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Research Article

(4+1)-Boyutlu Fokas Denkleminin Etkili Bir İntegrasyon Tekniği ile Tek Dalga Çözümleri

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Öz

Bu çalışmada yüksek boyutlu problemler içerisinde kendine özgü bir öneme sahip olan integrrallenebilir nonlineer (4+1)-boyutlu Fokas denkleminin soliton çözümleri son zamanlarda literature kazandırılmış olan yeni Kudryashov metodu ile incelenmektedir. Yapılan inceleme çerçevesinde (4+1)-boyutlu Fokas denkleminin temel soliton çözümlerinin elde edilmesine ek olarak, yöntemin yüksek boyutlu problemler için de etkin olarak kolaylıkla kullanılabileceği, aynı zamanda güvenilir olduğu da gösterilmektedir. Çalışmada elde edilen soliton çözümlerine ait grafiklerin 3D, 2D ve contour sunumları yapılarak ayrıca gerekli açıklamalar yapılmıştır.

Anahtar Kelimeler: (4+1)-Fokas denklemi, Yeni Kudryashov yöntemi, Parlak soliton, Soliton çözümü.

Solitary Wave Solutions of the (4+1)-Dimensional Fokas Equation Via an Efficient Integration Technique

Abstract

In this study, the soliton solutions of the integrable nonlinear (4+1)-dimensional Fokas equation, which has a unique importance in high-dimensional problems, are examined by the new Kudryashov method, which has recently been introduced into literature. In addition to obtaining the basic soliton solutions of the (4+1)-dimensional Fokas equation, it is showed that the method can be easily used effectively for high-dimensional problems and is also reliable. 3D, 2D and contour presentations of the graphs of the soliton solutions obtained in the study were made and the necessary explanations were also made.

Keywords: (4+1)-Fokas equation, The new Kudryashov Method, Bright soliton, Soliton solution.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) are widely used predominantly in the field of engineering to model many physical phenomena. Considering that there are very common branches of engineering today, this field of use covers many areas such as physics, biology, chemistry, genetics, statistics, earth science, medicine, meteorology, hydrodynamics, solid physics, quantum mechanics, wave dynamics, nonlinear optics [1-10]. Especially in the last 50 years, technological developments in the field of electronics and computers have brought about many advances in the software sector, some symbolic calculation tools (such as mathematica, matlab, maple, etc.) have been developed for both numerical and symbolic programming and thus great progress has been made in solving many non-linear problems waiting to be solved before. Many researchers started to work in this field and developed new equations (for example, Kadomtsev Petviasvili (KP) and its variants, Kdv forms, Boussinessq forms, Maccari systems, Kawahara equation, Vakhnenko-Parkes equation, Caudrey-Dodd-Gibbon-KP equation, Bogoyavlenskii-KP equation, KP-Joseph-Egri equation, Zoomeron equation and its variants, Kudryashov equation, Schrödinger equation and its more variants, Hirota equation, Schrödinger-Hirota equation, Hirota-Satsuma equation, Boussinesq-Peregrine equation, Davey Stewartson (DS), Fokas-Lenells, Ginzburg Landau, Landau-Lifschitz-Gilbert, Calogero-Bogoyavlenskii Schiff equation and its variants, Pochhammer-Chree, Newell-Whitehead, Fitzhugh-Nagumo, Fisher equation, Phi Four equation, Kolmogorov-Petrovskii-Piskunov equation, Drinfeld-Sokolov system, Drinfeld-Sokolov-Satsuma-Hirota system, Mikhailov-Novikov-Wang equation, Krichever–Novikov equation, Jaulent–Miodek evolution equation, Burgers-Huxley equation, Chen-Lee-Liu equation, Eckhaus-Kundu, Radhakrishnan-Kundu-Lakshmanan , Gerdjikov–Ivanov, Lakshmanan–Porsezian–Daniel, Zakharov-Kuznetsov, Benjamin-Bona-Mahony, Gardner-Ostrosvky, Bogoyavlensky Konopelchenko, Whitham-Broer-Kaup, Ablowitz–Ramani–Segur equation, Biswas-Milovic equation, Biswas-Arshed equation, Triki-Biswas model, Fornberg-Whitham equation, Camassa-Holm equation and its variants, Degasperis-Procesi equation, Van der Pol type Rayleigh wave equation, Sharma-Tasso-Olver equation and many more) [11-20]. The equations listed above reflect only a fraction of the equations developed in this field and studied by many researchers. Because, considering the different dimensions (1+1, 2+1, 3+1, 4+1), nonlinearity laws (Kerr, power, parabolic, dual power, log, polynomial, cubic, anti-cubic, quadratic-cubic, septic, sextic etc.), perturbation, and higher order forms [21-30] of these equations, it is better understood how current and important the studies on this field are. Obtaining integrable high-dimensional forms of some equations within these problems is both important in itself and also physically important. In this context, although the Kdv and Davey-Stewartson equations are the origin, one of these equations is the Fokas equation.

