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Constructing The Ellipse and Its Application in Analytical Fuzzy Plane Geometry

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Abstract — In this paper, we studied about a detailed analysis of fuzzy ellipse. In the previously studies, some methods for fuzzy parabola are discussed (Ghosh and Chakraborty, 2019). To define the fuzzy ellipse, it is necessary to modify the method applied for the fuzzy parabola. First, need to get five same points with the same membership grade to create crisp ellipse and the union of crisp ellipses passing through these points will form the fuzzy ellipse. Although it is difficult to determine the points with this property, it is important for constructing the fuzzy ellipse equation. In this study, we determine the points that satisfy this condition and prove the properties required to obtain the fuzzy ellipse to be formed by using these points. We have drawn a graph of a fuzzy ellipse and depicted the geometric location of fuzzy points with different membership grades on graph. We have also shown some geometric application on examples. In the third part of this study, it has been shown that the determinants defined in the calculation of the coefficients of the fuzzy ellipse can be calculated for different points and angles with the examples given, thus different fuzzy ellipses can be obtained.

Subject Classification (2020): 51H25, 51M05, 51M15, 53C80, 58A40, 58A05.

1. Introduction

In the case of crisp sets, a given object x may belong to a set A or not belong to this set and these two options are denoted by $x \in A$ or $x \notin A$, A classic set may be described by the characteristic function (χ_A) that takes two values:

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Fuzzy sets are introduced and described using membership functions by Zadeh in 1965 [10]. As opposed to crisp set, if $\bar{A}$ is a fuzzy set, we write its membership function as $\mu(x|\bar{A})$, $\mu(x|\bar{A})$ is in [0,1] for all x.

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Many studies are available to understand fuzzy logic [10 − 12]. Likewise, certain ideas in fuzzy plane geometry have been introduced and studied by Buckley and Eslami in the study [1] may be the first to analyze fuzzy sets. Fuzzy points and the fuzzy distance between fuzzy points was defined by Buckley an Eslami in [8]. And they showed it is a (weak) fuzzy metric and fuzzy point, fuzzy line segment, fuzzy distance and the angle between two fuzzy segments and same and inverse points are defined by Ghosh and Chakraborty [3]. Buckley and Eslami studied fuzzy points and fuzzy lines and gave the theorems about them in [1] and studied fuzzy circles, fuzzy rectangles, fuzzy triangles and fuzzy polygons and showed that the area and perimeter of a fuzzy circle and a fuzzy polygon are a fuzzy number in [2]. A fuzzy line passing through several fuzzy points whose cores are collinear and introduced four different forms of fuzzy lines were introduced by Ghosh and Chakraborty in the study [4]. Ghosh and Chakraborty constructed a fuzzy circle in a fuzzy geometrical plane and showed that the center of a fuzzy circle may not be a fuzzy point in [5]. Rosenfeld presented fuzzy geometry and fuzzy topology of image subsets [9]. The fuzzy triangle as the intersection of three fuzzy half-planes and computed area and perimeter of the fuzzy triangle were discussed by Rosenfeld in the study [8]. Zimmermann dealt with types of fuzzy sets, fuzzy measures, fuzzy functions, applications of fuzzy set theory and gave basic definitions and theorems about fuzzy sets [12]. A fuzzy parabola that passes through five fuzzy points are constructed by Ghosh and Chakraborty in the study [6]. Then Özekinci and Aycan introduced a method to construct a fuzzy hyperbola and made applications about fuzzy hyperbola [7].

Fuzzy set theory provides a convenient method that is easy to implement in real-time applications, and also enables designers and operators to transfer their knowledge to the dynamic control systems. Fuzzy logic is also used in different fields such as artificial intelligence, computers, face recognition systems, cybernetic internet technologies, space vehicles, robot and war technologies, the formation of the universe, etc. Fuzzy logic has been the subject of many studies since it is an approach that is not only theoretical but also practical. When all these studies are examined geometrically, it is seen that only fuzzy circle, fuzzy parabola and fuzzy hyperbola curves are studied from the conics. No study has got been to construct a fuzzy ellipse. Fuzzy systems are used in the planning of technological structures developing in the field of engineering nowadays. Then fuzzy ellipse can be use kidney stones crushing machines, billiard games, aerospace engineering and lazer technology etc. Therefore, in this study, we studied how to construct a fuzzy ellipse and to obtain the equation for the geometric location of a fuzzy ellipse by using the properties of conics. While we aimed to analyze how the fuzzy ellipse could be defined, calculated and graphed mathematically, we thought that it would be useful to work on combining the applications mentioned above. We examine and prove these calculations with the evaluation of previous studies.

