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The Wonder World of Complex Systems

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ABSTRACT

Complex systems pervade nature and form the core of many technological applications. An exciting feature of these systems is that they exhibit a wide range of temporal behaviors, ranging from collective motion, synchronization, pattern formation, and chaos, among others. This has not only caught the attention of scientists, but also the interest of a wider audience. Consequently, our goal in this work is to provide a simple but descriptive explanation of some concepts related to complex systems. Specifically, the reader embarks on a journey that begins in the 17th century with the discovery of synchronization by Dutch scientist Christiaan Huygens and ends in the chaotic world explored by meteorologist Edward Lorenz around 1963. The journey is filled with examples, including synchronized clocks and metronomes, electronic fireflies that flash harmoniously, and even a chaotic dress.

KEYWORDS
Complex systems
Synchronization
Chaos
Emergent behavior
Collective behavior
Chaotic dynamics

INTRODUCTION

The term *complex system* is increasingly used nowadays. It is is used for describing phenomena ranging from our daily lives to behaviors typical of the scientific field, and has implications for social sciences, anthropology, mathematics, and biology, to name a few (Ottino 2003; Larsen-Freeman and Cameron 2008). This interdisciplinary field of science aims to study, characterize, and understand complex systems, their interactions, physical/biological effects, and the mechanisms that produce their particular behaviors (Huerta-Cuéllar *et al.* 2022).

It is relatively easy to identify a complex system using climate as an example, but it is somewhat more complex to clearly define the concept itself. This is because this definition changes depending on the field of application and adapts to the research subject's own needs. For example, in computer science, a complex system may refer to the computational time required by the processor to estimate the solution, while in biology it may refer to the interactions between different species in a wild area.

Although different definitions can be found in the literature, complex systems have in common the fact that they consist of various interconnected, interdependent, adaptive, and temporally changing actors whose interactions lead to emergent phenomena (Ladyman *et al.* 2013). In general, we can define a complex system as an organized and inseparable entity that consists of different interconnected parts and, considered as a whole, exhibits properties and behaviors that do not result from the sum of the individual parts or behaviors of any of its elements.

In other words, it is possible to know each of the agents that are part of a Complex System (CS), as well as their independent dynamics, but since they are interconnected and interact with each other, behaviors arise that are not very obvious based on individual knowledge of each element. Because of this peculiarity, complex systems are studied as living entities where it is necessary to consider all the elements and interactions that make them up. Let us take as an example the flight of a bird compared to the flight of a flock (Wang and Lu 2019). We can study individually the behavior of a bird and the mechanisms it needs to take to the skies.

We are able to understand the mechanics of wing flapping, the dynamics of the airflow that allows it to fly, the density and distribution of its feathers, and the limitations of the bird when flying at higher altitudes or speeds. Knowing all this about a single bird, one cannot predict (without prior knowledge) that a flock of birds (of the same species) will behave in such a way and form the flight patterns necessary to fly long distances or to protect the young from predators. This lack of answers in extrapolating data is the prerequisite for studying complex systems as living entities, and it is the behavior that arises from the interactions between them that we call emergent behavior, i.e., it is impossible to obtain

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said result by studying and interacting only one of the entities that make up the CS.

Once the concept of complex system is defined, realize that they are present in many of the phenomena that surround us, that we are a part of them, and that this type of behavior exists in our bodies. Classic examples of complex systems can be found in something as mundane as the behavior of climate, which is one of the most studied complex systems. The behavior of stocks and all currencies, or the dynamics of planets and galaxies are examples of complex systems. The brain behavior and the transmission of information by neurons are examples of CS's which takes place in our bodies. The transmission of electrical energy, traffic in the air and on land are also clear examples of complex systems.

Note that in each of these examples it is possible to know the behavior of the individual elements that compose the CS, but we cannot estimate their behavior on the basis of individual dynamics. Take the example of land traffic in a city: it is possible to know the number of vehicles, the layout of roads and their traffic direction, the position of stop signs and traffic lights. But even with all this information and knowledge of the individual elements, it is impossible to predict the exact location and timing of a traffic jam. To understand the complex traffic system, one must study it as a living entity.

