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ON SPECIAL SINGULAR CURVE COUPLES OF FRAMED CURVES IN 3D LIE GROUPS

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Abstract. In this paper, we introduce Bertrand and Mannheim curves of framed curves, which are a special singular curve in 3D Lie groups. We explain the conditions for framed curves to be Bertrand curves and Mannheim curves in 3D Lie groups. We give relationships between framed curvatures and Lie curvatures of Bertrand and Mannheim curves of framed curves. In addition, we obtain the characterization of Bertrand and Mannheim curves according to the various frames of framed curves in 3D Lie groups.

1. Introduction

It is known that a moving frame cannot be installed for curves with singular points [1]. However, thanks to the recent studies for smooth singular curves, there are important developments and these studies have important contributions to the singularity theory. Framed curves defined by Honda and Takahashi are one of them [10]. Framed curves that can have singular points are actually smooth curves. Since they are the general form of Legendre curves on unit tangent bundles and of regular curves with linear independent conditions, they have a great contribution to the studies of singular curves. Some of the pioneering work on framed curves is given in [6, 8, 10, 11, 16].

Bertrand and Mannheim curves are special curve types in differential geometry [2, 12, 13]. For curves $\gamma_1, \gamma_2 : I \to \mathbb{R}$ and moving frames $\{\mathcal{T}_1, \mathcal{N}_1, \mathcal{B}_1\}$ and $\{\mathcal{T}_2, \mathcal{N}_2, \mathcal{B}_2\}$ respectively, if $\mathcal{N}_1 = \mathcal{N}_2$ then curves γ_1, γ_2 are called Bertrand couple, if $\mathcal{N}_1 = \mathcal{B}_2$ then curves γ_1, γ_2 are called Mannheim couple [2, 13]. Bertrand and Mannheim curves of singular curves have been given by Honda and Takahashi in

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recent years, as well as the studies of regular curves on Bertrand and Mannheim curves [11]. In addition, Honda and Takahashi added nondegenerate condition to Bertrand and Mannheim curves for regular curves in the literature. Also, they gave a theory that is not valid in the regular case. For framed curves, a curve can be both a Bertrand and Mannheim curve.

Lie groups given by bi invariant metric are a structure that has important results in physics as well as its importance in differential geometry. Lie groups have three different forms in mathematics such that S^3, $SO(3)$ and abelian Lie groups [5]. There are some pioneering studies on 3D Lie groups in differential geometry. As a generalization of the characterizations in Euclidean space, helices, slant helices, Bertrand and Mannheim curves have been introduced in 3D Lie groups in various studies [3, 9, 14, 15]. These studies are based on the condition that the curve is regular. Framed curves, a singular curve, were introduced in 3D Lie groups by Yazıcı, Okuyucu and Tosun [7]. They, gave a new perspective to both physical and geometrical forms of Lie groups. Then, they defined various frames of framed curves in 3D Lie groups.

In this study, we investigate Bertrand and Mannheim curves in 3D Lie groups of framed curves, which have an important place in singularity theory. We express the necessary and sufficient conditions for the framed curves to be Bertrand or Mannheim curves in 3D Lie groups.

2. Lie Groups

Let G be a Lie group with a bi-invariant metric $\langle , \rangle$ and ∇ be the Levi-Civita connection of Lie group G. $\mathfrak{g}$ is isomorphic to T_eG where e is neutral element of G and $\mathfrak{g}$ is Lie algebra of G. Since $\langle , \rangle$ is a bi-invariant metric on G, we have

$$\langle P, [Q, R] \rangle = \langle [P, Q], R \rangle$$

and

$$\nabla_P Q = \frac{1}{2}[P, Q].$$

for all $P, Q, R \in \mathfrak{g}$. On the other hand the Lie bracket of two vector fields W_1 and W_2 is given

$$[W_1, W_2] = \sum_{i=1}^{n} w_{1i} w_{2i} [Y_i, Y_j],$$

where $W_1 = \sum_{i=1}^{n} w_{1i} Y_i$ and $W_2 = \sum_{i=1}^{n} w_{2i} Y_i$ with orthonormal basis $\{Y_1, Y_2, ..., Y_n\}$ of $\mathfrak{g}$.