2. Preliminaries

In this section, we will mention the basic fuzzy definitions that will be used in this paper.

We will draw "a bar" over capital letters to denote a fuzzy subset of $R^n, i.e\ \bar{A}, \bar{B}, \bar{X}, \bar{Y}, \ldots$ and we will write membership of fuzzy set $\bar{A}$ as $\mu(x|\ \bar{A}\ ), x \in R^n$ and $\mu(R^n)$ is in [0,1].

Definition 2.1: (Fuzzy Set) The set of ordered pairs $\bar{A} = \{(x, \mu(x|\bar{A}\ )): x \in X\}$, where $\mu: X \to [0,1]$ is called a fuzzy set in X. The function $\mu: X \to [0,1]$ evaluates membership degree of x in the fuzzy set $\bar{A}$ [2].

Definition 2.2: For a fuzzy set $\bar{A}$ of R^n, its $\alpha - cut$ is denoted by $\bar{A}(\alpha)$ and it is defined by:

$$\bar{A}(\alpha) = \begin{cases} \{x, \mu(x|\bar{A}\ ) \geq \alpha\ \}\ if\ 0 < \alpha \leq 1 \\ \text{Clouse}\ \{x, \mu(x|\bar{A}\ ) > 0\}\ if\ \alpha = 0 \end{cases}$$

The set $\{x, \mu(x|\bar{A}\ ) > 0\ \}$ is called as support of the fuzzy set $\bar{A}$. The set $\bar{A}(0)$ ise often said as base of $\bar{A}$ and the set $\bar{A}(1) = \{x, \mu(x|\bar{A}\ ) = 1\}$ is said to be core of the fuzzy set $\bar{A}$. If the core is non-empty, the fuzzy set is called as a normal fuzzy set. A fuzzy set is said to be convex if all of its α-cuts are convex [3].

Definition 2.3 (Fuzzy Points): A fuzzy point at (a, b) in R^2, written as $\bar{P}(a, b)$ is defined by its membership function:

(i) $\mu((x, y)|\ \bar{P}(a, b))$ is upper semi-continuous,

(ii) $\mu((x, y)|\ \bar{P}(a, b)) = 1$ if and only if $(x, y) = (a, b)$,

(iii) $\bar{P}(a, b)(\alpha)$ is a compact, convex subset of R^2 of all α in [0,1].

The notations $\bar{P}_1(a, b), \bar{P}_2(a, b), \bar{P}_3(a, b), \ldots$ or $\bar{P}_1, \bar{P}_2, \bar{P}_3, \ldots$ are used to represent fuzzy points [2].

Definition 2.4 (Same points with respect to fuzzy points): Let take two points (x_1, y_1) and (x_2, y_2). Such that (x_1, y_1) is support of fuzzy point $\bar{P}(a, b)$ and similarly (x_2, y_2) is support of fuzzy point $\bar{P}(c, d)$. Let L_1 is a line joining (x_1, y_1) and (a, b). As $\bar{P}(a, b)$ is a fuzzy point, along L_1, a fuzzy number, $\bar{r}_1$ say, is situated on the support of $\bar{P}(a, b)$. The membership function of this fuzzy number $\bar{r}_1$ can be written as $\mu((x, y)|\bar{r}_1) = \mu((x, y)|\ \bar{P}(a, b))$ for (x, y) in L_1 , and 0 otherwise. Similarly, along a line, L_2 say, joining (x_2, y_2) and (c, d), the exists a fuzzy number, $\bar{r}_2$ say, on the support of $\bar{P}(c, d)$. The points (x_1, y_1) and (x_2, y_2) are said to be same points with respect to $\bar{P}(a, b)$ and $\bar{P}(c, d)$ if :

(i) (x_1, y_1) and (x_2, y_2) are same -points with respect to $\bar{r}_1$ and $\bar{r}_2$,

(ii) L_1, L_2 have equal angle with line joining (a, b) and (c, d) [3].