The rest of the article deals with two of the most common behaviors in complex systems: Synchronization and Chaos. Equations and proofs are deliberately omitted, and the text focuses on describing and explaining the main ideas about these behaviors. These are in turn illustrated with everyday references and illustrated with videos of simple experiments that the reader can consult on the Internet. The last part of the paper draws some preliminary conclusions.

SYNCHRONIZATION

In our time we speak more and more often of synchronization. We speak of synchrony between a user's cell phone, his TV and his computer. We also speak of synchronization in sports, for example synchronized swimming or rowing, and even in electronic transfers with dynamic keys linked to the cell phone number. Although the meaning of synchronization yields something obvious and commonplace, it can be defined as the coincidence in time of two or more events resulting from the interaction between two dynamic entities, which can be of almost any kind and nature. This makes synchronization an omnipresent behavior that can be found everywhere. The occurrence of synchronized behavior is very common in nature, with examples in biology, ecology, climatology, sociology, technology, and even art (Pikovsky *et al.* 2003; Strogatz 2004; Osipov *et al.* 2007).

To show how widespread this exciting phenomenon is, let us consider the universe, and in particular the Moon, which orbits our planet. The Moon spins on its own axis (rotation) at the same speed it spins around the Earth (translation), in other words, the Moon's rotation and translation speeds are synchronized. Because of this timing, we always see the same side of the moon. This behavior is also found in the animal kingdom. Have you ever observed birds flying in a "V" formation and flapping their wings at the same time? This allows them to use less energy and travel greater distances. Another example of synchronization in animals is a school of fish. A school is a group of synchronized fish that all move at the same speed and in the same direction as their nearest neighbors. Fish join together in schools for two main reasons: to protect themselves and to migrate. Just like birds that migrate or cyclists that group together, fish move in sync to move faster and expend less energy, which helps them survive.

People also synchronize, for example, the members of an orchestra playing in perfect synchronicity. A ballet performing a routine in time to the music, and even at the Olympic Games synchronization is present. There is also a symphony of synchronized rhythms in our bodies. For example, each beat of our heart is controlled by thousands of pacemaker cells that send out electrical impulses that stimulate the heart cells and cause them to contract or relax in a perfectly synchronized rhythm.

The first precursor in literature dealing with the concept of synchronization is the work of the extraordinary Dutch scientist Christiaan Huygens, mathematician, physicist and inventor of the pendulum clock (Pena Ramirez and Nijmeijer 2020). In 1665, Huygens noticed that two pendulum clocks suspended side by side from a crossbeam showed a kind of sympathy, that is, the pendulums of each clock swung at the same frequency, and when disturbed, they returned to the same rate of oscillation after about thirty minutes. In his writings Huygens points out that the main reason for this fact is the connection between the clocks, the crossbar being the said means of communication. Figure 1 shows the original hand drawing made by Huygens. It shows two pendulum clocks suspended from a wooden rod supported by two chairs.

