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On a Rational $(P+1)$th Order Difference Equation with Quadratic Term

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Abstract

In this paper, we derive the forbidden set and determine the solutions of the difference equation that contains a quadratic term

$$x_{n+1} = \frac{x_n x_{n-p}}{a x_{n-(p-1)} + b x_{n-p}}, \quad n \in \mathbb{N}_0,$$

where the parameters a and b are real numbers, p is a positive integer and the initial conditions $x_{-p}, x_{-p+1}, \cdots, x_{-1}, x_0$ are real numbers.

1. Introduction

In [1], the authors determined the forbidden set, introduced an explicit formula for the solutions and discussed the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k+1}}{b x_{n-k+1} + c x_{n-k}}, \quad n \in \mathbb{N}_0,$$

where a, b, c are positive real numbers and the initial conditions $x_{-k}, x_{-k+1}, \cdots, x_{-1}, x_0$ are real numbers.
In [2], the second author studied the global behavior and introduced an explicit formula for the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k}}{-b x_n + c x_{n-k-1}}, \quad n \in \mathbb{N}_0,$$

where a, b, c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \cdots, x_{-1}, x_0$ are real numbers.
In [3], the author determined the forbidden set, introduced an explicit formula for the solutions and discussed the global behavior of solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k}}{b x_n - c x_{n-k-1}}, \quad n \in \mathbb{N}_0,$$

where a, b, c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \cdots, x_{-1}, x_0$ are real numbers.
In [4], Abo-Zeid determined the forbidden set and studied the global behavior of the solutions of the difference equation

$$x_{n+1} = \frac{a x_n x_{n-k}}{b x_n + c x_{n-k-1}}, \quad n \in \mathbb{N}_0,$$

where a, b, c are positive real numbers and the initial conditions $x_{-k-1}, x_{-k}, \cdots, x_{-1}, x_0$ are real numbers.
For more on difference equations, one can see [5–28] and the references therein.

In this paper we generalize the solutions of the nonlinear rational difference equations presented in [5] and [10], which were established through a mere application of the induction principle.

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2. Main Results

In this section, we investigate the solutions of the difference equation

$$x_{n+1}=\frac{x_n x_{n-p}}{a x_{n-(p-1)}+b x_{n-p}}, \quad n\in\mathbb{N}_0, \tag{2.1}$$

where the parameters a and b are real numbers, p is a positive integer and the initial conditions $x_{-p}, x_{-p+1}, \cdots, x_{-1}, x_0$ are real numbers.
The transformation

$$u_n=\frac{x_{n-1}}{x_n}, \textit{ with } u_{-i}=\frac{x_{-i-1}}{x_{-i}}, \quad i=\overline{0,(p-1)}, \tag{2.2}$$

reduces equation (2.1) into the difference equation

$$u_{n+1}=\frac{a}{u_{n-p+1}}+b, \quad n\in\mathbb{N}_0.$$

Suppose that

$$u_m^{(j)}=u_{pm+j}, \ j=\overline{1,p} \text{ and } m\geq -1.$$

Then, we can write

$$u_m^{(j)}=\frac{a}{u_{m-1}^{(j)}}+b, \quad m\in\mathbb{N}_0. \tag{2.3}$$

Let

$$u_m^{(j)}=\frac{z_{m+1}}{z_m}, \quad m\geq -1. \tag{2.4}$$

Then, equation (2.3) becomes

$$z_{m+1}-b z_m-a z_{m-1}=0, \quad m\in\mathbb{N}_0. \tag{2.5}$$

with initial condition $z_{-1}=1$, $z_0=u_{-1}^{(j)}$.
Throughout this paper, we denote b^2+4a by Δ.

2.1. Case $\Delta>0$

In this subsection, we have that $b^2>-4a$. Suppose that

$$\phi_j=\frac{\lambda_+^j-\lambda_-^j}{\lambda_+-\lambda_-}, \quad j\in\mathbb{N}_0,$$

where λ_+ and λ_- are the roots of the equation $\lambda^2-b\lambda-a=0$.
Let

$$\gamma_{-i}(j)=a x_{-i}\phi_j+x_{-i-1}\phi_{j+1}, \quad i=\overline{0,(p-1)}.$$