The (4+1)-dimensional Fokas equation (FE) reads [31]:

$$4\frac{\partial^2\Theta}{\partial x\partial t}-\frac{\partial^4\Theta}{\partial x^3\partial y}+\frac{\partial^4\Theta}{\partial x\partial y^3}+12\left(\frac{\partial\Theta}{\partial x}\frac{\partial\Theta}{\partial y}+\Theta\frac{\partial^2\Theta}{\partial x\partial y}\right)-6\frac{\partial^2\Theta}{\partial\omega\partial z}=0, \tag{1.1}$$

$$\Theta(x,y,z,\omega,t)=\Theta(\eta) \quad , \quad \eta=\lambda_1 x+\lambda_2 y+\lambda_3 z+k\omega+ct \ . \tag{1.2}$$

This equation was obtained by Fokas as an extension of the integrable KP and DS equations to some high-order nonlinear wave equations using the Lax pair method [31]. As it is known, KP and DS equations are important wave models describing surface and interior waves in throats and channels of varying width and depth, as well as being used for three-dimensional models on finite depth water waves. In Eq.(1.2), $\lambda_1,\lambda_2,\lambda_3,k$ and c are real constants. To date, different studies have been conducted on Eq.(1.1) and examined with different methods. For example the the extended simplest equation and modified simple equation methods [32], improved F-expansion and generalized exp-(-$\phi(\xi)$)-expansion methods [33], the Hirota's bilinear and the KP hierarchy reduction methods [34], Jacobian-function method, He's semi-inverse variational technique and sine–cosine or triangle function approach the Padés type transformation [35], generalized exponential rational function [36], modified exp (-$\Omega(\xi)$)-expansion function [37], improved tanh-coth [38], extended version of exp($-\psi(\kappa)$)-expansion [39].

The manuscript has the following body: Section 2 is devoted to the introduction of the method, Section 3 to the application of the method, Section 4 to the results and discussion obtained in the article, and Section 5 to the conclusions.

2. Presentation of the new Kudryashov method

Step 1: Taken into account the Eq.(2) and Eq.(3):

$$Z\left(\Theta,\Theta_x,\Theta_y,\Theta_z,\Theta_\omega,\Theta_t,\Theta_{xx},...,\Theta_{tt},\Theta_{xy},...,\Theta_{\omega t},...\right)=0\,, \tag{2}$$

$$\Theta(x,y,z,t)=\kappa(\eta),\ \ \eta=\rho(x+y+z-\sigma t). \tag{3}$$

In Eq.(2), Z is a polynomial in Θ, ρ and σ are arbitrary constants ($\rho,\sigma\neq 0$). Plugging Eq.(3) into Eq.(2), we obtain the following nonlinear ordinary differential equation (NLODE):

$$P\left(\kappa(\eta),\kappa'(\eta),\kappa''(\eta),...\right)=0\,. \tag{4}$$

where P is a polynomial in $\kappa(\eta)$, $\kappa'(\eta)=d\kappa(\eta)/d\eta$, $\kappa''(\eta)=d^2\kappa(\eta)/d\eta^2$.