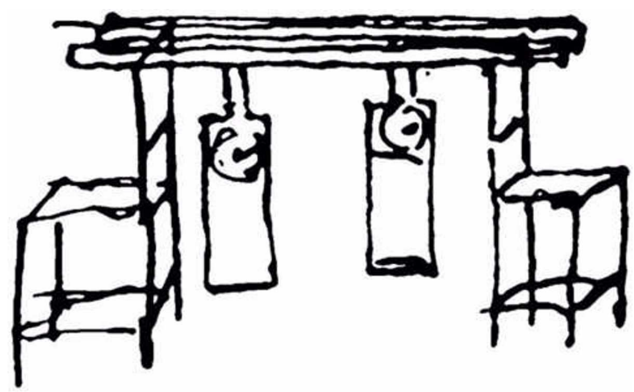

Figure 1 Synchronization scheme developed by Huygens for two pendulum clocks

To demonstrate in a simple way the synchronization between two inanimate objects, and inspired by the same Huygens experiment, we consider three monumental pendulum clocks (they are known as monumental clocks because this type of mechanism is used in towers, churches, obelisks and other monuments) (Pena Ramirez *et al.* 2016). Two of them are connected by a simple wooden rod, while the third has no connection with the other two, see Figure 2 and (Echenausía-Monroy 2022g). Some time after the clocks are set in motion, the two connected mechanisms synchronize, their pendulums working at the same time and in the same direction, while the third clock is free and never follows the rhythm or time of the two synchronized clocks. This illustrates three important points:

- *Synchronization is ubiquitous and can be found in living and inanimate systems;*
- *For synchronization you need at least two systems, agents, complex systems or dynamic units to be synchronized;*
- *For the phenomenon to appear, there must be a means of communication: physical, optical, acoustic, gravitational, electronic, etc.*

Regardless of the size of the object, synchronization is possible. Now consider metronomes, variable frequency pendulum

Figure 2 Modern version of the Huygens experiment, synchronizing two monumental clocks. Note that the means of communication between the two clocks is the orange wooden stick.

clocks that help music students keep time. These metronomes are mounted on a suspended floor structure, with the base free to move depending on the tension of the struts supporting it. When the metronomes are put into operation, they transmit their motion to the base on which they stand, which starts moving according to the number of metronomes. After a certain time, the base transmits this movement back to the metronomes and serves as a means of communication (coupling), giving the metronomes a synchronized response, see Figure 3 and (Echenausía-Monroy 2022c).

Figure 3 Hanging platform with synchronized metronomes (Echenausía-Monroy 2022c).

This phenomenon is similar to that observed when crossing a suspension bridge, where the movement of the pedestrians causes the bridge itself to vibrate so that it sways in the direction of travel, or to the phenomenon observed at the inauguration of the Millennium Bridge in London (2000), where the bridge swayed to the same extent as the pedestrians due to the lightness of the tensioners and the large number of visitors.

As with the size transition between monumental clocks and metronomes, it is possible to reduce the number of clocks and the size of the base to achieve a synchronous response. Now consider three metronomes on a 50 cm board standing on two cans of iced tea. If the clocks operate according to the scheme described, synchronization will occur between the metronomes as they transfer their motion to the table, causing the cans to move slightly and act in place of the struts. After a short time, this transfer of motion will cause the metronomes to operate at the same time and in the same direction. The experimental setup is shown in Figure 4, and the operation can be found in (Echenausía-Monroy 2022i).

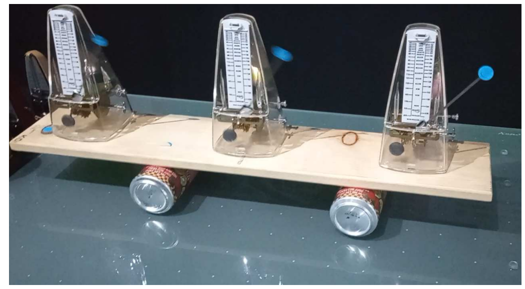

Figure 4 Experimental set-up to synchronize 3 metronomes.

As mentioned earlier, synchronization is not an unknown phenomenon in the animal kingdom; take fireflies, for example. These small insects, which are capable of biologically producing light (bioluminescence), are one of the many examples where synchronization occurs in animals (Buck and Buck 1976). During the breeding season, fireflies migrate to specific forested regions that meet certain climatic conditions. Once there, the males, like males of almost all species, try to attract the attention of females, in this case by the brightness and rhythm of their light. As expected, there is not just one male and one female, but hundreds of them, which leads to the males "seeing" each other. The fact that they see the light of the other male fireflies causes them to synchronize the rhythm of their blinking.

