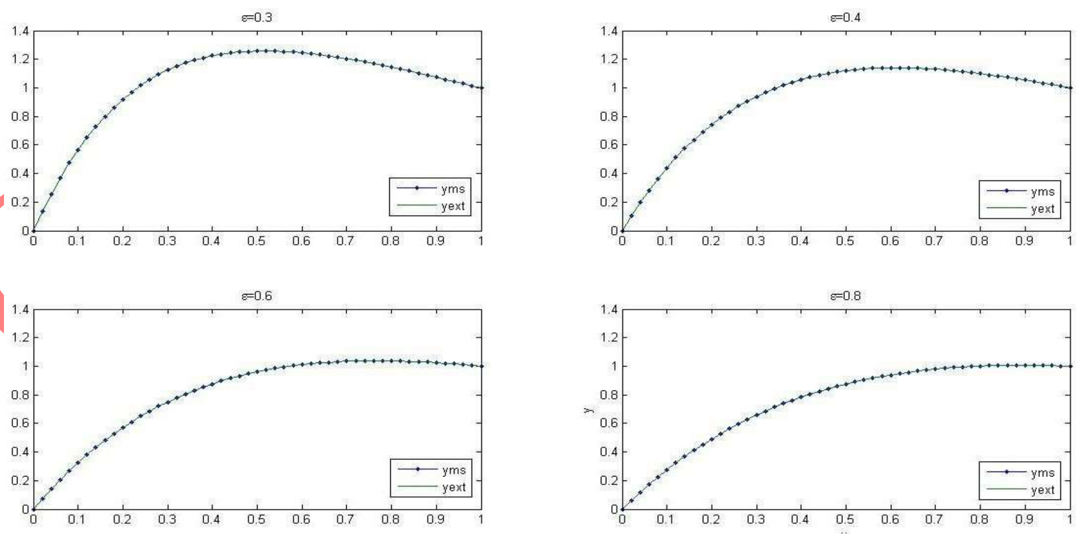

Figure 1. Comparison of multiple scales method and exact solution

On the other hand, the exact solution of the problem and numerical solution are illustrated in Figure 2.

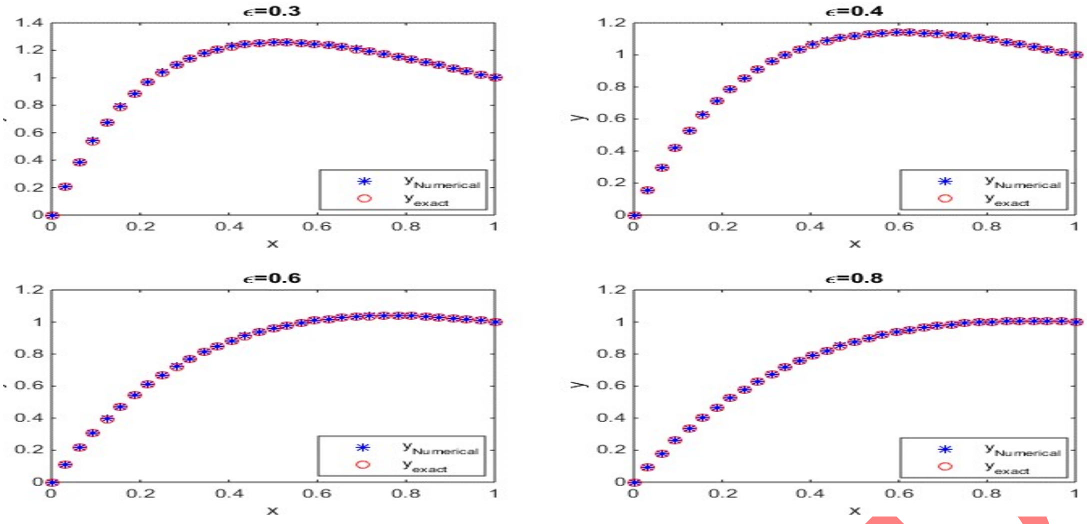

Figure 2. Comparison of numerical solution and exact solution

Furthermore, the computational results are presented on Tables (1-4) for different values of ε.