Suppose that $\beta : I \to$ G be an unit speed regular curve. Then the covariant derivative of X along the curve β is given as follows

$$\nabla_{\beta'} X = \nabla_T X = \dot{X} + \frac{1}{2}[T, X],$$

where T is tangent and $\dot{X} = \sum_{i=1}^{n} \frac{dx}{dt} Y_i$. Moreover, if W is the left-invariant vector field to the curve, then $\dot{X} = 0$ (see for details [4]).

The representation of the Frenet-Serret formulas in the 3D Lie group G with the covariant derivative is given as follows:

$$\begin{array}{lcl} \nabla_T T & = & \kappa_1 N_1, \\ \nabla_T N_1 & = & -\kappa_1 T + \kappa_2 N_2, \\ \nabla_T N_2 & = & -\kappa_2 N_1, \end{array}$$

where ∇ is connection of G and $\kappa_1 = \|\dot{T}\|$.

2.1. **Framed curves in 3D Lie groups.** In this part, framed curves, general and adapted frames in 3D Lie groups are discussed [7]. Obviously, framed curves in 3D Lie groups [7] are a generalization of framed curves in $\mathbb{R}^3$ [10].

Definition 1. *[7] A curve $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ in 3D Lie group G is a framed curve if $\langle \gamma'(s), \varrho_i(s) \rangle = 0$ for all $s \in I$ and $i = 1, 2$ where*

$$\Delta_{\mathrm{G}} = \{\varrho = (\varrho_1, \varrho_2) \in \mathrm{G} \times \mathrm{G} | \quad \langle \varrho_1, \varrho_1 \rangle = \langle \varrho_2, \varrho_2 \rangle = 1, \langle \varrho_1, \varrho_2 \rangle = 0\}.$$

A unit vector ω is defined by $\omega = \varrho_1 \times \varrho_2$. The covariant derivative of X along the framed curve $(\gamma, \varrho_1, \varrho_2)$ with the help of unit vector ω as follows

$$\nabla_\omega X = \dot{X} + \frac{1}{2}[\omega, X]. \tag{1}$$

A smooth function on I is given as $\gamma'(s) = \alpha(s)\omega(s)$ and it is clear that s_0 is a singular point if and only if $\alpha(s_0) = 0$. Then the representation with Levi-Civita connection of Frenet-Serret type formulas of $(\gamma, \varrho_1, \varrho_2)$ satisfies:

$$\begin{array}{lcl} \nabla_\omega \omega & = & -l_2(s)\varrho_1(s) - l_3(s)\varrho_2(s), \\ \nabla_\omega \varrho_1 & = & l_1(s)\varrho_2(s) + l_2(s)\omega(s), \\ \nabla_\omega \varrho_2 & = & -l_1(s)\varrho_1(s) + l_3(s)\omega(s), \end{array} \tag{2}$$

where ∇ is Levi-Civita connection of G and $\sqrt{l_2^2(s) + l_3^2(s)} = \|\dot{\omega}\|$. If ω is the left-invariant vector field to the framed curve, then $l_2(s) = l_3(s) = 0$ for every $s \in I$.

Proposition 1. *[7] Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ be a framed curve in 3D Lie groups. Then,*

$$[\omega, \varrho_1] = \langle [\omega, \varrho_1], \varrho_2 \rangle \varrho_2 = 2\delta_G \varrho_2,$$

$$[\omega, \varrho_2] = \langle [\omega, \varrho_2], \varrho_1 \rangle \varrho_1 = -2\delta_G \varrho_1.$$

is provided.