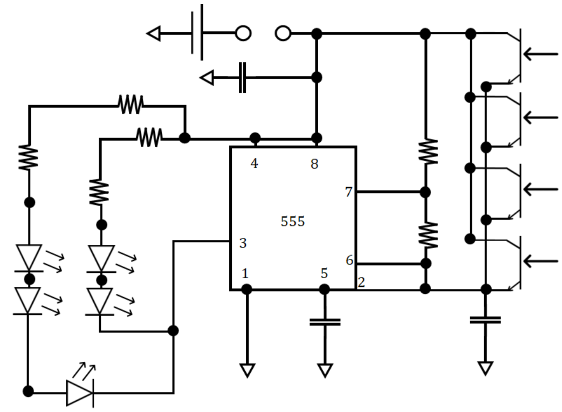

Figure 5 Electronic design of a firefly that can synchronize in four directions.

Using the electronic circuit shown in Figure 5, cf. (Arellano-Delgado *et al.* 2015), it is possible to electronically reproduce the behavior of a firefly. And in turn, it is possible to mimic the synchronization of these insects without having to enter their reproductive habitats. So these are friendly and didactic devices with which we show that synchronization is ubiquitous and that the type of communication between systems does not matter, as long as there is one, synchronization will emerge. In Figure 6 you can see the electronic firefly, and in (Echenausía-Monroy 2022h) you can see its operation in a beehive.

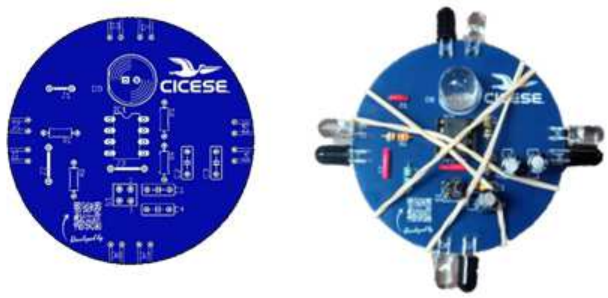

Figure 6 Electronic firefly that can synchronize in four directions. The picture shows the circuit without components and the final version.

It should be noted that in the above examples, the systems completely synchronize: the pendulums of the monumental clocks oscillate in harmony, moving in the same direction and with the same amplitudes; the electronic fireflies fire in unison, i.e., at the same frequency and with the same intensity, and also, the triplet of metronomes keep a rhythmic behavior such that their pendulums move with the same amplitude, frequency, and phase. However, many other types of synchronous motion can also be observed, like for example, the pendulum clocks moving at the same frequency but in opposite direction, a phenomenon called anti-phase synchronization, the electronic fireflies flashing at the unison but with different light intensities, which is referred to as frequency synchronization, and in the triplet of metronomes, they can produce a synchronized rotating wave: the metronomes oscillate a the same frequency and amplitude, but the pendulums of the metronomes have a phase difference of 120 degrees between them (Martens *et al.* 2013; Goldsztein *et al.* 2021).

Finally, it is important to note that in all the experiments described above, we can formally explain the onset of spontaneous synchronization using mathematical tools such as Lyapunov stability theory, the master stability function approach, or perturbation methods such as Poincaré's method (Ramirez and Nijmeijer 2016).

CHAOS

In the wonder world of complex systems (Cuesta-García 2022; Echenausía-Monroy 2022j), we are mostly dealing with nonlinear systems. This means that the dynamics of these systems are described by equations of motion with nonlinear terms, such as multiplications between variables of the same system, powers with degree greater than two, special nonlinear functions such as trigonometric or Piece Wise Linear (PWL) functions, to name a few (Drazin and Drazin 1992; Echenausía-Monroy *et al.* 2020). This type of complex systems does not respond to the superposition principle, where the system response cannot be decomposed as the sum of two or more responses corresponding to the number of system variables. In this type of system, it is possible to find chaotic behaviors or chaotic dynamics.

When we speak of "chaos" in science, we do not refer to the Greek cosmological stories that point to what existed before the existence of everything "*before the gods and the elemental forces there was CHAOS*". Nor do we refer to the absence of rules or order. Colloquially, chaos is often confused with examples such as a teenager's messy room, the actions of an angry mob, the behavior of an elementary school class when the teacher is absent for more than three minutes, or the mental disaster left behind after failing to conquer a summer love.