Step 2: The new Kudryashov scheme offers that the Eq.(5) is the solution of Eq.(4).

$$\Theta(\eta)=\sum_{i=0}^{r}\phi_i\kappa^i(\eta) \quad , \quad \phi_r\neq 0 . \tag{5}$$

In Eq.(5), $\phi_0,...,\phi_r$ are real values to be found. We obtain positive integer r which is the balancing term by balancing the highest derivative term and the highest power nonlinear term in Eq.(4). Furthermore, the function $\kappa(\eta)$ satisfies the following auxiliary equation [40]:

$$\left[\frac{d\kappa(\eta)}{d\eta}\right]^2=\kappa^2(\eta)\left[1-\lambda\kappa^2(\eta)\right] , \tag{6}$$

$$\kappa(\eta)=\mp\frac{4L}{4L^2e^{\eta}+\lambda e^{-\eta}} . \tag{7}$$

In Eq.(6) and Eq.(7), λ and L are arbitrary real constants to be determined.

Step 3: Plugging the Eq.(5) and its derivatives into Eq.(4) produces a polynomial form of $\kappa(\eta)$. Collecting the coefficients of $\kappa(\eta)$ with the same power and equating each coefficients to zero, we derive a algebraic equations for $\phi_{-r},...\phi_0,...,\phi_r$, ρ , σ and k.

Step 4: Solution of Step 3, serves the various sets. Combining the appropriate sets with Eq.(7), Eq.(3) and Eq.(5) the solutions of Eq.(1.1) are obtained.

3. Application of the new Kudryashov method

Applying the wave transformation in Eq.(1.2) to NLPDE equation in Eq.(1.1) we reduce to Eq.(1.1) to following NLODE form.

$$\left(4c\lambda_1-6k\lambda_3\right)\Theta+6\Theta^2\lambda_1\lambda_2+\lambda_1\lambda_2\left(\lambda_2^{\,2}-\lambda_1^{\,2}\right)\frac{d^2\Theta}{d\eta^2}=0 \tag{8}$$

where $\Theta=\Theta(\eta)$. Balancing Θ'' and Θ^2 , gives $r=2$. According to Eq.(5), this offers a truncated series as the form of :

$$\Theta(\eta)=a_0+a_1\kappa(\eta)+a_2\kappa^2(\eta) \ , \ a_2\neq 0 . \tag{9}$$

Substituting Eq.(9) and Eq.(6) into Eq.(8) and equating the coefficient of each power of $\kappa(\eta)$ to zero, we obtain following equations:

$$\begin{aligned}
&\kappa(\eta)^0: 6\lambda_1\lambda_2a_0^{\,2}+4a_0c\lambda_1-6a_0k\lambda_3=0,\\
&\kappa(\eta)^1: \left(-\lambda_1^{\,3}\lambda_2+\left(\lambda_2^{\,3}+12a_0\lambda_2+4c\right)\lambda_1-6k\lambda_3\right)a_1=0,\\
&\kappa(\eta)^2: -4a_2\lambda_1^{\,3}\lambda_2+\left(\left(4\lambda_2^{\,3}+12a_0\lambda_2+4c\right)a_2+6a_1^{\,2}\lambda_2\right)\lambda_1-6ka_2\lambda_3=0,\\
&\kappa(\eta)^3: 2\lambda_2a_1\left(\left(\lambda_1^{\,2}-\lambda_2^{\,2}\right)\lambda+6a_2\right)\lambda_1=0,\\
&\kappa(\eta)^4: 6\lambda_1a_2\lambda_2\left(\left(\lambda_1^{\,2}-\lambda_2^{\,2}\right)\lambda+a_2\right)=0,
\end{aligned} \tag{10}$$