3. Fuzzy Ellipse

In this section we will develop a method for obtaining a fuzzy ellipse. As its known the general conic equation has the form:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

If we divide both sides of this equation by a $(a \neq 0)$, this equation takes the form:

$$x^2 + b'xy + c'y^2 + d'x + e'y + f' = 0$$

Thus, the number of unknown coefficients, in the equation containing six terms, becomes five. Then the common solution of the five equations, will be obtained by substituting five different pairs for x and y, will be sufficient to find these unknowns. So, five points in the plane is enough to write a conic equation. Namely five points on the plane will denote a single conic.

In this study, since we will define an ellipse in fuzzy plane geometry, first of all, these five points must be points in the fuzzy space that ensure the necessary properties. It will also be seen that the ellipse in fuzzy space is formed by different crisp elliptic curves. Their combination will form the fuzzy ellipse. The curve of ellipse passing through the core of five fuzzy points will be called a crisp ellipse and be denoted by CE . However, since these points are fuzzy points, their membership degrees may change. Differences in membership degrees affect the drawing of the resulting ellipse curves. Therefore, calculating five different coefficients for five same-points in the conic equation. Calculation of these coefficients is possible with five by determinants. For this reason, five different curves emerge for the fuzzy ellipse that we want to reach in our study. Therefore, in terms of the importance of the fuzzy membership degree, the definite ellipse CE with membership degree one is taken. The other four curves are ellipse and the combination of all of them gives the fuzzy ellipse and is denoted by FE. The system formed by these curves can also be considered as a curvilinear system or distribution in mathematical applications.

Now, we will denote a method to create a fuzzy ellipse in a fuzzy plane by taking five fuzzy points. These points will be the same-points which we gave in Definition 2.4 in preliminaries section.

Necessary explanations and proofs are presented below.

Let $\bar{E}_i(a_i, b_i), i = 1,2, \ldots ,5$ be given five fuzzy points whose cores lie on a crisp ellipse CE . We will construct a fuzzy ellipse that passes through these five fuzzy points $\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_5$. We will denote fuzzy ellipse as $\overline{FE}$, briefly. Below are the steps of the method we used to create the fuzzy ellipse.

3.1. Construction of Fuzzy Ellipse $\overline{FE}$

In this section we construct the segment $\overline{FE}_{\ 1\ldots5}$ for the fuzzy ellipse. This segment defined as,

$$\overline{FE} = \bigvee_{\alpha\in[0,1]} \left\{ \begin{array}{c} FE_\alpha : Where\ FE_\alpha\ is\ a\ crisp\ ellipse\ that\ passes \\ through\ five\ same\ points\ an\ \bar{E}_i(a_i, b_i), \\ i = 1,2, \ldots, 5\ with\ membership\ value\ \alpha \end{array} \right\}$$

The ellipse $\overline{FE}$ can be defined by membership function below:

$$\mu\big((x,y)|\overline{FE}\big) = sup \left\{ \begin{array}{c} \alpha : Where\ (x,y)\ lies\ on\ FE_\alpha\ that \\ passes\ through\ five\ same\ points\ on \\ \bar{E}_i, i = 1,2, \ldots, 5\ with\ membership\ value\ \alpha \end{array} \right\}$$

As this definition show that the fuzzy elliptic-segment $\overline{FE}_{\ 1\ldots5}$ is a collection of crisp points with various membership degrees. However, the definition of membership function $\mu\left((x,y)|\overline{FE}_{\ 1\ldots5}\right)$ shows that a fuzzy ellipse is the union of all crisp ellipses that pass through five same-points on the supports of $\bar{E}_i, i = 1,2, \ldots, 5$.