Theorem 1. *[7] Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ be a framed curve. The Frenet-Serret type formulas of framed curves in 3D Lie groups are given by*

$$\begin{pmatrix} \dot{\omega} \\ \dot{\varrho_1} \\ \dot{\varrho_2} \end{pmatrix} = \begin{pmatrix} 0 & -l_2(s) & -l_3(s) \\ l_2(s) & 0 & (l_1(s) - \delta_{\mathrm{G}}) \\ l_3(s) & -(l_1(s) - \delta_{\mathrm{G}}) & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \varrho_1 \\ \varrho_2 \end{pmatrix}. \tag{3}$$

where $\delta_{\mathrm{G}} = \frac{1}{2}\langle[\omega, \varrho_1], \varrho_2\rangle$.

Corollary 1 ([7], **Bishop-type frame in 3D Lie groups**). *The under condition $l_1(s) - \delta_{\mathrm{G}} - \psi^{'}(s) = 0$, we have*

$$\begin{pmatrix} \dot{\omega} \\ \dot{\tilde{\varrho}}_1 \\ \dot{\tilde{\varrho}}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\tilde{l}_2(s) & -\tilde{l}_3(s) \\ \tilde{l}_2(s) & 0 & 0 \\ \tilde{l}_3(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \tilde{\varrho_1} \\ \tilde{\varrho_2} \end{pmatrix}, \tag{4}$$

where

$$\begin{pmatrix} \tilde{l}_2 \\ \tilde{l}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi(s) & -\sin\psi(s) \\ \sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} l_2 \\ l_3 \end{pmatrix}.$$

Corollary 2 ([7], **Frenet-type frame in 3D Lie groups**). *The under condition $l_2(s)\sin\psi(s) + l_3(s)\cos\psi(s) = 0$, we get*

$$\begin{pmatrix} \dot{\omega} \\ \dot{\tilde{\varrho}}_1 \\ \dot{\tilde{\varrho}}_2 \end{pmatrix} = \begin{pmatrix} 0 & p(s) & 0 \\ -p(s) & 0 & (q(s) - \delta_{\mathrm{G}}) \\ 0 & -(q(s) - \delta_{\mathrm{G}}) & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \tilde{\varrho_1} \\ \tilde{\varrho_2} \end{pmatrix}. \tag{5}$$

where $q(s) = l_1(s) - \psi^{'}(s)$ and $p(s) \neq 0$.

3. Bertrand Curves of Framed Curves in 3D Lie Groups

Definition 2. *The framed curves $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are called Bertrand couples if there exists a smooth function $\lambda : I \to \mathbb{R}$ where*

$$\overline{\gamma}(s) = \gamma(s) + \lambda(s)\varrho_1(s) \tag{6}$$

and

$$\varrho_1(s) = \overline{\varrho}_1(s)$$

for all $s \in I$.

Proposition 2. *If $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are Bertrand couples, $\lambda \neq 0$ is a constant.*

Proof. By differentiating equation (6) in 3D Lie groups and by using equation (3), we have

$$\frac{d\overline{\gamma}(s)}{ds} = \frac{d\gamma(s)}{ds} + \lambda^{'}(s)\varrho_1(s) + \lambda(s)\dot{\varrho}_1(s)$$

$$\overline{\alpha}(s)\overline{\omega}(s) = (\alpha(s) + \lambda(s)l_2(s))\omega(s) + \lambda^{'}(s)\varrho_1(s) + \lambda(s)(l_1(s) - \delta_{\mathrm{G}})\varrho_2(s) \tag{7}$$

Since $\varrho_1(s) = \overline{\varrho}_1(s)$, we get $\lambda^{'}(s) = 0$. That is, $\lambda(s)$ is a constant function on I. □

