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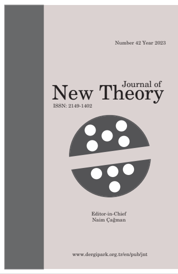

On the Hyperbolic Leonardo and Hyperbolic Francois Quaternions

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Research Article

Abstract — In this paper, we present a new definition, referred to as the Francois sequence, related to the Lucas-like form of the Leonardo sequence. We also introduce the hyperbolic Leonardo and hyperbolic Francois quaternions. Afterward, we derive the Binet-like formulas and their generating functions. Moreover, we provide some binomial sums, Honsberger-like, d'Ocagne-like, Catalan-like, and Cassini-like identities of the hyperbolic Leonardo quaternions and hyperbolic Francois quaternions that allow an understanding of the quaternions' properties and their relation to the Francois sequence and Leonardo sequence. Finally, considering the results presented in this study, we discuss the need for further research in this field.

Keywords *Fibonacci numbers, Leonardo numbers, Lucas numbers, Francois numbers, hyperbolic quaternions*

Mathematics Subject Classification (2020) 11B39, 05A15

1. Introduction

The algebra of hyperbolic quaternions in abstract algebra is a non-associative algebra over real numbers with elements of the form

$$q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3$$

where q_0, q_1, q_2, and q_3 are real numbers and e_0, e_1, e_2, and e_3 are the standard basis in $\mathbb{R}^4$. The hyperbolic quaternion multiplication is defined using the rules

$$e_0^2 = e_1^2 = e_2^2 = e_3^2 = 1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1, \quad \text{and} \quad e_3e_1 = -e_1e_3 = e_2$$

This algebra is also non-commutative. Let $q = q_0e_0+q_1e_1+q_2e_2+q_3e_3$ and $p = p_0e_0+p_1e_1+p_2e_2+p_3e_3$ be any two hyperbolic quaternions. Then, the addition and subtraction of the hyperbolic quaternions are

$$q \mp p = (q_0 \mp p_0)e_0 + (q_1 \mp p_1)e_1 + (q_2 \mp p_2)e_2 + (q_3 \mp p_3)e_3$$

and multiplication of the hyperbolic quaternions is

$$\begin{aligned} qp &= (q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3)(p_0e_0 + p_1e_1 + p_2e_2 + p_3e_3) \\ &= (q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3)e_0 + (q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2)e_1 \\ &+ (q_0p_2 - q_2p_0 + q_1p_3 + q_3p_1)e_2 + (q_0p_3 + q_3p_0 - q_1p_2 + q_2p_1)e_3 \end{aligned}$$

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Moreover, for $k \in \mathbb{R}$, the multiplication by scalar is

$$kq = kq_0e_0 + kq_1e_1 + kq_2e_2 + kq_3e_3$$

and conjugate and norm of the hyperbolic quaternion q are

$$\overline{q} = q_0e_0 - q_1e_1 - q_2e_2 - q_3e_3$$

and

$$||q|| = \sqrt{|q\overline{q}|} = \sqrt{q_0^2 - q_1^2 - q_2^2 - q_3^2}$$

respectively. One of the non-associative hyperbolic number systems, ideal for studying space-time theories in relativities, is the hyperbolic quaternions. Many studies have been published on hyperbolic quaternions. Macfarlane yields the hyperbolic counterpart of the spherical quaternions in [1]. Kösal introduces hyperbolic quaternions and their algebraic properties in [2]. The four-dimensional real algebra of bihyperbolic numbers is studied by Bilgin and Ersoy in [3]. An alternative representational method is proposed for the formulation of classical and generalized electromagnetism in the case of the existence of magnetic monopoles and massive photons after presenting the hyperbolic quaternion formalism by Demir et al. in [4]. Kuruz introduces hyperbolic matrices with hyperbolic number entries in [5]. Assis presents some properties of mathematical and physical interest in generalized algebras of two, three, and four dimensions in [6]. The Fibonacci and Lucas sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are defined by two order recurrences, respectively,

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n \tag{1}$$

and

$$L_0 = 2, \quad L_1 = 1, \quad \text{and} \quad L_{n+2} = L_{n+1} + L_n \tag{2}$$

Here, F_n and L_n are the nth Fibonacci and Lucas numbers. First few terms of these sequences are, respectively,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

and

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199$$

The Recurrences 1 or 2 involve the characteristic equation

$$x^2 - x - 1 = 0 \tag{3}$$

The roots of Equation 3 are

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} \tag{4}$$

Then, the following relations can be derived

$$\alpha + \beta = 1 \quad \alpha - \beta = \sqrt{5} \quad \alpha\beta = -1$$

Therefore, the Binet formulas for the Fibonacci and Lucas sequences are, respectively,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