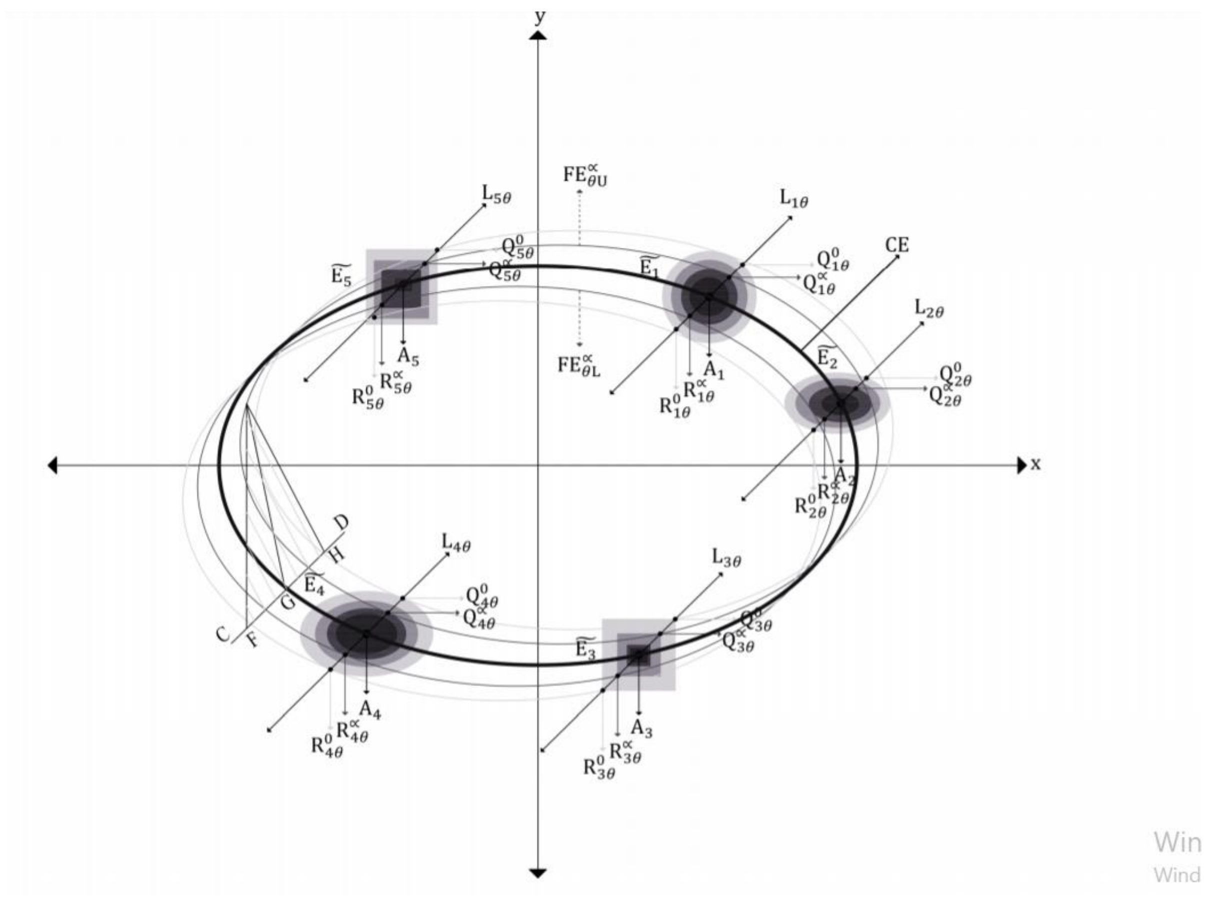

Figure.3.1. Construction of Fuzzy Ellipse in the Method

In Figure 3.1, we depict the fuzzy ellipse with membership degrees of given fuzzy points in detail. $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4$ and $\bar{E}_5$ are five fuzzy points. The regions under the circle centered at A_1, ellipse centered at A_2, square centered A_3, ellipse centered at A_4 and square centered at A_5 are the supports of the points $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4$ and $\bar{E}_5$, respectively. The grey -shaded regions inside the supports of the fuzzy points represent different α-cuts. The variation of the membership grades for fuzzy points is indicated by the intensity of the grey levels. The regions that, depicted in dark grey in the graph are formed by points with a high membership grade. Light grey regions on the graph are obtained as the membership grades approach 0. So, the membership grades of the centers of circle, squares and ellipses are one and it decreases gradually to zero on the periphery of the support of $\bar{E}_i$ for each $i = 1,2,3,4,5$.