Using equalities (2.2) and (2.4), we can write

$$\begin{aligned} x_{pm+p} &= \frac{1}{\prod_{i=1}^p u_{pm+i}} x_{pm} = x_0\prod_{i=1}^p \frac{\gamma_{-p+i}(0)}{\gamma_{-p+i}(m+1)} \\ &= \frac{\nu}{\prod_{i=1}^p \gamma_{-p+i}(m+1)}, \ m\in\mathbb{N}_0, \end{aligned}$$

where $\nu=\prod_{i=0}^p x_{-i}$.
It follows that

$$\begin{aligned} x_{pm+t} &= \frac{1}{\prod_{i=1}^t u_{pm+i}} x_{pm} = \frac{\nu}{\prod_{i=1}^p \gamma_{-p+i}(m)}\cdot\frac{\prod_{i=1}^t \gamma_{-p+i}(m)}{\prod_{i=1}^t \gamma_{-p+i}(m+1)} \\ &= \frac{\nu}{\prod_{i=1}^t \gamma_{-p+i}(m+1)\prod_{i=t+1}^p \gamma_{-p+i}(m)}, \ m\in\mathbb{N}_0, \text{ and } t=\overline{1,p}. \end{aligned}$$

Using the above arguments, we obtain the following result:

Theorem 2.1. *Let $\{x_n\}_{n=-p}^{\infty}$ be a well defined solution for equation (2.1). Then*

$$x_n=\begin{cases} \dfrac{\nu}{\gamma_{-p+1}(\frac{n+p-1}{p})\prod_{j=2}^p \gamma_{-p+j}(\frac{n-1}{p})}, & n=1,p+1,\ldots, \\ \dfrac{\nu}{\prod_{i=1}^2 \gamma_{-p+i}(\frac{n+p-2}{p})\prod_{j=3}^p \gamma_{-p+j}(\frac{n-2}{p})}, & n=2,p+2,\ldots, \\ \vdots & \vdots \\ \dfrac{\nu}{\prod_{i=1}^{p-1} \gamma_{-p+i}(\frac{n+1}{p})\gamma_0(\frac{n-p+1}{p})}, & n=p-1,2p-1,\ldots, \\ \dfrac{\nu}{\prod_{i=1}^p \gamma_{-p+i}(\frac{n}{p})}, & n=p,2p,\ldots, \end{cases}$$

where $\nu=\prod_{i=0}^p x_{-i}$, $\gamma_{-j}(m)=a x_{-j}\phi_m+x_{-j-1}\phi_{m+1}$, $j=\overline{0,(p-1)}$ and $m\geq -1$.

Consider the two sets

$$\mathbb{D}_1=\left\{(v_0,v_1,\cdots,v_p)\in\mathbb{R}^{p+1}:\frac{v_0}{(-1)^p(\lambda_+/a)^p}=\frac{v_1}{(-1)^{p-1}(\lambda_+/a)^{(p-1)}}=\cdots=\frac{v_{p-1}}{-\lambda_+/a}=v_p\right\},$$

$$\mathbb{D}_2=\left\{(v_0,v_1,\cdots,v_p)\in\mathbb{R}^{p+1}:\frac{v_0}{(-1)^p(\lambda_-/a)^p}=\frac{v_1}{(-1)^{p-1}(\lambda_-/a)^{(p-1)}}=\cdots=\frac{v_{p-1}}{-\lambda_-/a}=v_p\right\}.$$

Theorem 2.2. *The two sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are invariant sets for equation (2.1).*

Proof. Let $(x_0,x_{-1},\cdots,x_{-p})\in\mathbb{D}_2$. We show that $(x_n,x_{n-1},\cdots,x_{n-p})\in\mathbb{D}_2$ for each $n\in\mathbb{N}$. The proof is by induction on n. The point $(x_0,x_{-1},\cdots,x_{-p})\in\mathbb{D}_2$ implies

$$\frac{x_0}{(-1)^p\lambda_-^p/a^p}=\frac{x_{-1}}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}}=\cdots=\frac{x_{-(p-1)}}{(-1)\lambda_-/a}=x_{-p}.$$

Now for $n=1$, we have

$$\begin{aligned}x_1&=\frac{x_0x_{-p}}{ax_{-(p-1)}+bx_{-p}}=\frac{((-1)^{p-1}\lambda_-^{p-1}/a^{p-1})x_{-(p-1)}(-a/\lambda_-)x_{-(p-1)}}{ax_{-(p-1)}+b(-a/\lambda_-)x_{-(p-1)}}\\&=\frac{(-1)^p}{a^{p-1}}\frac{\lambda_-^{p-2}x_{-(p-1)}}{1-\frac{b}{\lambda_-}}=\frac{(-1)^p\lambda_-^p}{a^p}x_{-(p-1)}.\end{aligned}$$

Then we have

$$\frac{x_1}{(-1)^p\lambda_-^p/a^p}=\frac{x_0}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}}=\cdots=\frac{x_{-(p-2)}}{(-1)\lambda_-/a}=x_{-(p-1)}.$$