More information for the Fibonacci and Lucas numbers are given in [7, 8]. The Leonardo sequence $\{\mathcal{L}_n\}_{n\geq 0}$ is defined by recurrence

$$\mathcal{L}_0 = 1, \quad \mathcal{L}_1 = 1, \quad \text{and} \quad \mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n + 1$$

where $\mathcal{L}_n$ is the nth Leonardo number. An expression of the relationship between Leonardo and Fibonacci numbers is

$$\mathcal{L}_n = 2F_{n+1} - 1, \quad n \geq 0$$

The Binet-like formula for the Leonardo sequence is

$$\mathcal{L}_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1 \tag{5}$$

where α and β are given Equation 4. Other studies about Leonardo numbers can be listed in [9–15].

2. The Francois Numbers

This section presents a new definition, called the Francois sequence, related to the Lucas-like form of the Leonardo sequence as follows:

Definition 2.1. The Francois sequence $\{\mathcal{F}_n\}$ is defined by

$$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + 1, \quad n \geq 2 \tag{6}$$

with initial conditions $\mathcal{F}_0 = 2$ and $\mathcal{F}_1 = 1$. Here, $\mathcal{F}_n$ is the nth Francois number.

First few terms of this sequence are 2, 1, 4, 6, 11, 18, 30, 49, 80, 130, 211. The Recurrence 6 can also be written as follows

$$\mathcal{F}_{n+3} = 2\mathcal{F}_{n+2} - \mathcal{F}_n \tag{7}$$

In fact, by the equalities $\mathcal{F}_{n+3} = \mathcal{F}_{n+2} + \mathcal{F}_{n+1} + 1$ and $\mathcal{F}_{n+2} = \mathcal{F}_{n+1} + \mathcal{F}_n + 1$, we reach Equation 7.

Equation 7 satisfies the characteristic equation

$$t^3 - 2t^2 + 1 = 0 \tag{8}$$

The roots of Equation 8 are 1, α, and β. Here, the other roots except 1 are the same as those of Equation 3. Taking $\mathcal{F}_0 = 2$, $\mathcal{F}_1 = 1$, and $\mathcal{F}_2 = 4$, we can easily reach the following result.

Theorem 2.2. The Binet-like formula for the Francois sequence is

$$\mathcal{F}_n = \alpha^n + \beta^n + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} - 1, \quad n \geq 0 \tag{9}$$

where α and β are given in Equation 4.

PROOF.
Assume that $\mathcal{F}_n = a\alpha^n + b\beta^n + c$. Thus, we have

$$\mathcal{F}_0 = a + b + c = 2$$

$$\mathcal{F}_1 = a\alpha + b\beta + c = 1$$

and

$$\mathcal{F}_2 = a\alpha^2 + b\beta^2 + c = 4$$

By performing the solution with the Gaussian elimination method, we can find that

$$a = 1 + \frac{\alpha}{\alpha - \beta}, \quad b = 1 - \frac{\beta}{\alpha - \beta}, \quad \text{and} \quad c = -1$$

This proof is complete. $\square$

Theorem 2.3. For $n \geq 0$, the following identity is valid:

$$\mathcal{F}_n = L_n + F_{n+1} - 1, \quad n \geq 0$$

PROOF.
The proof is clear by Theorem 2.2. $\square$

Studies similar to the Leonardo and Francois numbers can be seen in [9, 16–18]. The hyperbolic Fibonacci and hyperbolic Lucas quaternions are defined as follows, respectively,

$$HF_n = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3$$

and

$$HL_n = L_n e_0 + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3$$

The Binet-like formulas for the hyperbolic Fibonacci and hyperbolic Lucas quaternions are as the form, respectively,

$$HF_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta} \tag{10}$$

and

$$HL_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n \tag{11}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$$

and

$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$$

The hyperbolic Fibonacci and hyperbolic Lucas quaternions and some of their generalizations are given in [19–23].

3. Hyperbolic Leonardo and Hyperbolic Francois Quaternions

In this section, we define the hyperbolic Leonardo and hyperbolic Francois quaternions, and we provide their Binet-like formulas and generating functions. Then, we obtain certain binomial sums, Honsberger-like, d'Ocagne-like, Catalan-like, and Cassini-like identities of the hyperbolic Leonardo quaternions.

Definition 3.1. The hyperbolic Leonardo quaternion sequence $\{\mathcal{HL}_n\}_{n\geq 0}$ is defined by

$$\mathcal{HL}_n = \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3 \tag{12}$$

where Le_n is the nth Leonardo number and e_0, e_1, e_2, and e_3 are units of the hyperbolic quaternions.

