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Completeness of the Category of Rack Crossed Modules

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Abstract — In this paper, we prove that the category of rack crossed modules (with a fixed codomain) is finitely complete. In other words, we construct the product, pullback and equalizer objects in the category of crossed modules of racks. We therefore unify the group-theoretical analogy of the completeness property in the sense of the functor **Conj**: **Grp** → **Rack**.

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1. Introduction

A rack is a set equipped with a binary operation satisfying two axioms that match to the second and third Reidemeister moves in knot theory. The most important and common rack which additionally satisfies an extra axiom analogues to the first Reidemeister move called quandle [11].

Crossed modules of groups are first introduced by Whitehead in [13] as models for homotopy 2-types. Afterwards, the notion of crossed module is also adapted to various algebraic structures such as algebras, Lie algebras, Hopf algebras; see [10] for more.

Crossed modules of racks are defined by Crans and Wagemann in [4]. They generalize the notion of crossed modules from the case of groups such that satisfying two Peiffer conditions as well. An interesting result of this notion is: the adjoint functors **As**: **Rack** → **Grp** and **Conj**: **Grp** → **Rack** between the categories of groups and racks are both preserving crossed module structures, see [4]. Therefore, one can consider them as (induced) functors between the category of group crossed modules **XGrp** and the category of rack crossed modules **XRack**; hence we get the following extended adjunction:

$$\mathrm{Hom}_{\mathrm{XGrp}}\big(\mathrm{As}^{\star}(\mathscr{X}),\mathscr{G}\big) \cong \mathrm{Hom}_{\mathrm{XRack}}\big(\mathscr{X},\mathrm{Conj}^{\star}(\mathscr{G})\big), \tag{1.1}$$

where $\mathscr{X}$ is a rack crossed module and $\mathscr{G}$ is a group crossed module [7].

A category $\mathscr{C}$ is said to be finitely complete if it has all (finite) limits. On the other hand, a category $\mathscr{C}$ has all finite limits iff one of the following conditions hold:

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- $\mathscr{C}$ has a terminal object and pullbacks,
- $\mathscr{C}$ has products and equalisers.

In this paper, we define such objects for rack crossed modules which will prove the completeness of the category of rack crossed modules. As another outcome, these constructions will be preserved under the functor **Conj** by using the properties of (1.1).

Many of these notions are examined for various algebraic structures such as (crossed modules and cat^1 objects of) groups, (associative) algebras, Lie algebras, etc. in [1–3, 5, 6, 8, 12].

2. Preliminaries

We recall some notions from [4, 9] which will be used in sequel.

2.1. Racks

Definition 2.1. A (right) rack consists of a set A equipped with a binary operation, satisfying:

$R1$) For each $a, a' \in A$, there exists a unique $a'' \in A$ such that:

$$a'' \lhd a = a',$$

$R2$) For all $a, a', a'' \in A$, we have:

$$\left(a \lhd a'\right) \lhd a'' = \left(a \lhd a''\right) \lhd \left(a' \lhd a''\right).$$

Definition 2.2. A pointed rack A is a rack equipped with a fixed element $1 \in A$ such that:

$$1 \lhd a = 1 \quad \text{and} \quad a \lhd 1 = a,$$

for all $a \in A$.

Remark 2.3. In this paper we only work with the pointed racks.

Definition 2.4. Let A and B be two racks. A rack morphism is a map:

$$f\colon A \to B$$

such that:

$$f\left(a \lhd a'\right) = f(a) \lhd f\left(a'\right) \qquad \left(\text{and } f(1) = 1\right)$$

for all $a, a' \in A$.

Thus we get the category of racks, denoted by **Rack**.

Some well-known examples of racks are given below:

1) If A is a group, one can define a rack (conjugation rack) with:

$$a \lhd a' = (a')^{-1} a a',$$

for all $a, a' \in A$. This yields a functor:

$$\mathbf{Conj} : \mathbf{Grp} \to \mathbf{Rack}$$

from the category of groups to the category of racks.

2) Another rack structure in a group A is defined by:

$$a \lhd a' = a' a^{-1} a',$$

for all $a, a' \in A$, which is called a core rack. But this is not functorial.