This implies that $(x_1,x_0,\cdots,x_{-p+1})\in\mathbb{D}_2$. Suppose now that $(x_n,x_{n-1},\cdots,x_{n-p})\in\mathbb{D}_2$. That is

$$\frac{x_n}{(-1)^p\lambda_-^p/a^p}=\frac{x_{n-1}}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}}=\cdots=\frac{x_{n-(p-1)}}{(-1)\lambda_-/a}=x_{n-p}.$$

Then

$$\begin{aligned}x_{n+1}&=\frac{x_nx_{n-p}}{ax_{n-(p-1)}+bx_{n-p}}=\frac{((-1)^{p-1}\lambda_-^{p-1}/a^{p-1})x_{n-(p-1)}(-a/\lambda_-)x_{n-(p-1)}}{ax_{n-(p-1)}+b(-a/\lambda_-)x_{n-(p-1)}}\\&=\frac{(-1)^p}{a^{p-1}}\frac{\lambda_-^{p-2}x_{n-(p-1)}}{1-\frac{b}{\lambda_-}}=\frac{(-1)^p\lambda_-^p}{a^p}x_{n-(p-1)}.\end{aligned}$$

This implies that

$$\frac{x_{n+1}}{(-1)^p\lambda_-^p/a^p}=\frac{x_n}{(-1)^{p-1}\lambda_-^{(p-1)}/a^{(p-1)}}=\cdots=\frac{x_{n-(p-2)}}{(-1)\lambda_-/a}=x_{n-(p-1)}.$$

That is $(x_{n+1},x_n,\cdots,x_{n-p+1})\in\mathbb{D}_2$. Then $(x_n,x_{n-1},\cdots,x_{n-p})\in\mathbb{D}_2$ for each $n\in\mathbb{N}$. Therefore, $\mathbb{D}_2$ is an invariant set for equation (2.1). By similar way, we can show that $\mathbb{D}_1$ is an invariant set for equation (2.1). This completes the proof. □

Theorem 2.3. *Assume that $\{x_n\}_{n=-p}^{\infty}$ is a well defined solution of equation (2.1). Then the following statements are true:*

1. *If $a+b>1$, then the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero.*
2. *If $a+b<1$, then the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.*

Proof. We can write $\phi_j=\lambda_+^j\frac{(1-(\frac{\lambda_-}{\lambda_+})^j)}{\sqrt{b^2+4a}}$.

1. If $a+b>1$, then $\lambda_+>1$. That is $\phi_m\to\infty$ as $m\to\infty$. Then $|\gamma_{-j}(m)|=|ax_{-j}\phi_j+x_{-j-1}\phi_{m+1}|\to\infty$ as $m\to\infty$, $j=\overline{0,(p-1)}$. This implies that for each $t=\overline{1,p}$, we have

$$|x_{pm+t}|=\left|\frac{v}{\prod_{i=1}^{t}\gamma_{-p+i}(m+1)\prod_{i=t+1}^{p}\gamma_{-p+i}(m)}\right|\to 0\text{ as }m\to\infty.$$

Therefore, the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero. For (2), it is enough to note that $\lambda_+<1$ when $a+b<1$.

This completes the proof. □

Theorem 2.4. *Assume that $a+b=1$, then every well defined solution $\{x_n\}_{n=-p}^{\infty}$ of equation (2.1) converges to a finite limit.*

Proof. When $a+b=1$, we have $\lambda_+ = 1$. Then

$$\gamma_{-p+i}(m) = ax_{-p+j}\phi_m + x_{-p+j-1}\phi_{m+1} \to \frac{ax_{-p+j}+x_{-p+j-1}}{1+a} \text{ as } m\to\infty,\ j=\overline{0,(p-1)}.$$

This implies that for each $t=\overline{1,p}$, we have

$$x_{pm+t} = \frac{\nu}{\prod_{i=1}^{t}\gamma_{-p+i}(m+1)\prod_{j=t+1}^{p}\gamma_{-p+j}(m)} \to \frac{(1+a)^p\nu}{\prod_{j=1}^{p}(ax_{-p+j}+x_{-p+j-1})} \text{ as } m\to\infty.$$

Therefore, the solution $\{x_n\}_{n=-p}^{\infty}$ of equation (2.1) converges to

$$\frac{(1+a)^p\nu}{\prod_{j=1}^{p}(ax_{-p+j}+x_{-p+j-1})} \text{ as } m\to\infty.$$