Definition 3.2. The hyperbolic Francois quaternion sequence $\{\mathcal{HF}_n\}_{n\geq 0}$ is defined by

$$\mathcal{HF}_n = \mathcal{F}_n e_0 + \mathcal{F}_{n+1} e_1 + \mathcal{F}_{n+2} e_2 + \mathcal{F}_{n+3} e_3$$

where $\mathcal{F}_n$ is the nth Francois number and e_0, e_1, e_2, and e_3 are units of the hyperbolic quaternions.

Theorem 3.3. (Binet-like Formula) The Binet-like formula for the hyperbolic Leonardo quaternions is

$$\mathcal{HL}_n = 2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \hat{1}, \quad n \geq 0 \tag{13}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$$

$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$$

and

$$\hat{1} = e_0 + e_1 + e_2 + e_3$$

PROOF.
From Identities 5 and 12,

$$\begin{aligned}
\mathcal{HL}_n &= \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3 \\
&= \left[2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1\right] e_0 + \left[2\left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right) - 1\right] e_1 + \left[2\left(\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\right) - 1\right] e_2 \\
&\quad + \left[2\left(\frac{\alpha^{n+4} - \beta^{n+4}}{\alpha - \beta}\right) - 1\right] e_3 \\
&= 2\left(\frac{\alpha^{n+1}}{\alpha - \beta}(e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3) - \frac{\beta^{n+1}}{\alpha - \beta}(e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3)\right) - (e_0 + e_1 + e_2 + e_3) \\
&= 2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \hat{1}
\end{aligned}$$

is obtained. □

Note that the hyperbolic Leonardo quaternion sequence can be expressed in terms of the hyperbolic Fibonacci quaternion as:

$$\mathcal{HL}_n = 2HF_{n+1} - \hat{1}, \qquad n \geq 0$$

where HF_n is nth the hyperbolic Fibonacci quaternion.

Theorem 3.4. (Binet-like Formula) The Binet-like formula for the hyperbolic Francois quaternions is

$$\mathcal{HF}_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta} - \hat{1}, \quad n \geq 0 \tag{14}$$

where

$$\hat{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3$$

$$\hat{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3$$

and

$$\hat{1} = e_0 + e_1 + e_2 + e_3$$

PROOF.
It is proved similarly to the proof of Theorem 3.3. □

Note that the hyperbolic Francois quaternion sequence can be expressed in terms of the hyperbolic Fibonacci and hyperbolic Lucas quaternion as:

$$\mathcal{HF}_n = HL_n + HF_{n+1} - \hat{1}, \quad n \geq 0$$

where HL_n and HF_n is nth the hyperbolic Lucas and hyperbolic Fibonacci quaternions, respectively.

Theorem 3.5. (Generating Function) The generating function for the hyperbolic Leonardo quaternions is

$$\mathcal{G}_{\mathcal{HL}}(x) = \frac{A - Bx + Cx^2}{1 - 2x + x^3}$$

where

$$A = e_0 + e_1 + 3e_2 + 5e_3$$

$$B = e_0 - e_1 + e_2 + e_3$$

and

$$C = e_0 - e_1 - e_2 - 3e_3$$

PROOF.
Let

$$\mathcal{G}_{\mathcal{HL}}(x) = \sum_{n=0}^{\infty} \mathcal{HL}_n x^n = \mathcal{HL}_0 + \mathcal{HL}_1 x + \mathcal{HH}_2 x^2 + \mathcal{HL}_3 x^3 + \ldots + \mathcal{HL}_n x^n + \ldots$$

be generating function of the hyperbolic Leonardo quaternions. Assume that multiply every side of the expansions above by $-2x$ and x^3 as follows:

$$-2x\mathcal{G}_{\mathcal{HL}}(x) = -2\mathcal{HL}_0 x - 2\mathcal{HL}_1 x^2 - 2\mathcal{HL}_2 x^3 - 2\mathcal{HL}_3 x^4 - \ldots - 2\mathcal{HL}_n x^{n+1} - \ldots$$

and

$$x^3\mathcal{G}_{\mathcal{HL}}(x) = \mathcal{HL}_0 x^3 + \mathcal{HL}_1 x^4 + \mathcal{HH}_2 x^5 + \mathcal{HL}_3 x^6 + \ldots + \mathcal{HL}_n x^{n+3} + \ldots$$

Then, we write

$$\begin{aligned}(1 - 2x + x^3)\mathcal{G}_{\mathcal{HL}}(x) = {} & \mathcal{HL}_0 + (\mathcal{HL}_1 - 2\mathcal{HL}_0)x + (\mathcal{HL}_2 - 2\mathcal{HL}_1)x^2 + (\mathcal{HL}_3 - 2\mathcal{HL}_2 + \mathcal{HL}_0)x^3 + \ldots \\ & + (\mathcal{HL}_n - 2\mathcal{HL}_{n-1} + \mathcal{HL}_{n-3})x^n + \ldots\end{aligned}$$