Solving the system by suitable method we obtain the following solution sets:

$$\begin{aligned}
&Kset^{1,1}=\left\{\lambda_3=\frac{\left(-2\lambda_1^{\,2}\lambda_2+2\lambda_2^{\,3}+2c\right)\lambda_1}{3k},a_0=0,a_1=0,a_2=-\lambda\lambda_1^{\,2}+\lambda\lambda_2^{\,2}\right\},\\
&Kset^{1,2}=\left\{\lambda_3=\frac{\left(2\lambda_1^{\,2}\lambda_2-2\lambda_2^{\,3}+2c\right)\lambda_1}{3k},a_0=-\frac{2\lambda_2^{\,2}}{3}+\frac{2\lambda_1^{\,2}}{3},a_1=0,a_2=-\lambda\lambda_1^{\,2}+\lambda\lambda_2^{\,2}\right\},\\
&Kset^{1,3}=\left\{k=\frac{\left(-2\lambda_1^{\,2}\lambda_2+2\lambda_2^{\,3}+2c\right)\lambda_1}{3\lambda_3},a_0=0,\ a_1=0,a_2=-\lambda\lambda_1^{\,2}+\lambda\lambda_2^{\,2}\right\},\\
&Kset^{1,4}=\left\{k=\frac{\left(2\lambda_1^{\,2}\lambda_2-2\lambda_2^{\,3}+2c\right)\lambda_1}{3\lambda_3},a_0=-\frac{2\lambda_2^{\,2}}{3}+\frac{2\lambda_1^{\,2}}{3},a_1=0,a_2=-\lambda\lambda_1^{\,2}+\lambda\lambda_2^{\,2}\right\},\\
&Kset^{1,5}=\left\{c=\frac{2\lambda_1^{\,3}\lambda_2-2\lambda_1\lambda_2^{\,3}+3k\lambda_3}{2\lambda_1},a_0=0,a_1=0,a_2=-\lambda\lambda_1^{\,2}+\lambda\lambda_2^{\,2}\right\},\\
&Kset^{1,6}=\left\{c=\frac{-2\lambda_1^{\,3}\lambda_2+2\lambda_1\lambda_2^{\,3}+3k\lambda_3}{2\lambda_1},a_0=-\frac{2\lambda_2^{\,2}}{3}+\frac{2\lambda_1^{\,2}}{3},a_1=0,a_2=-\lambda\lambda_1^{\,2}+\lambda\lambda_2^{\,2}\right\}.
\end{aligned} \tag{11}$$

According to Eq.(11) we construct the following solution functions for Eq.(1.1). (All functions satisfy Eq.(1.1))

$$\Theta_{1,1}(\mathrm{x,y,z},\omega,\mathrm{t})=\frac{16\lambda\left(\lambda_2^{\,2}-\lambda_1^{\,2}\right)L^2}{\left(4L^2e^{\chi_1}+\lambda e^{-\chi_1}\right)^2} \tag{12}$$

$$\Theta_{1,2}(\mathrm{x,y,z},\omega,\mathrm{t})=-\frac{2\lambda_2^{\,2}}{3}+\frac{2\lambda_1^{\,2}}{3}+\frac{16\lambda\left(\lambda_2^{\,2}-\lambda_1^{\,2}\right)L^2}{\left(4L^2e^{\chi_2}+\lambda e^{-\chi_2}\right)^2} \tag{13}$$

$$\Theta_{1,3}(\mathrm{x,y,z},\omega,\mathrm{t})=\frac{16\lambda\left(\lambda_2^{\,2}-\lambda_1^{\,2}\right)L^2}{\left(4L^2e^{\chi_3}+\lambda e^{-\chi_3}\right)^2} \tag{14}$$

$$\Theta_{1,4}(\mathrm{x,y,z},\omega,\mathrm{t})=-\frac{2\lambda_2^{\,2}}{3}+\frac{2\lambda_1^{\,2}}{3}+\frac{16\lambda\left(\lambda_2^{\,2}-\lambda_1^{\,2}\right)L^2}{\left(4L^2e^{\chi_4}+\lambda e^{-\chi_4}\right)^2} \tag{15}$$

$$\Theta_{1,5}(\mathrm{x,y,z},\omega,\mathrm{t})=\frac{16\lambda\left(\lambda_2^{\,2}-\lambda_1^{\,2}\right)L^2}{\left(4L^2e^{\chi_5}+\lambda e^{-\chi_5}\right)^2} \tag{16}$$