In the Figure 3.1, $L_{i\theta}$'s are five lines that passes through A_i, for each $i = 1,2,3,4,5$. These five lines have an angle θ with the positive x-axis. Because $\bar{E}_i(A_i)(\alpha)$, being α −cut of a fuzzy point, is convex and A_i is an interior point of $\bar{E}_i(A_i)(\alpha)$, the line $L_{i\theta}$ must intersect with the boundary of $\bar{E}_i(A_i)(\alpha)$ at exactly two points. Let these two intersecting points be $Q_{i\theta}^{\alpha}$ and $R_{i\theta}^{\alpha}$. Thus, $Q_{1\theta}^{\alpha}, Q_{2\theta}^{\alpha}, Q_{3\theta}^{\alpha}, Q_{4\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$ constitute a set of five same-points with membership degree α. And similarly, the collection of $R_{i\theta}^{\alpha}$'s are also represent the set of five same-points with membership degree α.

Let $FE_{\theta U}^{\alpha}$ is the ellipse that passes through the points $Q_{i\theta}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ is the ellipse that passes through the points $R_{i\theta}^{\alpha}$'s in Figure 3.1 Since membership degree of all the points $Q_{i\theta}^{\alpha}$ and $R_{i\theta}^{\alpha}$ is α, we put a membership degree of α to the ellipse $FE_{\theta U}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ on the fuzzy ellipse $\overline{FE}, i = 1,2,3,4,5$.
Trough varying θ in $[0,2\pi]$ and α in $[0,1]$, several ellipses such as $FE_{\theta U}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ will be obtained. According to the definition, the fuzzy ellipse $\overline{FE}$ is the collection of all the ellipses $FE_{\theta U}^{\alpha}$ and $FE_{\theta L}^{\alpha}$ with membership degree α.

Namely, we say

$$\overline{FE} = \bigvee_{\substack{\theta \in [0,2\pi] \\ \alpha \in [0,1]}} \{FE_{\theta U}^{\alpha}, FE_{\theta L}^{\alpha}\}$$

Let FE be any ellipse in the support of the fuzzy ellipse $\overline{FE}$. We define the membership degree of on ellipse FE in $\overline{FE}$ by

$$\mu(FE|\,\overline{FE}) = \min_{(x,y)\in FE} \mu((x,y)|\overline{FE}).$$

The underlying theorem shows how to obtain the membership degree ellipse FE in $\overline{FE}$ using the same -points in $\bar{E}_i$ $'s$, $i = 1,2,3,4,5$.

Theorem 3.1. Suppose that FE is an ellipse in $\overline{FE}$ and same -points $(x_i, y_i) \in \bar{E}_i(0)$ with $\mu\big((x_i, y_i)|\,\overline{FE}\big) = \alpha$ for all $i = 1,2,3,4,5$ such that FE is the ellipse that passes through the five (x_i, y_i) 's and $\mu(FE|\,\overline{FE}) = \alpha$.

Proof .

We examine the proof in two different cases that (i) $\mu(FE|\overline{FE}) \nless \alpha$ and (ii) $\mu(FE|\overline{FE}) \ngtr \alpha$.

(i) By contrast, let assume that $\mu(FE|\overline{FE}) < \alpha$. In that case, by the definition of $\mu(FE|\overline{FE})$, there exist (x_0, y_0) in $\overline{FE}$ such that $(x_0, y_0) \in FE$ and $\mu((x_0, y_0)|\overline{FE}) < \alpha$. Let say $\mu((x_0, y_0)|\overline{FE}) = \beta$. Since $(x_0, y_0) \in FE$ and FE is an ellipse that joins the five same-points with membership degree α,

$$\mu((x_0, y_0)|\overline{FE}) = sup\left\{\begin{matrix} \psi\text{: }where\ (x,y)\ lies\ on\ the\ ellipse \\ that\ joins\ the\ five\ same\ points \\ with\ membership\ degree\ \psi \end{matrix}\right\} \geq \alpha.$$

But this contradicts our acceptance $\beta < \alpha$. So, $\mu(FE|\overline{FE}) \nless \alpha$.

(ii) It is clear that $\mu(FE|\overline{FE}) \ngtr \alpha$. Since $\mu(FE|\overline{FE}) = \min\left\{\begin{matrix} \alpha\text{: where } (x,y) \text{ lies on} \\ FE \text{ and } \mu(FE|\overline{FE}) = \alpha \end{matrix}\right\}$, and all the points $(x_i, y_i), i = 1,2,3,4,5$ lie on FE.

Therefore $\mu(FE|\overline{FE}) = \alpha$ is obtained.