By using the values,

$$\mathcal{HL}_0 = e_0 + e_1 + 3e_2 + 5e_3$$

$$\mathcal{HL}_1 = e_0 + 3e_1 + 5e_2 + 9e_3$$

$$\mathcal{HL}_2 = 3e_0 + 5e_1 + 9e_2 + 15e_3$$

$$\mathcal{HL}_3 = 5e_0 + 9e_1 + 15e_2 + 25e_3$$

and

$$\mathcal{HL}_n - 2\mathcal{HL}_{n-1} + \mathcal{HL}_{n-3} = 0$$

are obtained. □

Theorem 3.6. (Generating Function) The generating function for the hyperbolic Francois quaternions is

$$\mathcal{G}_{\mathcal{HF}}(x) = \frac{E - Fx + Gx^2}{1 - 2x + x^3}$$

where

$$E = 2e_0 + e_1 + 4e_2 + 6e_3$$

$$F = 3e_0 - 2e_1 + 2e_2 + e_3$$

and

$$G = 2e_0 - 2e_1 - e_2 - 4e_3$$

PROOF.
The proof is similar to one of Theorem 3.5. □

Theorem 3.7. (Exponential Generating Function) The exponential generating function for the hyperbolic Leonardo quaternions is

$$\mathcal{E}_{\mathcal{HL}}(x) = 2\left(\frac{\hat{\alpha}\alpha e^{\alpha x} - \hat{\beta}\beta e^{\beta x}}{\alpha - \beta}\right) - \hat{1}e^{x}$$

PROOF.
Using Equation 13,

$$\begin{aligned}
\mathcal{E}_{\mathcal{HL}}(x) &= \sum_{n=0}^{\infty} \mathcal{HL}_n \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \hat{1}\right)\frac{x^n}{n!} \\
&= \frac{2\hat{\alpha}\alpha}{\alpha - \beta}\sum_{n=0}^{\infty}\frac{(\alpha x)^n}{n!} - \frac{2\hat{\beta}\beta}{\alpha - \beta}\sum_{n=0}^{\infty}\frac{(\beta x)^n}{n!} - \hat{1}\sum_{n=0}^{\infty}\frac{x^n}{n!} \\
&= 2\left(\frac{\hat{\alpha}\alpha e^{\alpha x} - \hat{\beta}\beta e^{\beta x}}{\alpha - \beta}\right) - \hat{1}e^{x}
\end{aligned}$$

is obtained. □

Theorem 3.8. (Exponential Generating Function) The exponential generating function for the hyperbolic Francois quaternions is

$$\mathcal{E}_{\mathcal{HF}}(x) = \hat{\alpha}\alpha e^{\alpha x} + \hat{\beta}\beta e^{\beta x} + \frac{\hat{\alpha}\alpha e^{\alpha x} - \hat{\beta}\beta e^{\beta x}}{\alpha - \beta} - \hat{1}e^{x}$$

PROOF.
The proof is similar to one of Theorem 3.7. □

Theorem 3.9. (Binomial Sum) Let m be a positive integer. Then,

$$\sum_{n=0}^{m}\binom{m}{n}\mathcal{HL}_n = \mathcal{HL}_{2m} + \hat{1}(1 - 2^m)$$

PROOF.
Considering Equations 3 and 13 and the binomial formula,

$$\begin{aligned}
\sum_{n=0}^{m}\binom{m}{n}\mathcal{HL}_n &= \sum_{n=0}^{m}\binom{m}{n}\left[2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \hat{1}\right] \\
&= \frac{2\hat{\alpha}\alpha}{\alpha - \beta}\sum_{n=0}^{m}\binom{m}{n}\alpha^n - \frac{2\hat{\beta}\beta}{\alpha - \beta}\sum_{n=0}^{m}\binom{m}{n}\beta^n - \hat{1}\sum_{n=0}^{m}\binom{m}{n} \\
&= \frac{2\hat{\alpha}\alpha}{\alpha - \beta}(\alpha + 1)^m - \frac{2\hat{\beta}\beta}{\alpha - \beta}(\beta + 1)^m - \hat{1}2^m \\
&= 2\left(\frac{\hat{\alpha}\alpha^{2m+1} - \hat{\beta}\beta^{2m+1}}{\alpha - \beta}\right) - \hat{1} + \hat{1} - \hat{1}2^m \\
&= \mathcal{HL}_{2m} + \hat{1}(1 - 2^m)
\end{aligned}$$

is obtained. □

Corollary 3.10. Let m be a positive integer. Then,

$$\sum_{k=0}^{m}\binom{m}{k}\mathcal{HL}_{n-k}=\mathcal{HL}_{n+m}+\hat{1}(1-2^m), \qquad n\geq 0$$