Theorem 2. *Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ be a framed curve with the curvature (l_1, l_2, l_3, α) and Lie curvature δ_{G}. Then $(\gamma, \varrho_1, \varrho_2)$ is a Bertrand curve if and only if there exist $\lambda \neq 0$=constant and a smooth function $\Phi : I \to \mathbb{R}$ where*

$$\lambda(l_1(s) - \delta_{\mathrm{G}})\cos\Phi(s) - (\alpha(s) + \lambda l_2(s))\sin\Phi(s) = 0 \tag{8}$$

Proof. Suppose that $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are Bertrand curve. Since $\varrho_1(s) = \overline{\varrho}_1(s)$, there exists a function Φ on I with

$$\overline{\varrho}_2(s) = \cos\Phi(s)\varrho_2(s) - \sin\Phi(s)\omega(s), \tag{9}$$

$$\overline{\omega}(s) = \sin\Phi(s)\varrho_2(s) + \cos\Phi(s)\omega(s). \tag{10}$$

If the equations (9) and (10) are substituted in the equation (7), we get

$$\overline{\alpha}(s)\sin\Phi(s) = \lambda(l_1(s) - \delta_{\mathrm{G}}), \tag{11}$$

$$\overline{\alpha}(s)\cos\Phi(s) = \alpha(s) + \lambda l_2(s). \tag{12}$$

Therefore, the equation (8) is found. Conversely, suppose that (8) is provided. If we define a mapping $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ with

$$\overline{\gamma}(s) = \gamma(s) + \lambda\varrho_1(s), \quad \varrho_1(s) = \overline{\varrho}_1(s)$$

and $\overline{\varrho}_2(s) = \cos\Phi(s)\varrho_2(s) - \sin\Phi(s)\omega(s)$, then $(\gamma, \varrho_1, \varrho_2)$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2)$ are Bertrand mates in 3D Lie groups. □

Proposition 3. *Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are Bertrand mates. Then,*

$$\overline{\delta}_{\mathrm{G}} = \delta_{\mathrm{G}}$$

where

$$\delta_{\mathrm{G}} = \frac{1}{2}\langle[\omega, \varrho_1], \varrho_2\rangle,$$

$$\overline{\delta}_{\mathrm{G}} = \frac{1}{2}\langle[\overline{\omega}, \overline{\varrho}_1], \overline{\varrho}_2\rangle.$$

Proof. Suppose that $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are Bertrand mates. By according to equations (9) and (10), we can write

$$\overline{\delta}_{\mathrm{G}} = \frac{1}{2}\langle[\overline{\omega}, \overline{\varrho}_1], \overline{\varrho}_2\rangle$$

$$= \frac{1}{2}\langle[\sin\Phi(s)\varrho_2(s)+\cos\Phi(s),\varrho_1(s)],\cos\Phi(s)\varrho_2(s)-\sin\Phi(s)\omega(s)\rangle$$
$$= \frac{1}{2}\langle\sin\Phi(s)[\varrho_2(s),\varrho_1(s)]+\cos\Phi(s)[\omega(s),\varrho_1(s)],\cos\Phi(s)\varrho_2(s)-\sin\Phi(s)\omega(s)\rangle$$

Hence, from Lie bracket properties, we get

$$\overline{\delta}_{\mathrm{G}} = \frac{1}{2}\langle[\overline{\omega},\overline{\varrho}_1],\overline{\varrho}_2\rangle = \frac{1}{2}\langle[\omega,\varrho_1],\varrho_2\rangle = \delta_{\mathrm{G}}.$$