Mathematical chaos, which is generated by deterministic equations, is bounded aperiodic behavior that cannot be predicted. Also, a particular feature of chaotic behavior, which in general tends to be of oscillatory nature, is a high sensitivity to initial conditions, i.e., for two arbitrarily close starting points, the distance between the generated trajectories will exponentially diverge in time, see e.g. (Sprott 2010; Devaney 2018). When we say it is aperiodic, it simply means that there is no recurrence pattern and it is not known when the event occurs. When we say it is sensitive to initial conditions, it means that a small change at the beginning can cause a very large change over time. The first person to discover chaos was the famous polymath Henry Poincaré when he was working on solving the three-body problem (Chenciner 2015). A cinematic allusion to chaos is found in the first Jurassic Park movie, where Dr. Ian Malcolm (played by Jeff Goldblum) explains that chaos is unpredictable, citing as an example the trajectory of two drops of water in the hand of a beautiful lady.

This unpredictable behavior is also known as the "butterfly effect," which also appears in pop culture in the movie of the same name (The Butterfly Effect, starring Ashton Kutcher). In this movie, the protagonist travels to his past and can change certain events. Changing a small event in his past causes very big changes in his future, which is the essence of the Butterfly Effect.

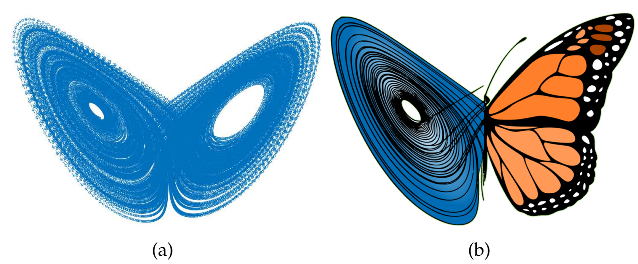

Figure 7 (a) Numerically determined Lorenz attractor for $\sigma = 10,\ r = 28,\ b = 8/3$ and all initial conditions are set to one. (b) Analogy of the Lorenz butterfly with the attractor formed by the system of the same name.

Edward Lorenz, an American mathematician and meteorologist, discovered this behavior in 1963 when he studied and reduced a system of twelve differential equations that described climatological behavior (Lorenz 1963). Lorenz programmed these equations into a computer and analyzed the results, which were accurate to six decimal places. He then took a value that the computer had already provided as a system solution and set it as the initial value so that the computer could "get on" with the simulation. After running the simulation again, Lorenz made himself a cup of coffee. When he returned, he hoped that the graphs he received were the same or very similar to the original ones. To his surprise, the results seemed to match at first, but after a while they diverged and no longer matched. Thus Lorenz proved sensitivity to initial conditions, and the analogy to the butterfly effect was born, summed up in Lorenz's maxim: "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?". The mathematical model of the Lorenz system is described by the folowing set of equations

$$
\begin{aligned}
\dot{x} &= -\sigma(x+y), \\
\dot{y} &= x(r-z) - y, \\
\dot{z} &= xy - bz,
\end{aligned}
\tag{1}
$$

which is derived from the simplified equations of convection rolls in the dynamical equations of the Earth's atmosphere (see (Lorenz 2000; Ambika 2015) and references therein for further information). The state variables (x, y, z) describe the behavior of the velocity and the direction of circulation of the convection rolls for state x, y is proportional to the temperature difference between the updrafts and downdrafts, and z is proportional to the deviation of the vertical temperature gradient from linearity. If the state variables of this system of equations are plotted on the $x - z$ plane, the result is a so-called attractor resembling the shape of a butterfly (Figure 7).