Proof.
Considering Equations 3 and 13 and the binomial formula,

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\mathcal{HL}_{n-k}&=\sum_{k=0}^{m}\binom{m}{k}\left[2\left(\frac{\hat{\alpha}\alpha^{n-k+1}-\hat{\beta}\beta^{n-k+1}}{\alpha-\beta}\right)-\hat{1}\right]\\&=\sum_{k=0}^{m}\binom{m}{k}\left[2\left(\frac{\hat{\alpha}\alpha^{m-k}\alpha^{n-m+1}-\hat{\beta}\beta^{m-k}\beta^{n-m+1}}{\alpha-\beta}\right)-\hat{1}\right]\\&=\frac{2\hat{\alpha}\alpha^{n-m+1}}{\alpha-\beta}\sum_{k=0}^{m}\binom{m}{k}\alpha^{m-k}1^k-\frac{2\hat{\beta}\beta^{n-m+1}}{\alpha-\beta}\sum_{k=0}^{m}\binom{m}{k}\beta^{m-k}1^k-\hat{1}\sum_{k=0}^{m}\binom{m}{k}\\&=\frac{2\hat{\alpha}\alpha^{n-m+1}}{\alpha-\beta}(\alpha+1)^m-\frac{2\hat{\beta}\beta^{n-m+1}}{\alpha-\beta}(\beta+1)^m-\hat{1}2^m\\&=2\left(\frac{\hat{\alpha}\alpha^{n+m+1}-\hat{\beta}\beta^{n+m+1}}{\alpha-\beta}\right)-\hat{1}+\hat{1}-\hat{1}2^m\\&=\mathcal{HL}_{n+m}+\hat{1}(1-2^m)\end{aligned}$$

is obtained. □

Some identities, such as Honsberger, dOcagne, Catalan, and Cassini identities for Fibonacci and its generating, have been studied by many authors (see [19,24,25]). Here, we obtain similar identities for the hyperbolic Leonardo quaternion.

Theorem 3.11. (Honsberger-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. The following relation is satisfied:

$$\mathcal{HL}_{n+1}\mathcal{HL}_m+\mathcal{HL}_n\mathcal{HL}_{m-1}=4\left(\frac{\hat{\alpha}^2\alpha^{n+m}-\hat{\beta}^2\beta^{n+m}}{\alpha-\beta}\right)-\hat{1}\left(\mathcal{HL}_{n+1}+\mathcal{HL}_m\right), \quad n,m\geq 0$$

Proof.
Using Equation 13,

$$\begin{aligned}\mathcal{HL}_{n+1}\mathcal{HL}_m+\mathcal{HL}_n\mathcal{HL}_{m-1}&=\left[2\left(\tfrac{\hat{\alpha}\alpha^{n+1}-\hat{\beta}\beta^{n+1}}{\alpha-\beta}\right)-\hat{1}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{m}-\hat{\beta}\beta^{m}}{\alpha-\beta}\right)-\hat{1}\right]\\&\quad+\left[2\left(\tfrac{\hat{\alpha}\alpha^{n}-\hat{\beta}\beta^{n}}{\alpha-\beta}\right)-\hat{1}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{m-1}-\hat{\beta}\beta^{m-1}}{\alpha-\beta}\right)-\hat{1}\right]\\&=4\left(\tfrac{(\hat{\alpha})^2\alpha^{n+m+1}-\hat{\alpha}\hat{\beta}\alpha^{n+1}\beta^{m}-\hat{\beta}\hat{\alpha}\beta^{n+1}\alpha^{m}+(\hat{\beta})^2\beta^{n+m+1}}{(\alpha-\beta)^2}\right)\\&\quad-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n+1}-\hat{\beta}\beta^{n+1}}{\alpha-\beta}\right)-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{m}-\hat{\beta}\beta^{m}}{\alpha-\beta}\right)+\hat{1}^2\\&\quad+4\left(\tfrac{(\hat{\alpha})^2\alpha^{n+m-1}-\hat{\alpha}\hat{\beta}\alpha^{n}\beta^{m-1}-\hat{\beta}\hat{\alpha}\beta^{n}\alpha^{m-1}+(\hat{\beta})^2\beta^{n+m-1}}{(\alpha-\beta)^2}\right)\\&\quad-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{n}-\hat{\beta}\beta^{n}}{\alpha-\beta}\right)-2\hat{1}\left(\tfrac{\hat{\alpha}\alpha^{m-1}-\hat{\beta}\beta^{m-1}}{\alpha-\beta}\right)+\hat{1}^2\\&=4\left(\frac{\hat{\alpha}^2\alpha^{n+m}-\hat{\beta}^2\beta^{n+m}}{\alpha-\beta}\right)-\hat{1}\left(\mathcal{HL}_{n+1}+\mathcal{HL}_m\right)\end{aligned}$$