This completes the proof. □

2.2. Case $\Delta = 0$

During this subsection, we assume that $b^2=-4a$. When $b^2=-4a$, the solution of equation (2.5) is

$$z_m = \frac{1}{2}\left(\frac{b}{2}\right)^m (2z_0(1+m)-bm),\ m\geq -1.$$

It follows that

$$\begin{aligned} u_{pm+j} &= \frac{b}{2}\frac{(m+1)b-2u_{-p+j}(2+m)}{mb-2u_{-p+j}(1+m)} \\ &= \frac{b}{2}\frac{(m+1)bx_{-p+j}-2x_{-p+j-1}(2+m)}{mbx_{-p+j}-2x_{-p+j-1}(1+m)}, \quad 1\leq j\leq p. \end{aligned}$$

If we set $\beta_{-p+j}(m) = mbx_{-p+j}-2x_{-p+j-1}(1+m)$, then we can write

$$u_{pm+j} = \frac{b}{2}\frac{\beta_{-p+j}(m+1)}{\beta_{-p+j}(m)}, \quad 1\leq j\leq p. \tag{2.6}$$

Using equalities (2.2) and (2.6), we obtain the following result:

Theorem 2.5. *Let $\{x_n\}_{n=-p}^{\infty}$ be a well defined solution of equation (2.1). If $b^2+4a=0$, then*

$$x_n = \begin{cases} (-2)^p(\frac{2}{b})^n\frac{\nu}{\beta_{-p+1}(\frac{n+p-1}{p})\prod_{j=2}^{p}\beta_{-p+j}(\frac{n-1}{p})}, & n=1,p+1,\dots, \\ (-2)^p(\frac{2}{b})^n\frac{\nu}{\prod_{i=1}^{2}\beta_{-p+i}(\frac{n+p-2}{p})\prod_{j=3}^{p}\beta_{-p+j}(\frac{n-2}{p})}, & n=2,p+2,\dots, \\ \vdots & \vdots \\ (-2)^p(\frac{2}{b})^n\frac{\nu}{\prod_{i=1}^{p-1}\beta_{-p+i}(\frac{n+1}{p})\beta_0(\frac{n-p+1}{p})}, & n=p-1,2p-1,\dots, \\ (-2)^p(\frac{2}{b})^n\frac{\nu}{\prod_{i=1}^{p}\beta_{-p+i}(\frac{n}{p})}, & n=p,2p,\dots, \end{cases} \tag{2.7}$$

where $\nu=\prod_{i=0}^{p}x_{-i}$, $\beta_{-j}(m) = mbx_{-j}-2x_{-j-1}(1+m)$, $j=\overline{0,(p-1)}$ and $m\geq -1$.

Theorem 2.6. *Assume that $\{x_n\}_{n=-p}^{\infty}$ is a well defined solution of equation (2.1). The following statements are true:*

1. *If $b\geq 2$ then the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero.*
2. *If $b<2$ then the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.*

Proof. The solution formula (2.7) can be written in the form

$$x_{pm+t} = (-2)^p\left(\frac{2}{b}\right)^{pm+t}\frac{\nu}{\prod_{i=1}^{t}\beta_{-p+i}(m+1)\prod_{j=t+1}^{p}\beta_{-p+j}(m)}, \quad t=\overline{1,p}. \tag{2.8}$$

Clear that $\beta_{-p+i}(m)$ are unbounded, $i=\overline{1,p}$.

1. If $b\geq 2$, then $\frac{2}{b}\leq 1$ and the result follows.
2. If $b<2$, then $(\frac{2}{b})^{pm+t}\to\infty$ as $m\to\infty$ for all $t=\overline{1,p}$.
   Using formula (2.8), we can write for $t=1$

$$\begin{aligned} \mid x_{pm+1}\mid &=\mid (-2)^p(\frac{2}{b})^{pm+1}\frac{\nu}{\beta_{-p+1}(m+1)\prod_{j=2}^{p}\beta_{-p+j}(m)}\mid \\ &=\mid (-2)^p\mid \frac{(\frac{2}{b})^{pm+1}}{m^p(1+\frac{1}{m})}\times\mid\frac{\nu}{(bx_{-p+1}-2x_{-p}\frac{2+m}{1+m})\prod_{j=2}^{p}(bx_{-p+j}-2x_{-p+j-1}\frac{1+m}{m})}\mid. \end{aligned}$$

Using L'Hospital's rule we can show that

$$\frac{(\frac{2}{b})^{pm+1}}{m^p(1+\frac{1}{m})} \to \infty \text{ as } m \to \infty.$$

This implies that $\mid x_{pm+1} \mid \to \infty$ as $m \to \infty$. Similarly, $\mid x_{pm+t} \mid \to \infty$ as $m \to \infty$, $2 \le t \le p$. Therefore, the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.