□

Proposition 4. *Let* $(\gamma,\varrho_1,\varrho_2): I \to \mathrm{G}\times\Delta_{\mathrm{G}}$ *and* $(\overline{\gamma},\overline{\varrho}_1,\overline{\varrho}_2): I \to \mathrm{G}\times\Delta_{\mathrm{G}}$ *are Bertrand mates. Then the curvatures* $(\bar{l}_1,\bar{l}_2,\bar{l}_3,\overline{\alpha})$ *of* $(\overline{\gamma},\overline{\varrho}_1,\overline{\varrho}_2)$ *are given by*

$$\begin{aligned}
\bar{l}_1(s) &= l_1(s)\cos\Phi(s)-l_2(s)\sin\Phi(s)+\delta_{\mathrm{G}}(1-\cos\Phi(s)),\\
\bar{l}_2(s) &= l_2(s)\cos\Phi(s)+l_1(s)\sin\Phi(s)-\delta_{\mathrm{G}}\sin\Phi(s),\\
\bar{l}_3(s) &= l_3(s)-\Phi^{'}(s),\\
\overline{\alpha}(s) &= \lambda(l_1(s)-\delta_{\mathrm{G}})\sin\Phi(s)+(\alpha(s)+\lambda l_2(s))\cos\Phi(s).
\end{aligned}$$

Proof. By differentiating equation (9), we have

$$\begin{aligned}
\bar{l}_3(s)\overline{\omega}(s)-(\bar{l}_1(s)-\overline{\delta}_{\mathrm{G}})\overline{\varrho}_1(s) &= (l_3(s)\cos\Phi(s)-\Phi^{'}(s)\cos\Phi(s))\omega(s)\\
&+ (-(l_1(s)-\delta_{\mathrm{G}})\cos\Phi(s)+l_2(s)\sin\Phi(s))\varrho_1(s)\\
&+ (-\sin\Phi(s)\Phi^{'}(s)+l_3(s)\sin\Phi(s))\varrho_2(s).
\end{aligned}$$

Since $\overline{\varrho}_1(s)=\varrho_1(s)$ and $\overline{\delta}_{\mathrm{G}}=\delta_{\mathrm{G}}$, we get

$$\bar{l}_1(s)=l_1(s)\cos\Phi(s)-l_2(s)\sin\Phi(s)+\delta_{\mathrm{G}}(1-\cos\Phi(s)).$$

By using equation (10), we have $\bar{l}_3(s)=l_3(s)-\Phi^{'}(s)$. Also, by differentiating equation (10), we get

$$\begin{aligned}
-\bar{l}_2(s)\overline{\varrho}_1(s)-\bar{l}_3(s)\overline{\varrho}_2(s) &= (l_3(s)\sin\Phi(s)-\Phi^{'}(s)\sin\Phi(s))\omega(s)\\
&+ (-(l_1(s)-\delta_{\mathrm{G}})\sin\Phi(s)-l_2(s)\cos\Phi(s))\varrho_1(s)\\
&+ (\cos\Phi(s)\Phi^{'}(s)-l_3(s)\cos\Phi(s))\varrho_2(s).
\end{aligned}$$

Since $\overline{\varrho}_1(s)=\varrho_1(s)$ and $\overline{\delta}_{\mathrm{G}}=\delta_{\mathrm{G}}$, we have

$$\bar{l}_2(s)=l_2(s)\cos\Phi(s)+l_1(s)\sin\Phi(s)-\delta_{\mathrm{G}}\sin\Phi(s).$$

On the other hand, If the equation (11) is multiplied by $\sin\Phi(s)$ on both sides, and the equation (12) is multiplied by $\cos\Phi(s)$ on both sides, then we get

$$\overline{\alpha}(s)=\lambda(l_1(s)-\delta_{\mathrm{G}})\sin\Phi(s)+(\alpha(s)+\lambda l_2(s))\cos\Phi(s).$$

□

Corollary 3. *Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ be a framed curve with the curvatures (l_1, l_2, l_3, α) and Lie curvature δ_{G}.*
(i). If $l_1(s) - \delta_{\mathrm{G}} = 0$ for every $s \in I$, then $(\gamma, \mu_1, \mu_2) : I \to G \times \Delta_G$ is a Bertrand curve.
(ii). If $\alpha(s) + \lambda l_2(s) = 0$ where $\lambda \neq 0$=constant, then $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ is a Bertrand curve.