is obtained. □

Theorem 3.12. (d'Ocagne-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. For $n, m \geq 0$,

$$\mathcal{HL}_m\mathcal{HL}_{n+1} - \mathcal{HL}_{m+1}\mathcal{HL}_n = \hat{\imath}\left(\mathcal{HL}_{m+1} + \mathcal{HL}_n - \mathcal{HL}_m - \mathcal{HL}_{n+1}\right) + 4\left(\frac{\hat{\alpha}\hat{\beta}\alpha^m\beta^n - \hat{\beta}\hat{\alpha}\beta^m\alpha^n}{\alpha-\beta}\right) + 2\hat{\imath}^2$$

Proof.
Using Equation 13,

$$\begin{aligned}
\mathcal{HL}_m\mathcal{HL}_{n+1} - \mathcal{HL}_{m+1}\mathcal{HL}_n &= \left[2\left(\frac{\hat{\alpha}\alpha^{m+1} - \hat{\beta}\beta^{m+1}}{\alpha-\beta}\right) - \hat{\imath}\right]\left[2\left(\frac{\hat{\alpha}\alpha^{n+2} - \hat{\beta}\beta^{n+2}}{\alpha-\beta}\right) - \hat{\imath}\right] \\
&\quad - \left[2\left(\frac{\hat{\alpha}\alpha^{m+2} - \hat{\beta}\beta^{m+2}}{\alpha-\beta}\right) - \hat{\imath}\right]\left[2\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha-\beta}\right) - \hat{\imath}\right] \\
&= 4\left(\frac{(\hat{\alpha})^2\alpha^{n+m+3} - \hat{\alpha}\hat{\beta}\alpha^{m+1}\beta^{n+2} - \hat{\beta}\hat{\alpha}\beta^{m+1}\alpha^{n+2} + (\hat{\beta})^2\beta^{n+m+3}}{(\alpha-\beta)^2}\right) \\
&\quad - 2\hat{\imath}\left(\frac{\hat{\alpha}\alpha^{m+1} - \hat{\beta}\beta^{m+1}}{\alpha-\beta}\right) - 2\hat{\imath}\left(\frac{\hat{\alpha}\alpha^{n+2} - \hat{\beta}\beta^{n+2}}{\alpha-\beta}\right) + \hat{\imath}^2 \\
&\quad - 4\left(\frac{(\hat{\alpha})^2\alpha^{n+m+3} - \hat{\alpha}\hat{\beta}\alpha^{m+2}\beta^{n+1} - \hat{\beta}\hat{\alpha}\beta^{m+2}\alpha^{n+1} + (\hat{\beta})^2\beta^{n+m+3}}{(\alpha-\beta)^2}\right) \\
&\quad + 2\hat{\imath}\left(\frac{\hat{\alpha}\alpha^{m+2} - \hat{\beta}\beta^{m+2}}{\alpha-\beta}\right) + 2\hat{\imath}\left(\frac{\hat{\alpha}\alpha^{n+1} - \hat{\beta}\beta^{n+1}}{\alpha-\beta}\right) - \hat{\imath}^2 \\
&= \hat{\imath}\left(\mathcal{HL}_{m+1} + \mathcal{HL}_n - \mathcal{HL}_m - \mathcal{HL}_{n+1}\right) + 4\left(\frac{\hat{\alpha}\hat{\beta}\alpha^m\beta^n - \hat{\beta}\hat{\alpha}\beta^m\alpha^n}{\alpha-\beta}\right) + 2\hat{\imath}^2
\end{aligned}$$

is obtained. □

Theorem 3.13. (Catalan-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. For $n \geq r \geq 0$, the following relation is satisfied:

$$\mathcal{HL}_{n-r}\mathcal{HL}_{n+r} - \mathcal{HL}_n^2 = (-1)^{n-r}\left(\frac{\hat{\alpha}\hat{\beta}\beta^r + \hat{\beta}\hat{\alpha}\alpha^r}{\alpha-\beta}\right)F_r + \hat{\imath}\left(2\mathcal{HL}_n - \mathcal{HL}_{n-r} - \mathcal{HL}_{n+r}\right) + 2\hat{\imath}^2$$