This completes the proof. □

2.3. Case $\Delta < 0$

During this subsection, we assume that $b^2 < -4a$. When $b^2 < -4a$, the solution of equation (2.5) is

$$z_m = \frac{(-a)^{\frac{m}{2}}}{\sin\theta}\left(z_0 \sin(m+1)\theta - \sqrt{-a}\sin m\theta\right), \quad m \ge -1.$$

It follows that

$$u_{pm+j} = \sqrt{-a}\frac{\alpha_{-p+j}(m+1)}{\alpha_{-p+j}(m)}, \quad j = \overline{1,p}, \tag{2.9}$$

where $\theta = \arctan\left(\frac{\sqrt{-b^2-4a}}{b}\right)$, $sin\theta = \frac{\sqrt{-b^2-4a}}{2\sqrt{-a}}$ and $\alpha_{-p+j}(m) = x_{-p+j}\sqrt{-a}\sin m\theta - x_{-p+j-1}\sin(m+1)\theta$, $j = \overline{1,p}$, and $m \ge -1$. Using equalities (2.2) and (2.9), we obtain the following result:

Theorem 2.7. *Let $\{x_n\}_{n=-p}^{\infty}$ be a well defined solution of equation (2.1). If $b^2+4a<0$, then*

$$x_n = \begin{cases} \frac{(-1)^p \sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\alpha_{-p+1}(\frac{n+p-1}{p})\prod_{j=2}^{p}\alpha_{-p+j}(\frac{n-1}{p})}, & n = 1, p+1, \ldots, \\ \frac{(-1)^p \sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\prod_{i=1}^{2}\alpha_{-p+i}(\frac{n+p-2}{p})\prod_{j=3}^{p}\alpha_{-p+j}(\frac{n-2}{p})}, & n = 2, p+2, \ldots, \\ \vdots & \vdots \\ \frac{(-1)^p \sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\prod_{i=1}^{p-1}\alpha_{-p+i}(\frac{n+1}{p})\alpha_0(\frac{n-p+1}{p})}, & n = p-1, 2p-1, \ldots, \\ \frac{(-1)^p \sin^p\theta}{(\sqrt{-a})^n}\frac{\nu}{\prod_{i=1}^{p}\alpha_{-p+i}(\frac{n}{p})}, & n = p, 2p, \ldots, \end{cases} \tag{2.10}$$

where $\nu = \prod_{i=0}^{p} x_{-i}$, $\alpha_{-j}(m) = x_{-j}\sqrt{-a}\sin m\theta - x_{-j-1}\sin(m+1)\theta$, $j = \overline{0,(p-1)}$ and $m \ge -1$.

Theorem 2.8. *Assume that $(x_n)_{n=-p}^{\infty}$ is a well defined solution of equation (2.1). The following statements are true:*

1. *Let $a=-1$ and if $\theta = \frac{l}{M}\pi$ is a rational multiple of π (with $0 < l < \frac{M}{2}$), then $\{x_n\}_{n=-p}^{\infty}$ is periodic with prime period pM (if lp is even) or prime period $2pM$ (if lp is odd).*
2. *If $-1 < a < 0$, then the solution $\{x_n\}_{n=-p}^{\infty}$ is unbounded.*
3. *If $a < -1$, then the solution $\{x_n\}_{n=-p}^{\infty}$ converges to zero.*

Proof. We can write the solution (2.10) as

$$x_{pm+t} = \frac{(-1)^p \sin^p\theta}{(\sqrt{-a})^{pm+t}}\frac{\nu}{\prod_{i=1}^{t}\alpha_{-p+i}(m+1)\prod_{j=t+1}^{p}\alpha_{-p+j}(m)}, \tag{2.11}$$

where $t = \overline{1,p}$ and $m \ge -1$.

1. Suppose that $a=-1$ and let $\theta = \frac{l}{M}\pi$ be a rational multiple of π (with $0 < l < \frac{M}{2}$). Then for each $i = \overline{1,p}$, we have

$$\begin{aligned} \alpha_{-i}(m+M) &= x_{-i}\sin(m+M)\theta - x_{-i-1}\sin(m+M+1)\theta, \\ &= x_{-i}\sin(m\theta+M\theta) - x_{-i-1}\sin((m+1)\theta+M\theta), \\ &= x_{-i}\sin(m\theta+l\pi) - x_{-i-1}\sin((m+1)\theta+l\pi), \\ &= (-1)^l\alpha_{-i}(m). \end{aligned}$$

Then for each $t = \overline{1,p}$, we have

$$\begin{aligned} x_{pm+pM+t} &= (-1)^p\sin^p\theta\frac{\nu}{\prod_{i=1}^{t}\alpha_{-p+i}(m+M+1)\prod_{j=t+1}^{p}\alpha_{-p+j}(m+M)} \\ &= (-1)^{pl}x_{pm+t}. \end{aligned}$$

Therefore, if lp is even, then the solution $\{x_n\}_{n=-p}^{\infty}$ is periodic with prime period pM and if lp is odd, then the solution $\{x_n\}_{n=-p}^{\infty}$ is periodic with prime period $2pM$. (2) and (3) are directly obtained using (2.11).