Table 1. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.3$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.10	0.56683630	0.57166634	0.56683630	0.00483004	0.00000000
0.20	0.91905065	0.92550574	0.91905065	0.00645508	0.00000000
0.30	1.12261488	1.12898982	0.12261488	0.00637494	0.00000000
0.40	1.22431137	1.22979802	1.22431137	0.00548664	0.00000000
0.50	1.25721917	1.26152632	1.25721917	0.00430715	0.00000000
0.60	1.24464049	1.24775846	1.24464049	0.00311797	0.00000000
0.70	1.20291024	1.20496837	1.20291024	0.00205812	0.00000000
0.80	1.14340543	1.14458862	1.14340543	0.00118319	0.00000000
0.90	1.07398176	1.07448391	1.07398176	0.00050215	0.00000000
1.00	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000

Table 2. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.4$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.10	0.44100450	0.44336159	0.44100449	0.00235709	0.00000000
0.20	0.74249202	0.74585401	0.74249202	0.00336199	0.00000000
0.30	0.93931731	0.94258805	0.93931731	0.00354073	0.00000000
0.40	1.05824523	1.06149247	1.05824522	0.00324724	0.00000000
0.50	1.11977637	1.12249081	1.11977636	0.00271444	0.00000000
0.60	1.13956546	1.14165652	1.13956546	0.00209106	0.00000000
0.70	1.12952287	1.13099090	1.12952287	0.00146803	0.00000000
0.80	1.14340565	1.09956688	1.09866965	0.00089723	0.00000000
0.90	1.05380088	1.05420560	1.05380088	0.00040472	0.00000000
1.00	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000

Table 3. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.6$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.10	0.32600247	0.32689788	0.32600246	0.00089541	0.00000000
0.20	0.57093436	0.57228989	0.57093436	0.00135553	0.00000000
0.30	0.75019375	0.75170825	0.75019375	0.00151450	0.00000000
0.40	0.87653387	0.87800673	0.87653386	0.00147286	0.00000000
0.50	0.96049589	0.96180091	0.96049589	0.00130502	0.00000000
0.60	1.01077271	1.01183790	1.01077271	0.00106519	0.00000000
0.70	1.03451458	1.03530666	1.03451457	0.00079209	0.00000000
0.80	1.03758572	1.03809833	1.03758571	0.00051262	0.00000000
0.90	1.02477970	1.02502449	1.02477969	0.00024479	0.00000000
1.00	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000

Table 4. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.8$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.10	0.27453902	0.27501941	0.27453902	0.00048039	0.00000000
0.20	0.49069302	0.49144053	0.49069302	0.00074751	0.00000000
0.30	0.65780861	0.65866703	0.65780861	0.00085842	0.00000000
0.40	0.78389757	0.78475561	0.78389757	0.00085803	0.00000000
0.50	0.87581619	0.87659759	0.87581619	0.00078139	0.00000000
0.60	0.93942141	0.94007694	0.93942141	0.00065553	0.00000000
0.70	0.97970670	0.98020772	0.97970669	0.00050102	0.00000000
0.80	1.00092017	1.00125345	1.00092017	0.00033328	0.00000000
0.90	1.00666729	1.00683088	1.00666728	0.00016360	0.00000000
1.00	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000

Example 2. We take into account another problem

$$\varepsilon u''(x) - u'(x) - (1+\varepsilon)u(x) = 0,$$

$$u(0) = 1 + \exp\left(\frac{-(1+\varepsilon)}{\varepsilon}\right), \qquad u(1) = 1 + {}^{1}/_{\varepsilon}.$$

The exact solution of the problem is $u(x) = e^{-x} + e^{(1+\varepsilon)^{(x-1)}/\varepsilon}$. The exact solution of the problem is compared with the solution which is obtained from multiple scales method in Figure 3.