$$\Theta_{1,6}(\mathrm{x,y,z},\omega,\mathrm{t})=-\frac{2\lambda_2^{\,2}}{3}+\frac{2\lambda_1^{\,2}}{3}+\frac{16\lambda\left(\lambda_2^{\,2}-\lambda_1^{\,2}\right)L^2}{\left(4L^2e^{\chi_6}+\lambda e^{-\chi_6}\right)^2} \tag{17}$$

where

$$\chi_1=ct+k\omega+\lambda_1x+\lambda_2y+2\frac{\left(-\lambda_1^{\,2}\lambda_2+\lambda_2^{\,3}+c\right)\lambda_1z}{3k},$$

$$\chi_2=ct+k\omega+\lambda_1x+\lambda_2y+2\frac{\left(\lambda_1^{\,2}\lambda_2-\lambda_2^{\,3}+c\right)\lambda_1z}{3k},$$

$$\chi_3=ct+2\frac{\left(-\lambda_1^{\,2}\lambda_2+\lambda_2^{\,3}+c\right)\lambda_1\omega}{3\lambda_3}+\lambda_1x+\lambda_2y+\lambda_3z,$$

$$\chi_4=ct+2\frac{\left(\lambda_1^{\,2}\lambda_2-\lambda_2^{\,3}+c\right)\lambda_1\omega}{3\lambda_3}+\lambda_1x+\lambda_2y+\lambda_3z,$$

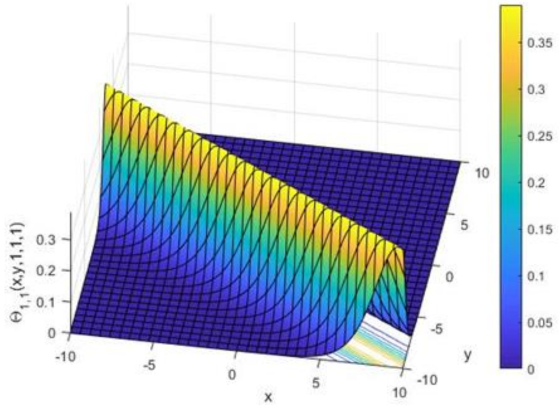
$$\chi_5=\frac{\left(2\lambda_1^{\,3}\lambda_2-2\lambda_1\lambda_2^{\,3}+3k\lambda_3\right)t}{2\lambda_1}+k\omega+\lambda_1x+\lambda_2y+\lambda_3z,$$

$$\chi_6=\frac{\left(-2\lambda_1^{\,3}\lambda_2+2\lambda_1\lambda_2^{\,3}+3k\lambda_3\right)t}{2\lambda_1}+k\omega+\lambda_1x+\lambda_2y+\lambda_3z.$$

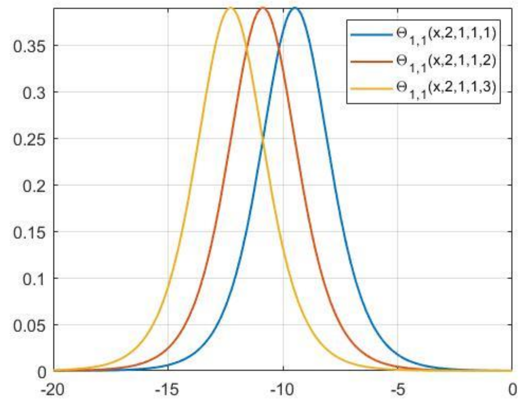
4. Results and Discussion

In this section, graphic presentations are made about the soliton solutions obtained in the article.