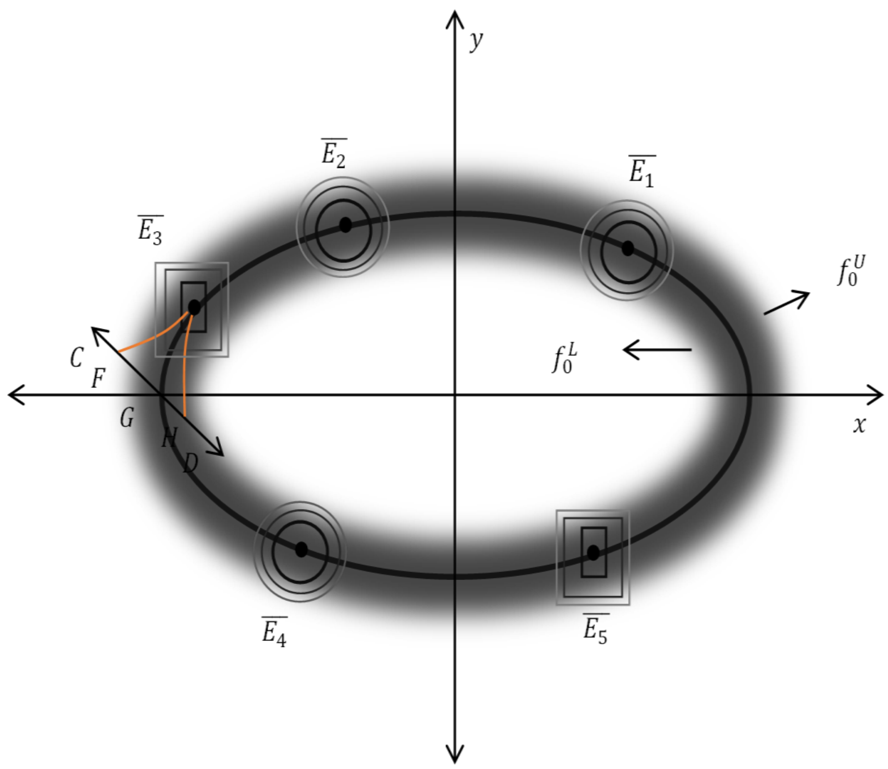

Figure 3.2 Fuzzy Ellipse (towards completing the fuzzy ellipse in the Figure 3.1)

The complete fuzzy ellipse $\overline{FE}$ is depicted in Figure 3.2. The region between the curves f_0L and f_0U is the support of the $\overline{FE}$. The core ellipse is the curve CE on which the five core points A_i of the fuzzy points $\overline{E_i}$ lies.

Let mention the line perpendicular to $CE \equiv \overline{FE}(1)$ that we take as the CD line in Figure 3.2. Along the CD, there exist a LR type fuzzy number that we denoted by $(F/G/H)_{LR}$. If we explain LR type fuzzy number like this, L and R are reference functions $L\ and\ R: [0,+\infty) \to [0,1]$ that doesn't decrease and satisfies two conditions $L(x) = L(-x)$ and $L(0) = 1$. Where α and β are positive and $\bar{A}$ is a fuzzy number, $\mu(x|\bar{A}\,)$ can be written as:

$$\mu(x|\bar{A}\,) = \begin{cases} L\left(\dfrac{m-x}{\alpha}\right) & if\ x \leq m \\ R\left(\dfrac{x-m}{\beta}\right) & if\ x \geq m \end{cases}$$

The notation $(m-\alpha/\ m/m+\beta)_{LR}$ is used to represent an LR-type fuzzy number. That is, all fuzzy ellipses can be visualized as a three-dimensional figure. (a subset of $(x,y) \times [0,1]$) whose cross-section across $\overline{FE}$ is a fuzzy number such as $(F/G/H)_{LR}$.

Let $(F/G/H)_{LR}$ is a fuzzy number on fuzzy ellipse $\overline{FE}$ and take a convex region on $\overline{FE}(0)$ such that, except F and H, all points on the line segment $[FH]$ are inner of convex region. When we take a fuzzy point $\bar{E}$ such that the membership function is $\mu\big((x,y)|\,\bar{E}\big) = \mu\big((x,y)|(F/G/H)_{LR}\big)$, if $(x,y) \in [FH]$, $\mu\big((x,y)|\,\bar{E}\big) \leq \mu\big((x,y)|\,\overline{FE}\big)$. Only at G, $\mu\big((x,y)|\,\bar{E}\big) = 1$. Membership degree decreases gradually to '0' that approach F or H.