3) If A and B are two racks, then the set

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

which is the cartesian product of A and B, defines a rack structure with: $(a, a' \in A,\ b, b' \in B)$

$$(a, b) \lhd \left(a', b'\right) = \left(a \lhd a', b \lhd b'\right).$$

Notice that $A \times B$ is the product object in **Rack**.

Definition 2.5. Let A be a rack and B be a non empty subset of A . We say that B is a subrack of A if $b \lhd b' \in B$ for each $b, b' \in B$.

Definition 2.6. For a given rack A, a normal subrack N is a subrack if it further satisfies $n \lhd a \in N$, for all $n \in N$ and $a \in A$.

2.2. Rack Action

Definition 2.7. Let A be a rack and S be a set. We say that S is an A-set when there are bijections $(\cdot\ a) : S \to S$ for each $a \in A$ such that:

$$(s \cdot a) \cdot a' = \left(s \cdot a'\right) \cdot \left(a \lhd a'\right),$$

for all $s \in S$ and $a' \in A$.

Definition 2.8. Let A be a rack and S be a A-set. We say that the hemi-semi-direct product $S \rtimes A$ is a rack with:

$$(s, a) \lhd \left(s', a'\right) = \left(s \cdot a', a \lhd a'\right)$$

for all $a, a' \in A$, $s, s' \in S$.

Definition 2.9. Let A, S be two racks. We say that S acts on A by automorphisms when there is a (right) rack action of S on A and:

$$(a \lhd a') \cdot s = (a \cdot s) \lhd (a' \cdot s)$$

for all $a, a' \in A$, $s \in S$.

3. Crossed Modules of Racks

Definition 3.1. A rack crossed module [4] is a rack morphism $\partial\colon A \to B$ together with a (right) rack action of B on A such that satisfying:

X1) $\partial(a \cdot b) = \partial(a) \lhd b$,

X2) $a \cdot \partial(a') = a \lhd a'$,

for all $a, a' \in A$ and $b \in B$. We denote any crossed module by (A, B, ∂).

If (A, B, ∂) and (A', B', ∂') are two rack crossed modules, a crossed module morphism:

$$(f_1, f_0) : (A, B, \partial) \to (A', B', \partial')$$

is a tuple which consists of rack morphisms $f_1 : A \to A'$, $f_0 : B \to B'$ such that making the following diagram commutative:

$$\begin{array}{ccc} A & \xrightarrow{\partial} & B \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ A' & \xrightarrow[\partial']{} & B' \end{array}$$

and:

$$f_1(a \cdot b) = f_1(a) \cdot f_0(b)$$

for all $a \in A$, $b \in B$.

Thus we get the category of rack crossed modules, denoted by **XRack**. We also have a full subcategory **XRack/B** where the codomain B is fixed for any rack crossed module.

Some examples of rack crossed modules are given below:

1) For any normal subrack N of S, the inclusion map $N \to S$ defines a rack crossed module with: $(n \in N, s \in S)$

$$n \cdot s = n \lhd s.$$

2) If (X, Y, μ) is a group crossed module, we obtain a rack crossed module by passing to the associated conjugation racks of X and Y as being:

$$\mathbf{Conj}(\mu)\colon \mathbf{Conj}(\mathbf{X}) \to \mathbf{Conj}(\mathbf{Y}).$$

3) If (X, Y, μ) and (X', Y', μ') are rack crossed modules, then:

$$(X \times X', Y \times Y', \mu \times \mu')$$

defines a rack crossed module where the action is defined in a natural way.

Definition 3.2. Let $\lambda\colon A \to C$ and $\theta\colon B \to C$ be two rack morphisms. The fiber product is the subrack of $A \times B$

defined by:

$$A \times_C B = \{(a,b) \mid \lambda(a) = \theta(b)\}.$$

From the categorical point of view, the fiber product is the equalizer of the two parallel rack morphisms:

$$A \times B \underset{\theta \circ \pi_2}{\overset{\lambda \circ \pi_1}{\rightrightarrows}} C\ .$$