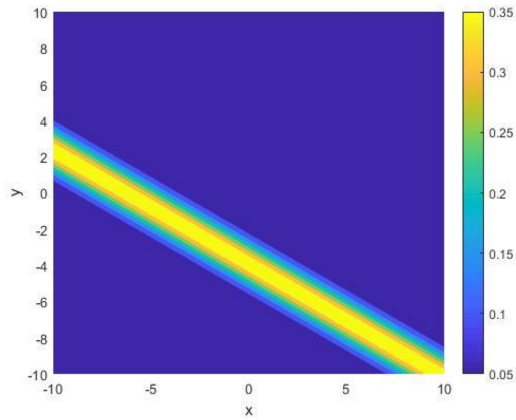
Fig. 1 belongs to $\Theta_{1,1}(\mathrm{x,y,z},\omega,\mathrm{t})$ given by Eq. (12), and its 3D, 2D, contour figures are given in (a), (b), (c) respectively. For this, the $\mathrm{KSet}^{1,1}$ given by Eq. (11) and the parameter values $\lambda=1, L=1.5, \lambda_1=0.5,\ \lambda_2=0.8, k=1, c=0.7,\ z=\omega=t=1$ are chosen. Fig.1a represents the bright soliton, one of the basic soliton types, and Fig.1b shows the state of the wave at t=1,2,3 time values and $y=2$. The wave has the character of a walking wave. Fig.1c reflects the contour graph of the projection of the wave on the xy plane and reinforces the bright soliton character of the wave.



(a) 3D profile



(b) 2D projection

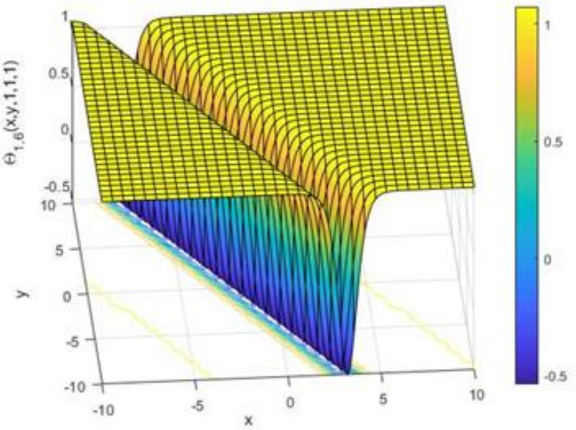


(c) Contour view

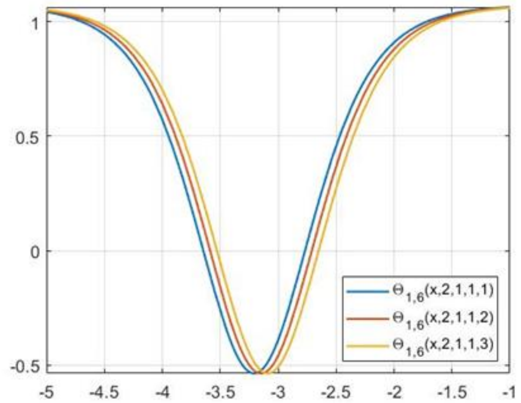
Fig. 1 The some illustrations of $\Theta_{1,1}(x, y, z, \omega, t)$ in Eq.(12) with $KSet^{1,1}$ in Eq.(11)

and $\lambda = 1, L = 1.5, \lambda_1 = 0.5, \lambda_2 = 0.8, c = 0.7, k = z = \omega = t = 1$

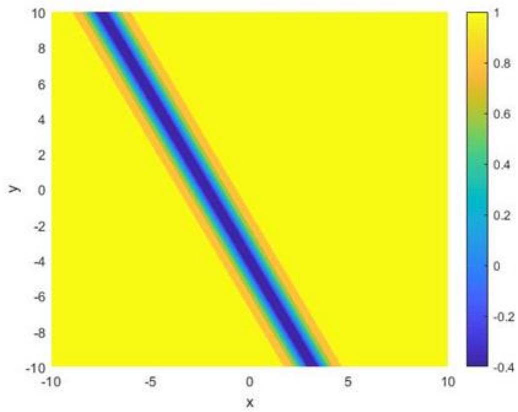
Fig. 2 belongs to $\Theta_{1,6}(x, y, z, \omega, t)$ in Eq.(17), and its 3D, 2D, contour figures are given in (a), (b), (c) respectively. For this, the $KSet^{1,6}$ given by Eq. (11) and the parameter values $\lambda = 1, L = 1.5, \lambda_1 = 1.5, \lambda_2 = 0.8, \lambda_3 = 1.2, k = 1, z = \omega = t = 1$ are chosen. Fig.2a represents the dark soliton, one of the basic soliton types, and Fig.2b illustrates the state of the wave at t=1,2,3 values and $y = 2$. The wave has the character of a traveling wave. Fig. 2c reflects the contour graph of the projection of the wave on the xy plane and reinforces the dark soliton character of the wave.