Proof.
Using Equality 13,

$$\begin{aligned}
\mathcal{HL}_{n-r}\mathcal{HL}_{n+r} - \mathcal{HL}_n^2 &= \left[2\left(\tfrac{\hat{\alpha}\alpha^{n-r} - \hat{\beta}\beta^{n-r}}{\alpha-\beta}\right) - \hat{\imath}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha-\beta}\right) - \hat{\imath}\right] \\
&\quad - \left[2\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) - \hat{\imath}\right]\left[2\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) - \hat{\imath}\right] \\
&= 4\left(\tfrac{(\hat{\alpha})^2\alpha^{2n} - \hat{\alpha}\hat{\beta}\alpha^{n-r}\beta^{n+r} - \hat{\beta}\hat{\alpha}\beta^{n-r}\alpha^{n+r} + (\hat{\beta})^2\beta^{2n}}{(\alpha-\beta)^2}\right) \\
&\quad - 2\hat{\imath}\left(\tfrac{\hat{\alpha}\alpha^{n-r} - \hat{\beta}\beta^{n-r}}{\alpha-\beta}\right) - 2\hat{\imath}\left(\tfrac{\hat{\alpha}\alpha^{n+r} - \hat{\beta}\beta^{n+r}}{\alpha-\beta}\right) + \hat{\imath}^2 \\
&\quad - 4\left(\tfrac{(\hat{\alpha})^2\alpha^{2n} - \hat{\alpha}\hat{\beta}\alpha^{n}\beta^{n} - \hat{\beta}\hat{\alpha}\beta^{n}\alpha^{n} + (\hat{\beta})^2\beta^{2n}}{(\alpha-\beta)^2}\right) \\
&\quad + 2\hat{\imath}\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) + 2\hat{\imath}\left(\tfrac{\hat{\alpha}\alpha^{n} - \hat{\beta}\beta^{n}}{\alpha-\beta}\right) - \hat{\imath}^2 \\
&= (-1)^{n-r}\left(\frac{\hat{\alpha}\hat{\beta}\beta^r + \hat{\beta}\hat{\alpha}\alpha^r}{\alpha-\beta}\right)F_r + \hat{\imath}\left(2\mathcal{HL}_n - \mathcal{HL}_{n-r} - \mathcal{HL}_{n+r}\right) + 2\hat{\imath}^2
\end{aligned}$$

is obtained. □

Corollary 3.14. (Cassini-like Identity) Let $\mathcal{HL}_n$ be nth hyperbolic Leonardo quaternion. The following relation is satisfied:

$$\mathcal{HL}_{n-1}\mathcal{HL}_{n+1} - \mathcal{HL}_n^2 = (-1)^{n-1}\left(\frac{\hat{\alpha}\hat{\beta}\beta + \hat{\beta}\hat{\alpha}\alpha}{\alpha-\beta}\right) + \hat{1}\left(2\mathcal{HL}_n - \mathcal{HL}_{n-1} - \mathcal{HL}_{n+1}\right) + 2\hat{1}^2, \quad n \geq 0$$

Proof.
We take 1 instead of r in Theorem 3.13 to prove this theorem. □

Proofs of the following propositions can be easily proved using Equations 5, 9–11, 13, and 14.

Proposition 3.15. For $n \geq 0$, the following identities are valid:

i. $\mathcal{HL}_{n+1}e_0 - \mathcal{HL}_{n+2}e_1 - \mathcal{HL}_{n+3}e_2 - \mathcal{HL}_{n+4}e_3 = -2\mathcal{L}_{n+5} - 3$

ii. $\mathcal{HL}_{n+1}e_0 + \mathcal{HL}_{n+2}e_1 + \mathcal{HL}_{n+3}e_2 + \mathcal{HL}_{n+4}e_3 = 2\mathcal{HL}_n + 2\mathcal{L}_{n+5} + 3$

Proposition 3.16. The following identities are valid:

i. $\mathcal{HL}_{n+r}F_{n+r} = \frac{2}{5}\left(HL_{2n+2r+1} - (-1)^{n+r}HL_1\right) - \hat{1}F_{n+r}, \quad n, r \geq 0$

ii. $\mathcal{HL}_{n-r}F_{n-r} = \frac{2}{5}\left(HL_{2n-2r+1} - (-1)^{n-r}HL_1\right) - \hat{1}F_{n-r}, \quad n \geq r \geq 0$

iii. $\mathcal{HL}_{n-r}F_{n+r} = \frac{2}{5}\left(HL_{2n+1} - (-1)^{n+r}HL_{1-2r}\right) - \hat{1}F_{n-r}, \quad n \geq r \geq 0$

iv. $\mathcal{HL}_{n+r}F_{n-r} = \frac{2}{5}\left(HL_{2n+1} - (-1)^{n-r}HL_{2r+1}\right) - \hat{1}F_{n+r}, \quad n \geq r \geq 0$

v. $\mathcal{HL}_{n+r}L_{n+r} = 2\left(HF_{2n+2r+1} + (-1)^{n+r}HF_1\right) - \hat{1}L_{n+r}, \quad n, r \geq 0$