Proposition 3.3. Let (A, C, λ) and (B, C, θ) be two rack crossed modules. The map:

$$\partial\colon\ A \times_C B \to C$$

defined by:

$$\partial(a,b) = \lambda(a) = \theta(b)$$

yields a crossed module $(A \times_C B, C, \partial)$ with the rack action:

$$\begin{array}{ccc} (A \times_C B) \times C & \rightarrow & A \times_C B \\ ((a,b), c) & \mapsto & (a,b) \cdot c = (a \cdot c, b \cdot c) \end{array}$$

Proposition 3.4. Let (A, C, μ) and (B, C, λ) be two rack crossed modules. Then we have the following natural rack crossed module morphisms:

$$\left(p_1, id_C\right) : (A \times_C B, C, \partial) \to \left(A, C, \mu\right),$$
$$\left(p_2, id_C\right) : (A \times_C B, C, \partial) \to (B, C, \lambda).$$

4. Some Categorical Constructions in XRack/C

Recall that, the trivial rack is the zero object in the category of racks. Moreover, for given two rack morphisms $f\colon A \to C$ and $g\colon B \to C$, we have the rack $A \times_C B$ which is the pullback. Therefore, we say that the category of racks **Rack** is finitely complete.

In this section, we give some categorical constructions for the case of rack crossed modules which will prove the completeness of **XRack/C**.

Theorem 4.1. The category **XRack/C** has products.

Proof.
Let (A, C, μ) and (B, C, λ) be two rack crossed modules. Define

$$\partial\colon\ A \times_C B \to C$$

where:

$$\partial(a,b) = \mu(a) = \lambda(b).$$

We already know from Proposition 3.3 that ∂ is a crossed module. Also we have crossed module morphisms $\left(p_1, id_C\right)$ and $\left(p_2, id_C\right)$ from Proposition 3.4. Now we need to check the universal property.

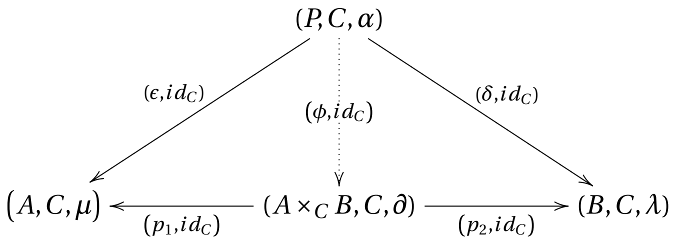
Let (P, C, α) be a crossed module with two crossed module morphisms:

$$(\epsilon, id_C) : (P, C, \alpha) \to (A, C, \mu),$$
$$(\delta, id_C) : (P, C, \alpha) \to (B, C, \lambda).$$

Then there must be a unique crossed module morphism:

$$(\phi, id_C) : (P, C, \alpha) \to (A \times_C B, C, \partial)$$

such that the diagram:

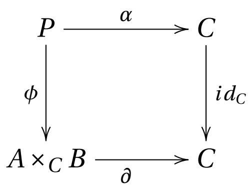


(4.1)

commutative. Define:

$$\phi(p) = (\epsilon(p), \delta(p)),$$

for all $p \in P$. By the diagram:



(ϕ, id_C) becomes a crossed module morphism since:

$$\begin{aligned}\phi(p \cdot c) &= (\epsilon(p \cdot c), \delta(p \cdot c)) \\ &= (\epsilon(p) \cdot id_C(c), \delta(p) \cdot id_C(c)) \\ &= (\epsilon(p), \delta(p)) \cdot id_C(c) \\ &= \phi(p) \cdot id_C(c),\end{aligned}$$

and

$$\begin{aligned}\partial\phi(p) &= \partial(\epsilon(p), \delta(p)) \\ &= \mu(\epsilon(p)) \\ &= id_C \alpha(p),\end{aligned}$$

for all $p \in P$ and for all $c \in C$.

Furthermore diagram (4.1) is commutative because:

$$\begin{aligned} p_1\phi\left(p\right) &= p_1\left(\epsilon\left(p\right),\delta\left(p\right)\right) \\ &= \epsilon\left(p\right), \end{aligned}$$

$$\begin{aligned} p_2\phi\left(p\right) &= p_2\left(\epsilon\left(p\right),\delta\left(p\right)\right) \\ &= \delta\left(p\right), \end{aligned}$$

for all $p \in P$.