This completes the proof. □

2.4. The forbidden sets

In this subsection, we introduce the forbidden sets of equation (2.1).

Theorem 2.9. *The following statements are true:*

1. *If $b^2+4a>0$, then the forbidden set of equation (2.1) can be written as*

$$
\begin{aligned}
F_1 = &\bigcup_{i=0}^{p}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-i}=0\right\}\cup\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+1}=-\frac{1}{a}\frac{\phi_{m+1}}{\phi_m}u_{-p}\right\}\cup\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+2}=-\frac{1}{a}\frac{\phi_{m+1}}{\phi_m}u_{-p+1}\right\}\cup\\
&\vdots\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_0=-\frac{1}{a}\frac{\phi_{m+1}}{\phi_m}u_{-1}\right\}.
\end{aligned}
$$

2. *If $b^2+4a=0$, then the forbidden set of equation (2.1) can be written as*

$$
\begin{aligned}
F_2 = &\bigcup_{i=0}^{p}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-i}=0\right\}\cup\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+1}=\frac{2(1+m)}{mb}u_{-p}\right\}\cup\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+2}=\frac{2(1+m)}{mb}u_{-p+1}\right\}\cup\\
&\vdots\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_0=\frac{2(1+m)}{mb}u_{-1}\right\}.
\end{aligned}
$$

3. *If $b^2+4a<0$, then the forbidden set of equation (2.1) can be written as*

$$
\begin{aligned}
F_3 = &\bigcup_{i=0}^{p}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-i}=0\right\}\cup\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+1}=\frac{\sin(m+1)\theta}{\sqrt{-a}\sin m\theta}u_{-p}\right\}\cup\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_{-p+2}=\frac{\sin(m+1)\theta}{\sqrt{-a}\sin m\theta}u_{-p+1}\right\}\cup\\
&\vdots\\
&\bigcup_{m=1}^{\infty}\left\{(u_0,u_{-1},...,u_{-p})\in\mathbb{R}^{p+1}: u_0=\frac{\sin(m+1)\theta}{\sqrt{-a}\sin m\theta}u_{-1}\right\}.
\end{aligned}
$$

3. Illustrative Examples

Example 3.1. *Figure 3.1 shows that, if $p=7$, $a=0.2$ and $b=1\,(\Delta>0\, and\, a+b>1)$, then a solution $\{x_n\}_{n=-7}^{\infty}$ of equation (2.1) with $x_{-7}=-4, x_{-6}=-5, x_{-5}=-3$, $x_{-4}=-8.2$, $x_{-3}=5$, $x_{-2}=3$, $x_{-1}=6.2$ and $x_0=-7$ converges to zero.*

Example 3.2. *Figure 3.2 shows that, if $p=4$, $a=0.1$ and $b=0.7\,(\Delta>0\, and\, a+b<1)$, then a solution $\{x_n\}_{n=-4}^{\infty}$ of equation (2.1) with $x_{-4}=-1, x_{-3}=-3, x_{-2}=-5.9, x_{-1}=-3$ and $x_0=-12.2$ is unbounded.*

Example 3.3. *Figure 3.3 shows that, if $p=7$, $a=-1$ and $b=2\,(\Delta=0)$, then a solution $\{x_n\}_{n=-7}^{\infty}$ of equation (2.1) with $x_{-7}=-2$, $x_{-6}=-5$, $x_{-5}=-3$, $x_{-4}=-12.2$, $x_{-3}=5$, $x_{-2}=3$, $x_{-1}=6.2$ and $x_0=-5$ converges to zero.*

Example 3.4. *Figure 3.4 shows that, if $p=7$, $a=-1/4$ and $b=1\,(\Delta=0\, and\, b<2)$, then a solution $\{x_n\}_{n=-7}^{\infty}$ of equation (2.1) with $x_{-7}=-4$, $x_{-6}=-5.3$, $x_{-5}=-1.3$, $x_{-4}=-9.2$, $x_{-3}=6$, $x_{-2}=13$, $x_{-1}=6.2$ and $x_0=-5$ is unbounded.*

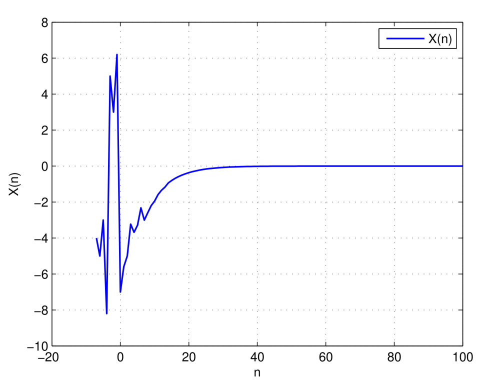

Figure 3.1: Equation $x_{n+1}=\frac{x_n x_{n-7}}{0.2x_{n-6}+x_{n-7}}$.