vi. $\mathcal{HL}_{n-r}L_{n-r} = 2\left(HF_{2n-2r+1} + (-1)^{n-r}HF_1\right) - \hat{1}L_{n-r}, \quad n \geq r \geq 0$

vii. $\mathcal{HL}_{n-r}L_{n+r} = 2\left(HF_{2n+1} + (-1)^{n-r}HF_{2r+1}\right) - \hat{1}L_{n+r}, \quad n \geq r \geq 0$

viii. $\mathcal{HL}_{n+r}L_{n-r} = 2\left(HF_{2n+1} + (-1)^{n+r}HF_{1-2r}\right) - \hat{1}L_{n-r}, \quad n \geq r \geq 0$

ix. $\mathcal{HF}_{n+r}F_{n+r} = HF_{2n+2r} - (-1)^{n+r}HF_0 + \frac{1}{5}\left(HL_{2n+2r+1} - (-1)^{n+r}HL_1\right) - \hat{1}F_{n+r}, \quad n, r \geq 0$

x. $\mathcal{HF}_{n-r}F_{n-r} = HF_{2n-2r} - (-1)^{n-r}HF_0 + \frac{1}{5}\left(HL_{2n-2r+1} - (-1)^{n-r}HL_1\right) - \hat{1}F_{n+r}, \quad n \geq r \geq 0$

xi. $\mathcal{HF}_{n-r}F_{n+r} = HF_{2n} - (-1)^{n+r}HF_{-2r} + \frac{1}{5}\left(HL_{2n+1} - (-1)^{n+r}HL_{1-2r}\right) - \hat{1}F_{n+r}, \quad n \geq r \geq 0$

xii. $\mathcal{HF}_{n+r}F_{n-r} = HF_{2n} - (-1)^{n-r}HF_{2r} + \frac{1}{5}\left(HL_{2n+1} - (-1)^{n-r}HL_{2r+1}\right) - \hat{1}F_{n-r}, \quad n \geq r \geq 0$

xiii. $\mathcal{HF}_{n+r}L_{n+r} = HL_{2n+2r} + (-1)^{n+r}HL_0 + HF_{2n+2r+1} - (-1)^{n+r}HF_1 - \hat{1}L_{n+r}, \quad n, r \geq 0$

xiv. $\mathcal{HF}_{n-r}L_{n-r} = HL_{2n-2r} + (-1)^{n-r}HL_0 + HF_{2n-2r+1} - (-1)^{n-r}HF_1 - \hat{1}L_{n+r}, \quad n \geq r \geq 0$

xv. $\mathcal{HF}_{n-r}L_{n+r} = HL_{2n} + (-1)^{n+r}HL_{-2r} + HF_{2n+1} - (-1)^{n+r}HF_{1-2r} - \hat{1}L_{n+r}, \quad n \geq r \geq 0$

xvi. $\mathcal{HF}_{n+r}L_{n-r} = HL_{2n} + (-1)^{n-r}HL_{2r} + HF_{2n+1} - (-1)^{n-r}HF_{2r+1} - \hat{1}L_{n-r}, \quad n \geq r \geq 0$

Proposition 3.17. For $n \geq 0$, the following identities are valid:

i. $\mathcal{HL}_n + \mathcal{HF}_n = 3HF_{n+1} + HL_n - 2\hat{1}$

ii. $\mathcal{HF}_n - \mathcal{HL}_n = HF_{n+1} + HL_n$

iii. $\mathcal{HL}_nL_n + \mathcal{HF}_nF_n = 2\left(HF_{2n+1} + (-1)^nHF_1\right) + HF_{2n} - (-1)^nHF_0 + \frac{1}{5}\left(HL_{2n+1} - (-1)^nHL_1\right) - \hat{1}\left(L_n + F_n\right)$

iv. $\mathcal{HL}_nL_n - \mathcal{HF}_nF_n = 2\left(HF_{2n+1} + (-1)^nHF_1\right) - HF_{2n} + (-1)^nHF_0 - \frac{1}{5}\left(HL_{2n+1} - (-1)^nHL_1\right) - \hat{1}\left(L_n - F_n\right)$

4. Conclusion

In the present study, we consider the Leonardo and Francois numbers related to the Fibonacci and Lucas numbers, respectively. We define and investigate the hyperbolic Leonardo and hyperbolic Francois quaternions. We derive the Binet-like formulas, generating and exponential generating functions for these new quaternions. We provide certain binomial sums. Finally, we establish well-known identities for these quaternions, such as the Honsberger-like, d'Ocagne-like, Catalan-like, and Cassini-like identities. In the future, researchers may examine many more identities of the hyperbolic Leonardo and Francois quaternions. In addition, these quaternions can be used in interdisciplinary studies.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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