(a) 3D profile



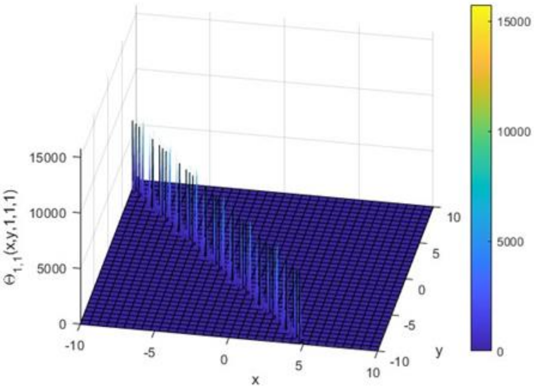
(b) 2D projection



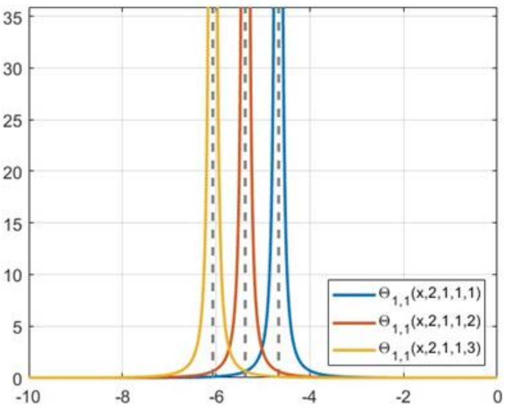
(c) Contour view

Fig. 2 The some depictions of $\Theta_{1,6}(x, y, z, \omega, t)$ in Eq.(17) with $KSet^{1,6}$ in Eq.(11) and $\lambda = 1,\ L = 1.5,\ \lambda_1 = 1.5,\ \lambda_2 = 0.8,\ \lambda_3 = 1.2,\ k = z = \omega = t = 1$

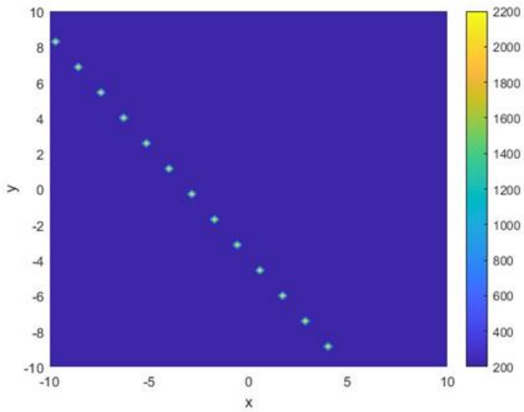
Fig. 3 belongs to $\Theta_{1,1}(x, y, z, \omega, t)$ in Eq. (12), and its 3D, 2D, contour figures are given in (a), (b), (c) respectively. For this, the $KSet^{1,1}$ given by Eq. (11) and the parameter values $\lambda = -1, L = 1.5, \lambda_1 = 1,\ \lambda_2 = 0.8, k = 1, c = 0.7,\ \ z = \omega = t = 1$ are chosen. Fig.3a represents the singular solution (bright-singular), one of the basic solution types, and Fig.3b shows the state of the wave at t=1,2,3 values and $y = 2$. Fig. 3c depicts the contour view of the projection of the wave on the xy plane and reinforces the singular solution of the wave.