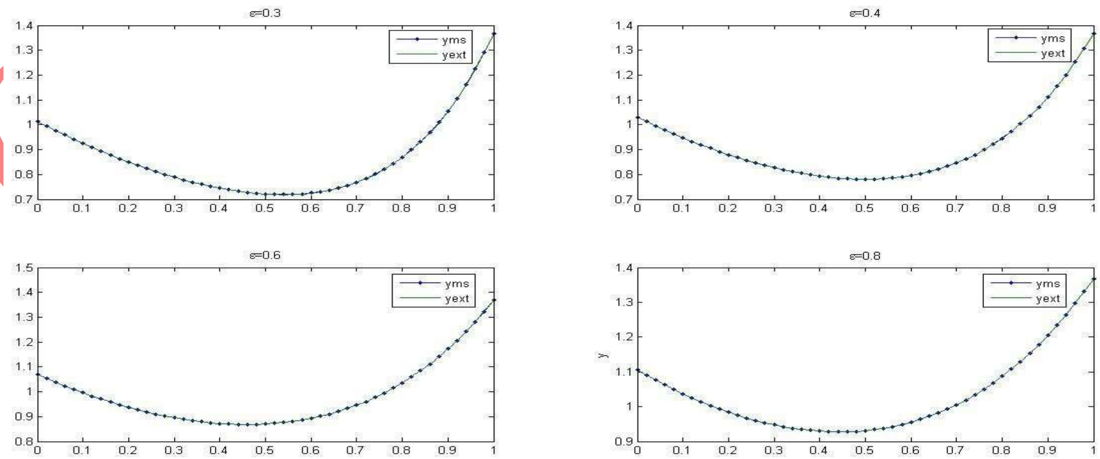

Figure 3. Comparison of multiple scales method and exact solution

Moreover, the exact solution of problem and numerical solution are shown in Figure 4.

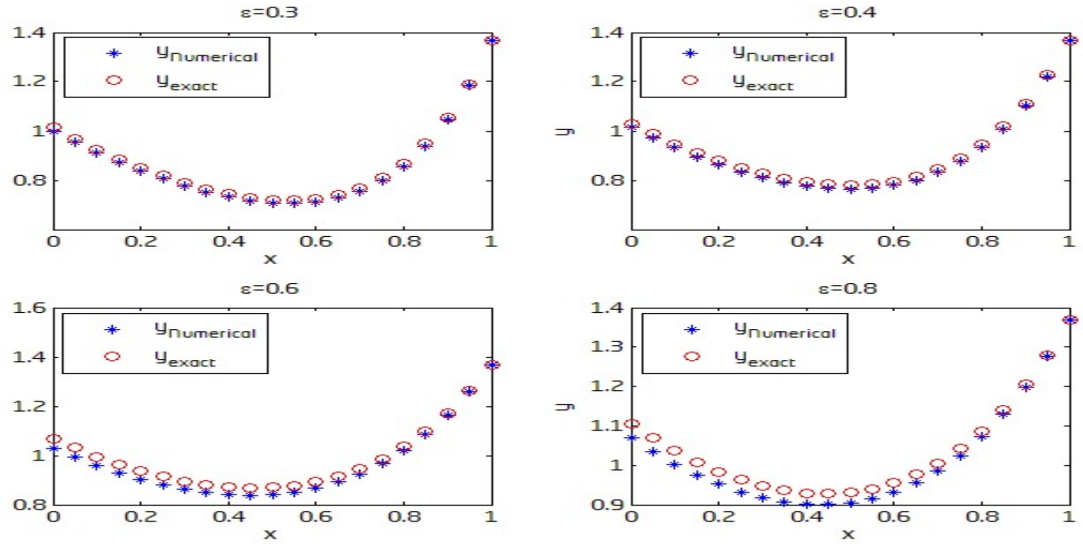

Figure 4. Comparison of numerical solution and exact solution

The obtained results are presented on Tables (5-8).