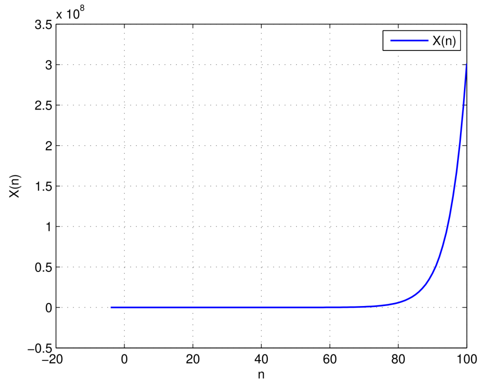

Figure 3.2: Equation $x_{n+1}=\frac{x_n x_{n-4}}{0.1x_{n-3}+0.7x_{n-4}}$.

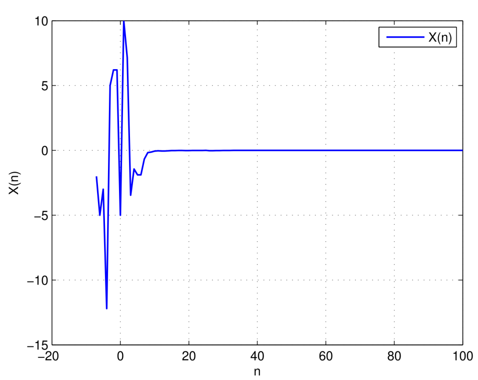

Figure 3.3: Equation $x_{n+1}=\frac{x_n x_{n-7}}{-x_{n-6}+2x_{n-7}}$.

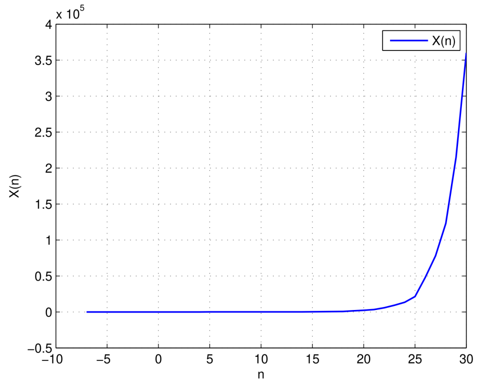

Figure 3.4: Equation $x_{n+1}=\frac{x_n x_{n-7}}{-0.25x_{n-6}+x_{n-7}}$.

Example 3.5. *Figure 3.5 shows that, if $p=4$, $a=-1$ and $b=\sqrt{3}\,(\Delta<0\,\text{and}\,l\,p\,\text{is}\,\text{even})$, then a solution $\{x_n\}_{n=-4}^{\infty}$ of equation (2.1) with $x_{-4}=-2$, $x_{-3}=-5$, $x_{-2}=3$, $x_{-1}=2.2$ and $x_0=5$ is periodic with prime period 24.*

Example 3.6. *Figure 3.6 shows that, if $p=7$, $a=-1$ and $b=1\,(\Delta<0\,\text{and}\,l\,p\,\text{is}\,\text{odd})$, then a solution $\{x_n\}_{n=-7}^{\infty}$ of equation (2.1) with $x_{-7}=-1$, $x_{-6}=-7$, $x_{-5}=-4$, $x_{-4}=-12.2$, $x_{-3}=5$, $x_{-2}=3$, $x_{-1}=6.2$ and $x_0=-5$ is periodic with prime period 42.*

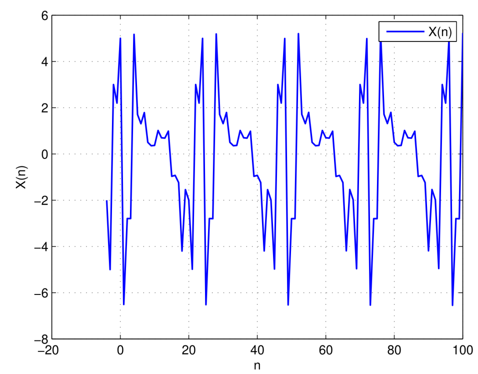

Figure 3.5: Equation $x_{n+1}=\frac{x_n x_{n-4}}{-x_{n-3}+\sqrt{3}x_{n-4}}$.