Proof. (i). If we assume that $\Phi(s) = 0$, it is clear that equation (7) is realized.
(ii). If we assume that $\Phi(s) = \dfrac{\pi}{2}$, it is clear that equation (7) is realized. □

Corollary 4. *For an adapted frame (Bishop-type frame) in 3D Lie groups, the framed curve is always a Bertrand curve.*

Corollary 5. *For an adapted frame (Frenet-type frame) in 3D Lie groups, the curves are Bertrand couples if and only if there exists λ=constant where $\Phi(s)$ is a constant. Because, the curvature $\bar{l}_3(s) = l_3(s) = 0$ for Frenet-type framed curve and by using equation $\bar{l}_3(s) = l_3(s) - \Phi^{'}(s)$, we have Φ is a constant.*

Corollary 6. *In the Propositions and Theorems obtained, if $\delta_{\mathrm{G}} = 0$, the results correspond to the study [11]. Therefore, these results are a generalization of both study [11] and [15].*

4. Mannheim Curves of Framed Curves in 3D Lie Groups

Definition 3. *The framed curves $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are called Mannheim couples if there exists a smooth function $\lambda : I \to \mathbb{R}$ where*

$$\overline{\gamma}(s) = \gamma(s) + \lambda(s)\varrho_1(s) \tag{13}$$

and

$$\varrho_1(s) = \overline{\varrho}_2(s)$$

for all $s \in I$.

Proposition 5. *If $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are Mannheim couples, then $\lambda \neq 0$ is a constant.*

Proof. Firstly, by differentiating equation (13) in 3D Lie groups and by using equation (3), we have

$$\overline{\alpha}(s)\overline{\omega}(s) = (\alpha(s) + \lambda(s)l_2(s))\omega(s) + \lambda^{'}(s)\varrho_1(s) + \lambda(s)(l_1(s) - \delta_{\mathrm{G}})\varrho_2(s) \tag{14}$$

Since $\varrho_1(s) = \overline{\varrho}_2(s)$, we get $\lambda^{'}(s) = 0$. That is, $\lambda(s)$ is a constant function on I. □

Theorem 3. *Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ be a framed curve with the curvature (l_1, l_2, l_3, α) and Lie curvature δ_{G}. Then $(\gamma, \varrho_1, \varrho_2)$ is a Mannheim curve if and only if there exist $\lambda \neq 0$=constant and a smooth function $\theta : I \to \mathbb{R}$ where*

$$\lambda(l_1(s) - \delta_{\mathrm{G}})\sin\theta(s) + (\alpha(s) + \lambda l_2(s))\cos\theta(s) = 0 \tag{15}$$

Proof. Assume that $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ is a Mannheim curve. Since $\varrho_1(s) = \overline{\varrho}_2(s)$, there exists a function θ on I with

$$\overline{\varrho}_1(s) = \sin\theta(s)\varrho_2(s) + \cos\theta(s)\omega(s), \tag{16}$$

$$\overline{\omega}(s) = \cos\theta(s)\varrho_2(s) - \sin\theta(s)\omega(s). \tag{17}$$

If the equations (16) and (17) are substituted in the equation (14), we get

$$-\overline{\alpha}(s)\sin\theta(s) = \alpha(s) + \lambda l_2(s) \tag{18}$$

$$\overline{\alpha}(s)\cos\theta(s) = \lambda(l_1(s) - \delta_{\mathrm{G}}) \tag{19}$$

Consequently, we have equation (15). Conversely, suppose that (15) is provided. If we define a mapping $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ with

$$\overline{\gamma}(s) = \gamma(s) + \lambda\varrho_1(s), \quad \varrho_1(s) = \overline{\varrho}_2(s)$$

and $\overline{\varrho}_1(s) = \sin\theta(s)\varrho_2(s) + \cos\theta(s)\omega(s)$, then $(\gamma, \varrho_1, \varrho_2)$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2)$ are Mannheim mates. □