Let $\left(\phi', id_C\right)$ be a crossed module with:

$$\begin{aligned} \left(p_1, id_C\right)\left(\phi', id_C\right) &= \left(\epsilon, id_C\right), \\ \left(p_2, id_C\right)\left(\phi', id_C\right) &= \left(\delta, id_C\right). \end{aligned}$$

Define $(a,b) \in A \times_C B$ by $\phi'\left(p\right) = (a,b)$. Then we get:

$$\begin{aligned} p_1\phi'\left(p\right) = \epsilon\left(p\right) &\Leftrightarrow p_1(a,b) = \epsilon\left(p\right) \\ &\Leftrightarrow a = \epsilon\left(p\right), \\ p_2\phi'\left(p\right) = \delta\left(p\right) &\Leftrightarrow p_2(a,b) = \delta\left(p\right) \\ &\Leftrightarrow b = \delta\left(p\right) \end{aligned}$$

for all $p \in P$ that proves the uniqueness of ϕ by:

$$\phi'\left(p\right) = (a,b) = \left(\epsilon\left(p\right),\delta\left(p\right)\right) = \phi\left(p\right).$$

Theorem 4.2. The category **XRack/C** has pullbacks.

Proof.
Let $\left(f, id_C\right): \left(A, C, \mu\right) \to (D, C, \theta)$ and $\left(g, id_C\right): (B, C, \lambda) \to (D, C, \theta)$ be two crossed module morphisms. We already know from Proposition 3.3 that,

$$\partial: A \times_D B \to C$$

is a crossed module. Also we have crossed module morphisms $\left(p_1, id_C\right)$ and $\left(p_2, id_C\right)$ from Proposition 3.4. Then we get the diagram:

$$\begin{array}{ccc} (A \times_D B, C, \partial) & \xrightarrow{(p_2, id_C)} & (B, C, \lambda) \\ \Big\downarrow{\scriptstyle (p_1, id_C)} & & \Big\downarrow{\scriptstyle (g, id_C)} \\ \left(A, C, \mu\right) & \xrightarrow[(f, id_C)]{} & (D, C, \theta) \end{array}$$

which is commutative. Let (P,C,δ) be a crossed module with the following crossed module morphisms:

$$(\alpha, id_C) : (P,C,\delta) \to \left(A,C,\mu\right),$$
$$\left(\beta, id_C\right) : (P,C,\delta) \to (B,C,\lambda),$$

where:

$$\left(f, id_C\right)(\alpha, id_C) = \left(g, id_C\right)\left(\beta, id_C\right).$$

Then there must be a unique crossed module morphism:

$$\left(\phi, id_C\right) : (P,C,\delta) \to (A \times_D B, C, \partial)$$

such that the diagram:

$$\begin{array}{ccccc} (P,C,\delta) & & & & \\ & \overset{(\phi,id_C)}{\searrow} & & \overset{(\beta,id_C)}{\longrightarrow} & \\ & & (A\times_D B, C, \partial) & \xrightarrow[(p_2,id_C)]{} & (B,C,\lambda) \\ & {\scriptstyle(\alpha,id_C)} \searrow & \Big\downarrow{\scriptstyle(p_1,id_C)} & & \Big\downarrow{\scriptstyle(g,id_C)} \\ & & \left(A,C,\mu\right) & \xrightarrow[(f,id_C)]{} & (D,S,\theta). \end{array} \tag{4.2}$$

commutes. Define

$$\phi(p) = (\alpha(p), \beta(p))$$

for all $p \in P$. By the diagram:

$$\begin{array}{ccc} P & \xrightarrow{\delta} & C \\ {\scriptstyle\phi}\Big\downarrow & & \Big|{\scriptstyle id_C} \\ A \times_D C & \xrightarrow[\partial]{} & C \end{array}$$

(ϕ, id_C) becomes a crossed module morphism since:

$$\begin{aligned} \phi\left(p\cdot c\right) &= \left(\alpha\left(p\cdot c\right), \beta\left(p\cdot c\right)\right) \\ &= \left(\alpha\left(p\right)\cdot id_C\left(c\right), \beta\left(p\right)\cdot id_C\left(c\right)\right) \\ &= (\alpha(p), \beta(p))\cdot id_C\left(c\right) \\ &= \varphi(p)\cdot id_C\left(c\right), \end{aligned}$$

and

$$\begin{aligned}\partial\phi(p) &= \partial(\alpha(p), \beta(p)) \\ &= \mu\alpha\left(p\right) \\ &= id_C\delta\left(p\right),\end{aligned}$$

for all $p \in P$ and for all $c \in C$.