Table 5. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.3$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	1.01310000	1.01312373	1.01312372	0.00002373	0.00002372
0.10	0.92504273	0.92393266	0.92507932	0.00111007	0.00003659
0.20	0.84989523	0.84757939	0.84995168	0.00231584	0.00005644
0.30	0.78888601	0.78524860	0.78897307	0.00363741	0.00008706
0.40	0.74445933	0.73936235	0.74459362	0.00509698	0.00013429
0.50	0.72088237	0.71421814	0.72108950	0.00666427	0.00020713
0.60	0.72518660	0.71698920	0.72550608	0.00819740	0.00031947
0.70	0.76862434	0.75928991	0.76911709	0.00933443	0.00049275
0.80	0.86891932	0.85962174	0.86967934	0.00929758	0.00076002
0.90	1.05374174	1.04719280	1.05491400	0.00654894	0.00117225
1.00	1.36607316	1.36787944	1.36787944	0.00180808	0.00180807

Table 6. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.4$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	1.03020000	1.03019738	1.03019738	0.00000262	0.02616577
0.10	0.94769325	0.93932533	0.94768954	0.00836792	0.03713100
0.20	0.87954608	0.86142483	0.87954081	0.01812125	0.05269140
0.30	0.82711928	0.79765264	0.82711180	0.02946664	0.07477266
0.40	0.79278708	0.75039715	0.79277647	0.04238993	0.10610745
0.50	0.78031966	0.72391214	0.78030460	0.05500889	0.15057364
0.60	0.79542996	0.72531077	0.79540860	0.07011919	0.21367417
0.70	0.84655337	0.76612089	0.84652305	0.08043248	0.30321809
0.80	0.94595729	0.86471566	0.94591426	0.08124163	0.43028695
0.90	1.11131881	1.05010894	1.11125774	0.06120987	0.61060625
1.00	1.36796609	1.36787944	1.36787944	0.00008665	0.86649152

Table 7. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.6$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	1.06948345	1.06838342	1.06948346	0.00110003	0.16548777
0.10	0.99555537	0.97474348	0.99555538	0.02081189	0.21606169
0.20	0.93717258	0.89328284	0.93717259	0.04388974	0.02820912
0.30	0.89545648	0.82619405	0.89545649	0.06926243	0.03682998
0.40	0.87221656	0.77578796	0.87221657	0.00964286	0.04808541
0.50	0.87012779	0.74621782	0.87012781	0.12390997	0.06278056
0.60	0.89296542	0.74445849	0.89296544	0.14850693	0.08196663
0.70	0.94591426	0.78183882	0.94591428	0.16407544	0.10701605
0.80	1.03597518	0.87643667	1.03597519	0.15953851	0.13972071
0.90	1.17249799	1.05681891	1.17249802	0.11567908	0.18242008
1.00	1.36777946	1.36787944	1.36787946	0.00009998	0.23816861

Table 8. Comparison of multiple scale method and finite difference method for $\varepsilon = 0.8$

x	Exact Solution	FDM	MS	Errors of FDM	Errors of MS
0.00	1.10540000	1.10539922	1.10539922	0.00000788	0.07754381
0.10	1.03683223	1.00712317	1.03683126	0.02970906	0.09710987
0.20	0.98403085	0.92240780	0.98402964	0.06162305	0.12161290
0.30	0.94782729	0.85228694	0.94782577	0.09554035	0.15229860
0.40	0.92956221	0.79900052	0.92956030	0.01305617	0.19072700
0.50	0.93118551	0.76660992	0.93118312	0.16457559	0.23885176
0.60	0.95538428	0.76196355	0.95538129	0.19342073	0.29911948
0.70	1.00574547	0.79620833	1.00574172	0.20953714	0.37459412
0.80	1.08696180	0.88715214	1.08695711	0.19980966	0.46911273
0.90	1.20509175	1.06295325	1.20508587	0.14213850	0.58748053
1.00	1.36788679	1.36787944	1.36787944	0.00000735	0.73571521

4. Discussion and Conclusion

In this paper, singularly perturbed convection-diffusion equations are treated. We obtain the multiple scale approximation solution that it's convergence rate is $O(\varepsilon^2)$. Moreover, exponentially fitted difference scheme is constructed on uniform mesh. It is obtained that the approximate rate of difference scheme is $O(h)$. Computational results are presented on the Tables (1-8) for both of methods. Multiple scale method produced better results for small values of ε. For finite difference method, optimal results are obtained when $\frac{h}{\varepsilon} \cong 1$. Numerical investigations can be carried out for various types such as delay, partial derivative forms with different physical properties of these equations .

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