Remark 1. *Similar to Proposition 3, by using equations $\varrho_1(s) = \overline{\varrho}_2(s)$, (16) and (17), it can be seen that the Lie curvature of the framed curve and the Lie curvature of the Mannheim curve are the same.*

Proposition 6. *Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ and $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ are Mannheim mates. Then the curvatures $(\bar{l}_1, \bar{l}_2, \bar{l}_3, \overline{\alpha})$ of $(\overline{\gamma}, \overline{\varrho}_1, \overline{\varrho}_2)$ are given by*

$$\begin{aligned}
\bar{l}_1(s) &= -l_1(s)\sin\theta(s) - l_2(s)\cos\theta(s) + \delta_{\mathrm{G}}(1 + \sin\theta(s)),\\
\bar{l}_2(s) &= -l_3(s) + \theta'(s),\\
\bar{l}_3(s) &= l_1(s)\cos\theta(s) - l_2(s)\sin\theta(s) - \delta_{\mathrm{G}}\cos\theta(s),\\
\overline{\alpha}(s) &= \lambda(l_1(s) - \delta_{\mathrm{G}})\cos\theta(s) - (\alpha(s) + \lambda l_2(s))\sin\theta(s).
\end{aligned}$$

Proof. By differentiating equation (16), we have

$$\begin{aligned}
\bar{l}_2(s)\overline{\omega}(s) + (\bar{l}_1(s) - \overline{\delta}_{\mathrm{G}})\overline{\varrho}_2(s) &= (l_3(s)\sin\theta(s) - \theta'(s)\sin\theta(s))\omega(s)\\
&+ (-(l_1(s) - \delta_{\mathrm{G}})\sin\theta(s) - l_2(s)\cos\theta(s))\varrho_1(s)\\
&+ (\cos\theta(s)\theta'(s) - l_3(s)\cos\theta(s))\varrho_2(s).
\end{aligned}$$

Since $\overline{\varrho}_2(s) = \varrho_1(s)$ and $\overline{\delta}_{\mathrm{G}} = \delta_{\mathrm{G}}$, we get

$$\bar{l}_1(s) = -l_1(s)\sin\theta(s) - l_2(s)\cos\theta(s) + \delta_{\mathrm{G}}(1 + \sin\theta(s)).$$

By using equation (17), we have $\bar{l}_2(s) = -l_3(s) + \theta'(s)$. Moreover, by differentiating equation (17), we get

$$\begin{aligned}
-\bar{l}_2(s)\overline{\varrho}_1(s) - \bar{l}_3(s)\overline{\varrho}_2(s) &= (l_3(s)\cos\theta(s) - \theta'(s)\cos\theta(s))\omega(s)\\
&+ (-(l_1(s) - \delta_{\mathrm{G}})\cos\theta(s) + l_2(s)\sin\theta(s))\varrho_1(s)\\
&+ (-\sin\theta(s)\theta'(s) + l_3(s)\sin\theta(s))\varrho_2(s).
\end{aligned}$$

Since $\overline{\varrho}_2(s) = \varrho_1(s)$ and $\overline{\delta}_{\mathrm{G}} = \delta_{\mathrm{G}}$, we have

$$\overline{l}_3(s) = l_1(s)\cos\theta(s) - l_2(s)\sin\theta(s) - \delta_{\mathrm{G}}\cos\theta(s).$$

On the other hand, If the equation (18) is multiplied by $-\sin\theta(s)$ on both sides, and the equation (19) is multiplied by $\cos\theta(s)$ on both sides, then we get

$$\overline{\alpha}(s) = \lambda(l_1(s) - \delta_{\mathrm{G}})\cos\theta(s) - (\alpha(s) + \lambda l_2(s))\sin\theta(s).$$

□

Corollary 7. *Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ be a framed curve with the curvature (l_1, l_2, l_3, α) and Lie curvature δ_{G}.*
(i). If $l_1(s) - \delta_{\mathrm{G}} = 0$ for all $s \in I$, then $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ is a Mannheim curve.
(ii). If $\alpha(s) + \lambda l_2(s) = 0$ where $\lambda \neq 0$=constant, then $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ is a Mannheim curve.