Furthermore we get:

$$\begin{aligned}p_1\phi\left(p\right) &= p_1\left(\alpha(p), \beta(p)\right) \\ &= \alpha(p),\end{aligned}$$

and

$$\begin{aligned}p_2\phi\left(p\right) &= p_2\left(\alpha(p), \beta(p)\right) \\ &= \beta(p),\end{aligned}$$

for all $p \in P$ that proves the commutativity of diagram (4.2).

Consider $\left(\phi', id_C\right)$ with the same property as $\left(\phi, id_C\right)$, i.e. the following conditions hold:

$$\begin{aligned}\left(p_1, id_C\right)\left(\phi', id_C\right) &= (\alpha, id_C), \\ \left(p_2, id_C\right)\left(\phi', id_C\right) &= \left(\beta, id_C\right).\end{aligned}$$

Define $(a,b) \in A \times_D B$ by $\phi'(p) = (a,b)$. We get:

$$\begin{aligned}p_1\phi'\left(p\right) = \alpha\left(p\right) &\Leftrightarrow p_1(a,b) = \alpha\left(p\right) \\ &\Leftrightarrow a = \alpha\left(p\right) \\ p_2\phi'\left(p\right) = \beta\left(p\right) &\Leftrightarrow p_2\left(a,b\right) = \beta\left(p\right) \\ &\Leftrightarrow b = \beta\left(p\right)\end{aligned}$$

for all $p \in P$ that proves the uniquness of ϕ by:

$$\begin{aligned}\phi'(p) &= (a,b) \\ &= \left(\alpha\left(p\right), \beta\left(p\right)\right) \\ &= \phi(p).\end{aligned}$$

Theorem 4.3. The category **XRack/C** has equalizers.

Proof.

Let we have two parallel crossed module morphisms:

$$\left(A,C,\mu\right)\underset{(g,id_C)}{\overset{(f,id_C)}{\rightrightarrows}}(B,C,\lambda).$$

Define:

$$P=\left\{a\in A\mid f(a)=g(a)\right\}.$$

P is a subrack of A, since:

$$\begin{aligned}f\left(a\lhd a'\right)&=f(a)\lhd f\left(a'\right)\\&=g(a)\lhd g\left(a'\right)\\&=g\left(a\lhd a'\right),\end{aligned}$$

for all $a,a'\in P$.

Define $\partial:P\to C$ by $\partial(a)=\mu(a)$, for all $a\in P$. Here ∂ becomes a rack morphism since:

$$\begin{aligned}\partial\left(a\lhd a'\right)&=\mu\left(a\lhd a'\right)\\&=\mu(a)\lhd\mu\left(a'\right)\\&=\partial(a)\lhd\partial\left(a'\right),\end{aligned}$$

for all $a,a'\in P$.

Moreover, (P,C,∂) is a rack crossed module since:

XM1)

$$\begin{aligned}\partial(a\cdot c)&=\mu(a\cdot c)\\&=\mu(a)\lhd c\\&=\partial(a)\lhd c,\end{aligned}$$

XM2)

$$\begin{aligned}a\cdot\partial\left(a'\right)&=a\cdot\mu\left(a'\right)\\&=a\lhd a',\end{aligned}$$

for all $a,a'\in P$ and $c\in C$.

The tuple:

$$(i,id_C):(P,C,\partial)\to\left(A,C,\mu\right)$$

where i is the inclusion map is a crossed module morphism since:

$$\begin{aligned} i\,(a\cdot c) &= a\cdot c \\ &= i\,(a)\cdot c \\ &= i\,(a)\cdot id_C\,(c)\,, \end{aligned}$$

and

$$\begin{aligned} \mu i\,(a) &= \mu\,(a) \\ &= \partial\,(a) \\ &= id_C \partial\,(a)\,, \end{aligned}$$

for all $a \in P$ and $c \in C$.