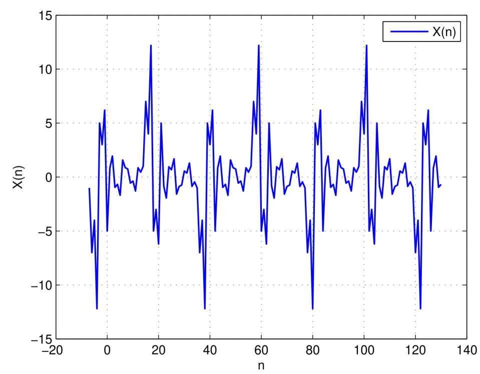

Figure 3.6: Equation $x_{n+1}=\frac{x_n x_{n-7}}{-x_{n-6}+x_{n-7}}$.

Example 3.7. *Figure 3.7 shows that, if $p=3$, $a=0.3$ and $b=0.7\,(\Delta>0\,\text{and}\,a+b=1)$, then a solution $\{x_n\}_{n=-3}^{\infty}$ of equation (2.1) with initial conditions $x_{-3}=1$, $x_{-2}=-2$, $x_{-1}=1$ and $x_0=0.7$ converges to*

$$\frac{(1.3)^3((1)(-2)(1)(0.7))}{\prod_{j=1}^{3}(0.3x_{-3+j}+x_{-4+j})}\simeq 3.738.$$

Example 3.8. *Figure 3.8 shows that, if $p=5$, $a=0.2$ and $b=0.8\,(\Delta>0\,\text{and}\,a+b=1)$, then a solution $\{x_n\}_{n=-5}^{\infty}$ of equation (2.1) with initial conditions $x_{-5}=-2$, $x_{-4}=-1$, $x_{-3}=0.5$, $x_{-2}=0.8$, $x_{-1}=0.7$ and $x_0=-0.8$ converges to*

$$\frac{(1.2)^5((-2)(-1)(0.5)(0.8)(0.7)(-0.8))}{\prod_{j=1}^{5}(0.2x_{-5+j}+x_{-6+j})}\simeq -1.681.$$

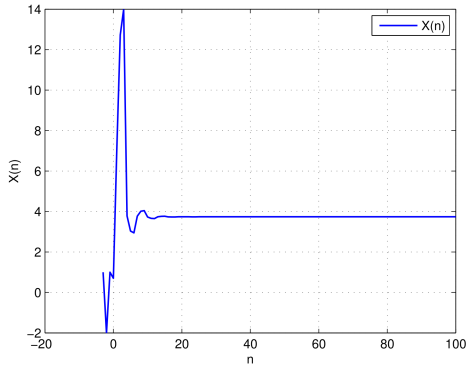

Figure 3.7: Equation $x_{n+1} = \frac{x_n x_{n-3}}{0.3x_{n-2}+0.7x_{n-3}}$.

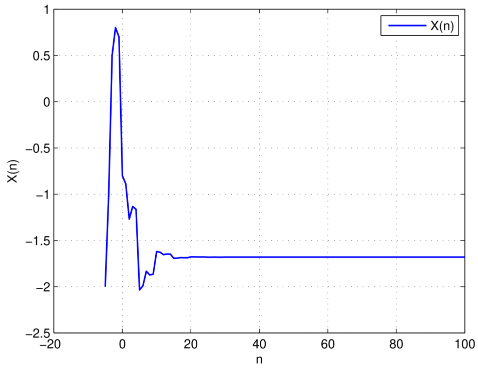

Figure 3.8: Equation $x_{n+1} = \frac{x_n x_{n-7}}{0.2x_{n-6}+0.8x_{n-7}}$.

Conclusion

In this study, we mainly obtained the solutions and introduced the forbidden sets of the difference equation that contains a quadratic term

$$x_{n+1} = \frac{x_n x_{n-p}}{a x_{n-(p-1)} + b x_{n-p}}, \quad n \in \mathbb{N}_0,$$

where the parameters a and b are real numbers, p is a positive integer and the initial conditions $x_{-p}, x_{-p+1}, \cdots, x_{-1}, x_0$ are real numbers. Also, we showed that the behavior of the solutions depends on the relation between a and b. That is if $\{x_n\}_{n=-p}^{\infty}$ is a solution of that equation, it may be converge to finite limit, unbounded or periodic with a certain period that depends on p. The mentioned difference equation may be generalized to a more complicated one that may has a complicated behavior.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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