3.2. Construction of Membership Function

The membership degree $\mu\big((x,y)|\,\overline{FE}\,\big)$ might not always be simple to evaluate. Furthermore, it is really a difficult task to obtain the closed form of the membership function of $\overline{FE}$. Because, the membership degree at a particular point is the supremum of a set of real numbers that is obtained by solving a set of nonlinear equations. First, we get the closed form of membership function of $\overline{FE}$.

We note that the definition of fuzzy ellipse implies

$$\mu\big((x,y)|\,\overline{FE}\,\big) = sup\left\{\begin{array}{c} \alpha: where\ (x,y)\ lies\ in\ an\ ellipse\ that \\ passes\ through\ five\ same\ points\ in\ \bar{E}_i, \\ i = 1,2,3,4,5\ with\ membership\ \text{degree}\ \alpha \end{array}\right\}.$$

For obtaining $\mu\big((x,y)|\,\overline{FE}\big)$, first we must find five same-points with membership degree $\alpha \in [0,1]$. Then, all possible values of α are identified for which (x,y) lies on the ellipse that joins five same-points with membership degrees. The evaluation of α may require to solving a nonlinear equation. From the solution of the equation, there may be real values between 0 and 1. The supremum of all these real α values is the membership degree of $\mu\big((x,y)|\,\overline{FE}\,\big)$. We refer the ellipse for which the supremum is attained as the adjoining ellipse of the points (x,y).

Now, we obtain a systematic procedure to identify the membership degree of a point (x_0,y_0) in a fuzzy ellipse $\overline{FE}$ which passes through five fuzzy points $\bar{E}_i,\ i=1,2,3,4,5$ We show the expansion of the same-points on $\bar{E}_i$'s as $(x_{i\theta}^{\alpha},y_{i\theta}^{\alpha}),\, i=1,2,3,4,5$ $(0\le\theta\le 2\pi\,,\alpha\in[0,1])$

As a result, we will have to examine the existence of solution of non-linear equations by giving various values to θ and determining the α membership degrees according to the angle θ.

Let the angle $\theta=\theta_0$ $(\,0\le\theta\le 2\pi)$ and S_{θ_0}'s are the sets of membership degrees that can be compatible with respect to the various angle θ_0.

We assume that the supremum of the set S_{θ_0} as s_{θ_0}. It can be seen from the given examples that non-linear equation systems may not have a solution for some θ_0. Fuzzy ellipse $\overline{FE}$ are obtained by determining and giving appropriate values. Then the membership degree of (x_0,y_0) in the $\overline{FE}$ fuzzy ellipse is given by

$$\mu\big((x_0,y_0)|\overline{FE}\big)=\sup_{\theta} s_{\theta_0}.$$

The explanation of this part is given also in this section where the membership function is explained. Let give the application of the procedure with following examples.

The following examples illustrate the procedure numerically.

Example 3.1: Let $\overline{E_1}(0,1),\overline{E_2}\left(\frac{1}{5},\frac{4\sqrt{6}}{5}\right),\overline{E_3}\left(-\frac{1}{4},\frac{\sqrt{15}}{2}\right),\overline{E_4}\left(-\frac{1}{2},-\sqrt{3}\right)$ and $\overline{E_5}\left(\frac{1}{3},\frac{4\sqrt{2}}{3}\right)$ be five fuzzy points. Let's get the fuzzy ellipse that passes through these points. The equation of the core ellipse through the points is

$$\left\{(x,y):x^2+\frac{y^2}{4}=1\right\}$$

In this example we take core of that the points are in different regions on the curve.