Furthermore, we get:

$$\begin{aligned} \left(fi\right)(a) &= f\,(i\,(a)) \\ &= f\,(a) \\ &= g\,(a) \\ &= g\,(i\,(a)) \\ &= \left(gi\right)(a)\,, \end{aligned}$$

for all $a \in P$ that proves the commutativity of diagram:

$$(P, C, \partial) \xhookrightarrow{(i, id_C)} \left(A, C, \mu\right) \underset{(g, id_C)}{\overset{(f, id_C)}{\rightrightarrows}} (B, C, \lambda)\,.$$

Let

$$(h, id_C) : (Q, C, \delta) \rightarrow \left(A, C, \mu\right)$$

be any crossed module morphism such that the diagram:

$$(Q, C, \delta) \xrightarrow{(h, id_C)} \left(A, C, \mu\right) \underset{(g, id_C)}{\overset{(f, id_C)}{\rightrightarrows}} (B, C, \lambda)$$

commutes. Then there must be a unique crossed module morphism:

$$\left(\varphi, id_C\right) : (Q, C, \delta) \rightarrow (P, C, \partial)\,,$$

such that the diagram:

$$\begin{array}{ccccc} (P,C,\partial) & \xrightarrow{(i,id_C)} & (A,C,\mu) & \overset{(f,id_C)}{\underset{(g,id_C)}{\rightrightarrows}} & (B,C,\lambda) \\ \uparrow{\scriptstyle (\varphi,id_C)} & \nearrow{\scriptstyle (h,id_C)} & & & \\ (Q,C,\delta) & & & & \end{array} \tag{4.3}$$

commutes. We can say that $h(q) \in P$ since:

$$f(h(q)) = g(h(q)),$$

for all $q \in Q$. Define φ by $\varphi(q) = h(q)$ for all $q \in Q$. Then we get:

$$\begin{aligned} i\varphi(q) &= ih(q) \\ &= h(q), \end{aligned}$$

for all $q \in Q$ proves the commutativity of (4.3).

Consider (φ', id_C) with the same property as (φ, id_C), i.e. the following condition hold:

$$(i, id_C)(\varphi', id_C) = (h, id_C)$$

Define $q \in Q$ by $\varphi'(q) = a$. We get:

$$\begin{aligned} i\varphi'(q) = h(q) &\Leftrightarrow i(q) = h(q) \\ &\Leftrightarrow a = h(q) \end{aligned}$$

for all $q \in Q$ that proves the uniquness of φ by:

$$\begin{aligned} \varphi'(q) &= a \\ &= \varphi(q). \end{aligned}$$

Therefore, we have proved the following:

Theorem 4.4. The category of rack crossed modules **XRack/C** is (finitely) complete.

5. Conclusion

We already know that, we have the adjunction:

$$\mathrm{Hom}_{\mathrm{XGrp}}(\mathrm{As}^{\star}(\mathscr{X}), \mathscr{G}) \cong \mathrm{Hom}_{\mathrm{XRack}}(\mathscr{X}, \mathrm{Conj}^{\star}(\mathscr{G})),$$

between the category of rack crossed modules and the category of group crossed modules. As a result of this adjunction, we can say that the functor $\mathrm{Conj}^{\star}$ preserve limits and $\mathrm{As}^{\star}$ preserve colimits. Therefore, all the constructions given in the previous section which are the certain cases of limits are preserved under the functor $\mathrm{Conj}^\star$.

For instance, let $\mathscr{A}$ and $\mathscr{B}$ be two group crossed modules with the same codomain. Their product is the crossed module $\mathscr{D}$ which is defined by using the fiber product of groups. By using the adjunction above, we can say that the product of rack crossed modules $\mathrm{Conj}^\star(\mathscr{A})$ and $\mathrm{Conj}^\star(\mathscr{B})$ is $\mathrm{Conj}^\star(\mathscr{D})$. As we mentioned above, not only the product object, we can also give the similar properties for all of the notions we have defined.

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