(a) 3D profile



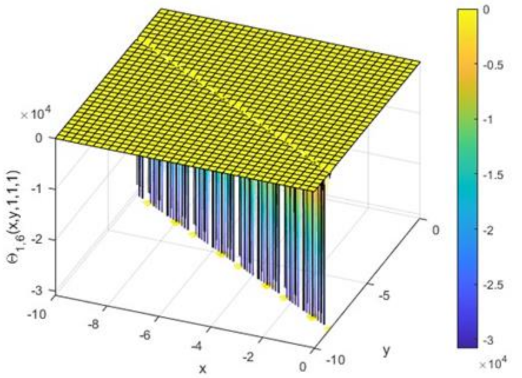
(b) 2D projection



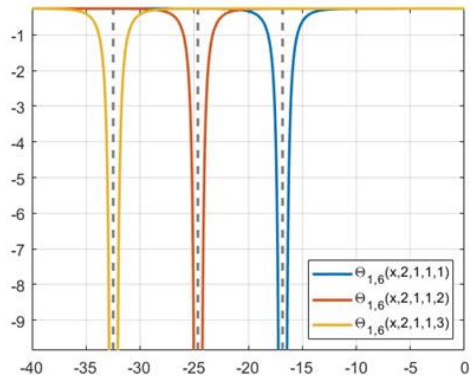
(c) Contour view

Fig. 3 The some illustrations of $\Theta_{1,1}(x, y, z, \omega, t)$ in Eq.(12) with $KSet^{1,1}$ in Eq.(11) and $\lambda = -1,\ L = 1.5,\ \lambda_1 = 1,\ \lambda_2 = 0.8,\ c = 0.7,\ k = z = \omega = t = 1$

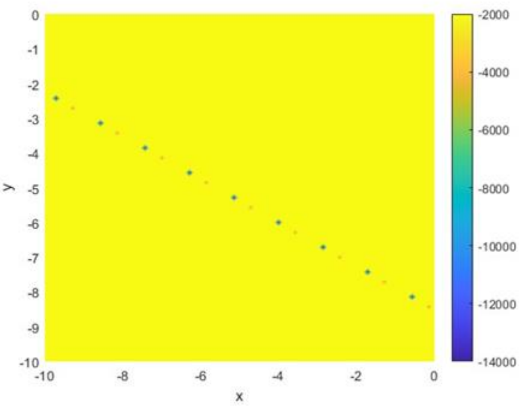
Fig. 4 is the portrait of the $\Theta_{1,6}(x, y, z, \omega, t)$ given by Eq.(17), and its 3D, 2D, contour figures are given in (a), (b), (c) respectively. For this, the $KSet^{1,6}$ given by Eq. (11) and the parameter values $\lambda = -0.25, L = 0.5, \lambda_1 = 0.5,\ \lambda_2 = 0.8,\ \lambda_3 = 1.2,\ z = \omega = t = k = 1$ are chosen. Fig.4a represents the singular solution (dark-singular), one of the basic solution types, and Fig.4b shows the state of the wave at t=1,2,3 values and $y = 2$. Fig. 4c projects the contour view of the projection of the wave on the xy plane and reinforces the singular solution of the wave.



(a) 3D profile



(b) 2D projection



(c) Contour view

Fig. 4 The some depictions of $\Theta_{1,6}(x,y,z,\omega,t)$ in Eq.(17) with $KSet^{1,6}$ in Eq.(11) and $\lambda=-0.25,\ L=0.5,\ \lambda_1=0.5,\ \lambda_2=0.8,\ \lambda_3=1.2,\ k=z=\omega=t=1$

5. Conclusions

In this study, a study was conducted to examine the limited number of high-dimensional nonlinear equations in the literature. The soliton solutions of the (4+1)-dimensional Fokas equation, which has an important place in terms of wave dynamics, were solved by the new Kudryashov method, which has recently entered the literature and has become widely used. Bright, dark and singular soliton solutions of the examined problem were obtained and graphic presentations were made of them. The examined method was applied for the first time on the problem and soliton solutions and graphical representations of the basic soliton types were successfully made. Apart from showing that the (4+1)-dimensional Fokas equation is a model that produces the basic soliton types, the study also shows the new Kudryashov method is a method that can be used effectively in solving high-dimensional problems. The next target studies include different solution techniques for the solution of such high-dimensional problems, expanding the physical interpretations of these equations, researching other possible soliton wave types and their interactions. We believe that both the study methodology and the results of the study will contribute to those who conduct research in this field.

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