To easily introduce and demonstrate the concept of the butterfly effect, imagine a touch screen onto which a numerical simulation of the Lorenz attractor is projected. Since the butterfly effect is based on changing the initial conditions of the system, the simulation considers the touch point (on the touch screen) as the initial condition, so that when you repeatedly touch "the same point", you get different trajectories of the Lorenz system under different colors, which initially agree in their behavior, but diverge over time and follow completely different paths. Figure 8 shows an experiment based on the demonstration of the butterfly effect, the video of this experiment is available on (Echenausía-Monroy 2022f).

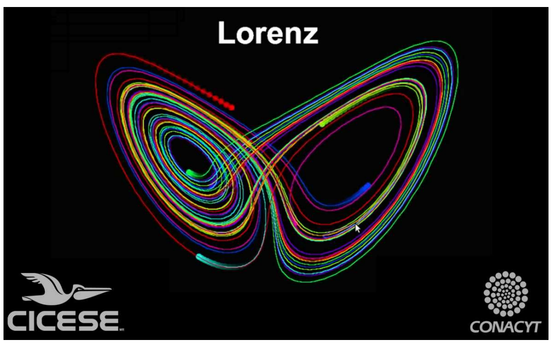

Figure 8 Experimental setup of the touchscreen to illustrate the butterfly effect of the Lorenz attractor.

Chaos is not only found in systems as complex as climate, but can also be observed in relatively simple models. Consider the behavior of a pendulum, like that of a wall clock, which behaves in a completely predictable, periodic, and monotonic manner; it always moves from left to right as long as the clock has a battery. Now, if the pendulum is disconnected from the clock, it will only move from left to right for a certain amount of time until it loses its energy and stops moving. If you attach another pendulum to the end of the system, you get a double pendulum. Since you know the behavior of a simple pendulum, you can assume that the new system will behave similarly to the first one. Surprisingly, the double pendulum follows unpredictable paths that change depending on the starting point of the pendulum, i.e., it shows chaotic behavior.

To observe the behavior of a double pendulum, consider its construction attached to an ultraviolet light-sensitive screen with a UV LED at the bottom. This allows visualization of the trajectories of the system when the pendulum is started in very similar positions, and the effects of initial conditions. Figure 9 shows the trajectory of the photoluminescent double pendulum, and the video of the experiment in operation can be found at (Echenausía-Monroy 2022b).

Figure 9 Double pendulum working with UV led placed on the tip of the second join. The image was taken with ISO 125 and a shutter speed of 4 seconds.

Chaos is not a phenomenon unique to weather or mechanical systems; rather, it is a quantifiable property (see (Wolff 1992; Abraham *et al.* 2013) and the references therein). As mentioned earlier, it is a phenomenon that surrounds us and that we can take advantage of. Take, for example, the logistic map, described by Eq. (2):

$$x_{n+1} = r x_n (1 - x_n), \tag{2}$$

which is one of the most studied complex systems in discrete time and has been applied in studying the dynamics of population growth (see (May 2004) for more information). In 2003, Professor Kazuyuki Aihara, a professor emeritus at the University of Tokyo, found that the bifurcation diagram of the logistic map (the behavior over time when a parameter changes) has a shape that resembles the silhouette of a dress, as shown in Figure 10 (a). This result was presented at Tokyo Fashion Week later that year ((Bulletin 2019)) and gave us a new perspective on the applications of chaos in our lives. A version of Aihara's chaotic dress can be seen in Figure 10 (b). For a 360° view, see (Echenausía-Monroy 2022e).

It has already been mentioned that there are chaotic dynamics in our body, which include the behavior of neurons. An example of this is the Hindmarsh-Rose model (HR), which describes the behavior of a single neuron in terms of axion potentials and the sodium-calcium channels that activate them (see (Shilnikov and Kolomiets 2008; Barrio *et al.* 2017) for more information). Although this complex system describes the behavior of a neuron and its excitatory agents, it is possible to take this model as a basis and use it to improve daily life.