Proof. (i). If we assume that $\theta(s) = \dfrac{\pi}{2}$, it is clear that equation (15) is realized.
(ii). If we assume that $\theta(s) = 0$, it is clear that equation (15) is realized. □

Corollary 8. *For an adapted frame (Bishop-type frame) in 3D Lie groups, the framed curve is always a Mannheim curve.*

Corollary 9. *For an adapted frame (Frenet-type frame) in 3D Lie groups, since $\overline{l}_3(s) = l_3(s) = 0$, by using Proposition (6), the curves are Mannheim couples if and only if there exist $\lambda \neq 0$=constant and a smooth function θ where*

$$\begin{aligned}
&\overline{p}(s) = -\theta^{'}(s),\\
&\overline{q}(s) = -(q - \delta_{\mathrm{G}})\sin\theta(s) + p(s)\cos\theta(s) + \delta_{\mathrm{G}},\\
&\overline{\alpha}(s) = -(\alpha(s) - \lambda p(s))\sin\theta(s) + \lambda(q(s) - \delta_{\mathrm{G}})\cos\theta(s),\\
&p(s)\sin\theta(s) + (q(s) - \delta_{\mathrm{G}})\cos\theta(s) = 0.
\end{aligned}$$

Let us now give a theorem that is not valid for regular Bertrand and Mannheim curves in both Euclidean space and 3D Lie groups:

Theorem 4. *Let $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ be a framed curve with the curvature (l_1, l_2, l_3, α) and Lie curvature δ_{G}. Then $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ is a Bertrand curve in 3D Lie groups if and only if $(\gamma, \varrho_1, \varrho_2) : I \to \mathrm{G} \times \Delta_{\mathrm{G}}$ is a Mannheim curve in 3D Lie groups.*

Proof. Assume that $(\gamma, \varrho_1, \varrho_2)$ is a Bertrand curve. Then, there exist $\lambda \neq 0$=constant and a smooth function Φ such that

$$\lambda(l_1(s) - \delta_{\mathrm{G})}\cos\Phi(s) - (\alpha(s) + \lambda l_2(s))\sin\Phi(s) = 0.$$

If $\Phi(s) = \theta(s) - \dfrac{\pi}{2}$, we have

$$\lambda(l_1(s) - \delta_{\mathrm{G})})\sin\theta(s) + (\alpha(s) + \lambda l_2(s))\cos\theta(s) = 0.$$

Then, $(\gamma, \varrho_1, \varrho_2)$ is a Mannheim curve. Conversely, suppose that $(\gamma, \varrho_1, \varrho_2)$ is a Mannheim curve. Then, there exist a constant $\lambda \neq 0$ and a smooth function θ such that

$$\lambda(l_1(s) - \delta_{\mathrm{G})}) \sin\theta(s) + (\alpha(s) + \lambda l_2(s)) \cos\theta(s) = 0.$$

If $\theta(s) = \Phi(s) + \dfrac{\pi}{2}$, then we have,

$$\lambda(l_1(s) - \delta_{\mathrm{G})}) \cos\Phi(s) - (\alpha(s) + \lambda l_2(s)) \sin\Phi(s) = 0.$$

Consequently, $(\gamma, \varrho_1, \varrho_2)$ is a Bertrand curve. □

Corollary 10. *In the Propositions and Theorems obtained, if $\delta_{\mathrm{G}} = 0$, the results correspond to the study [11]. Therefore, these results are a generalization of both study [11] and [9].*

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