Imagine a homemade blender spinning at a certain speed in the same direction. If we give the same blender a chaotic behavior, that is, it spins randomly in one direction or another and for different periods of time, it is possible to obtain a much faster homogeneous shake. This is exactly what Ricardo Núñez, an experimentalist researched based at CICESE, did when he developed a chaotic stirrer based on the Chua system. His idea, like Lorenz's discovery, was based on the morning coffee in the office and the time it takes to dissolve the different ingredients we add to the invigorating drink. As a result, he obtained a stirrer that homogenizes solutions

(a) (b)

Figure 10 (a) Section of interest from the bifurcation diagram of the logistic map by varying parameter r in Eq. (2). (b) Dress based on the logistic map designed by the authors. The original dress presented by Prof. Ahihara can be found at (Bulletin 2019).

more efficiently and in less time than a conventional one, Figure 11. For a video of the chaotic mixer in action, see (Echenausía-Monroy 2022d), and for more information, see (Núñez-Pérez 2022).

(a)

(b)

Figure 11 Stirrers mixing honey in water. The periodic shaker is shown on the left and the shaker based on a chaotic system is shown on the right. (a) 4 seconds of shaking versus (b) 12 seconds (Núñez-Pérez 2022).

In the last section, we described examples of how chaos occurs in our environment, how we can observe it, and how it can even help us in our daily lives. But for those of us who explore this exciting area of mathematics and physics, chaos has a beauty all its own. Each of the behaviors and models described can be represented by systems of equations. These, in turn, can be observed geometrically through so-called "attractors" that can be interpreted as the face of any system. In these chaotic attractors the whole beauty of chaos is shown. For example, consider the work of the Swiss artist "*Chaotic Atmospheres*", who has projected in his portfolio various chaotic attractors as graphic works of art, which you can find at (Atmospheres 2022). With the same idea and using 3D printing, it is possible to turn a system of differential equations into something tangible and bring chaos to the real plane, as seen in Figure 12, where four 3D-printed chaotic attractors are shown. This is not only a clear example of the use of technology to provide new educational tools, but also serves to explain and teach complex concepts such as chaos to people with visual impairments. Printed attractors can be found at (Echenausía-Monroy 2022a).

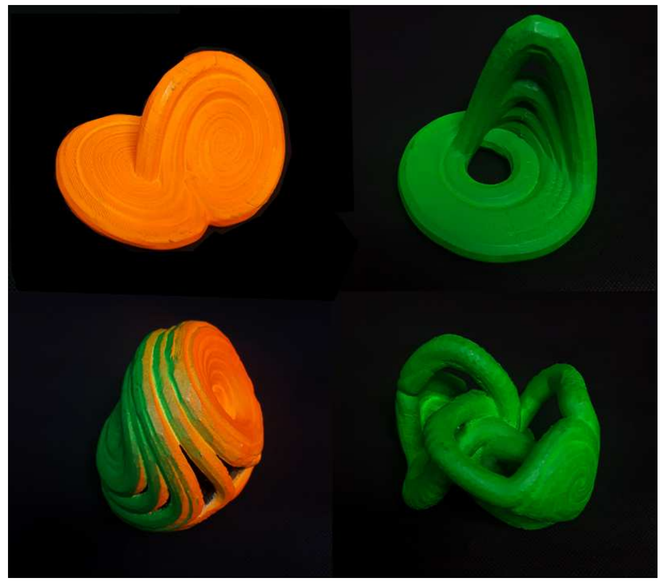

Figure 12 Chaotic attractors printted in 3D. (a) Lorenz, (b) Rössler, (c) Dequan Li, and (d) Thomas attractor.

Finally, it should be noted that the chaotic behavior discussed in this section and the phenomenon of synchronization presented in the previous section are two related concepts. Indeed, a pair or network of chaotic systems can synchronize provided they are suitably coupled, as shown in the pioneering work of Fujisaka and Yamada (Fujisaka and Yamada 1983).

CONCLUSION

It is our believe that the examples presented in this work may be useful for introducing concepts from complex systems like synchronization, emergent behavior and chaos, to non specialist and to further motivate the excitement for investigating these systems in the new generations.

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Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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