The membership function of these five fuzzy points are circular and elliptical cones with bases, respectively.

$$\{(x,y):(x-1)^2+y^2\le 1\}\text{ (circular)}$$

$$\left\{(x,y):\left(x-\frac{1}{5}\right)^2+4\left(y-\frac{4\sqrt{6}}{5}\right)^2\le 1\right\}\text{ (eliptical)}$$

$$\left\{(x,y):\left(x+\frac{1}{4}\right)^2+\left(y-\frac{\sqrt{15}}{2}\right)^2\leq 1\right\}\text{ (circular)}$$

$$\left\{(x,y):\left(x+\frac{1}{2}\right)^2+\left(y+\sqrt{3}\right)^2\leq 1\right\}\text{ (eliptical)}$$

$$\left\{(x,y):\left(x-\frac{1}{3}\right)^2+\left(y-\frac{4\sqrt{2}}{3}\right)^2\leq 1\right\}\text{ (circular)}$$

The vertices of the membership functions are $(1,0),\left(\frac{1}{5},\frac{4\sqrt{6}}{5}\right),\left(-\frac{1}{4},\frac{\sqrt{15}}{2}\right),\ \left(-\frac{1}{2},-\sqrt{3}\right)$ and $\left(\frac{1}{3},\frac{4\sqrt{2}}{3}\right)$ respectively.

Now, for $\alpha\in[0,1]$, we may find the same-points with membership degree α on $\bar{E}_1,\bar{E}_2,\bar{E}_3,\bar{E}_4$ and $\bar{E}_5$ as below;

$$Q_{1\theta}^{\alpha}:(x_{1\theta}^{\alpha},y_{1\theta}^{\alpha})=(1+(1-\alpha)\cos\theta\,,(1-\alpha)\sin\theta)$$

$$Q_{2\theta}^{\alpha}:(x_{2\theta}^{\alpha},y_{2\theta}^{\alpha})=\left(\frac{1}{5}+(1-\alpha)\frac{\cos\theta}{\sqrt{1+3\sin^2\theta}},\frac{4\sqrt{6}}{5}+(1-\alpha)\frac{\sin\theta}{\sqrt{1+3\sin^2\theta}}\right)$$

$$Q_{3\theta}^{\alpha}:(x_{3\theta}^{\alpha},y_{3\theta}^{\alpha})=\left(-\frac{1}{4}+(1-\alpha)\cos\theta\,,\frac{\sqrt{15}}{2}+(1-\alpha)\sin\theta\right)$$

$$Q_{4\theta}^{\alpha}:(x_{4\theta}^{\alpha},y_{4\theta}^{\alpha})=\left(-\frac{1}{2}+(1-\alpha)\frac{\cos\theta}{\sqrt{1+3\sin^2\theta}},-\sqrt{3}+(1-\alpha)\frac{\sin\theta}{\sqrt{1+3\sin^2\theta}}\right)$$

$$Q_{5\theta}^{\alpha}:(x_{5\theta}^{\alpha},y_{5\theta}^{\alpha})=\left(\frac{1}{3}+(1-\alpha)\cos\theta\,,\frac{4\sqrt{2}}{3}+(1-\alpha)\sin\theta\right)\tag{1}$$

The ellipse E_{θ}^{α} that passes through $Q_{1\theta}^{\alpha},Q_{2\theta}^{\alpha},Q_{3\theta}^{\alpha},Q_{4\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$ can be determinant by the equation

$$a_{\theta}^{\alpha}x^2+\ 2h_{\theta}^{\alpha}xy+\ b_{\theta}^{\alpha}y^2+2g_{\theta}^{\alpha}x+2f_{\theta}^{\alpha}y+c_{\theta}^{\alpha}=0\tag{2}$$

with ${h_{\theta}^{\alpha}}^2<a_{\theta}^{\alpha}.b_{\theta}^{\alpha}$ where

$$a_{\theta}^{\alpha}=\frac{2h_{\theta}^{\alpha}}{k_{\theta}^{\alpha}}\begin{vmatrix}-x_{1\theta}^{\alpha}y_{1\theta}^{\alpha} & {y_{1\theta}^{\alpha}}^2 & x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & 1\\ -x_{2\theta}^{\alpha}y_{2\theta}^{\alpha} & {y_{2\theta}^{\alpha}}^2 & x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & 1\\ -x_{3\theta}^{\alpha}y_{3\theta}^{\alpha} & {y_{3\theta}^{\alpha}}^2 & x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & 1\\ -x_{4\theta}^{\alpha}y_{4\theta}^{\alpha} & {y_{4\theta}^{\alpha}}^2 & x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & 1\\ -x_{5\theta}^{\alpha}y_{1\theta}^{\alpha} & {y_{5\theta}^{\alpha}}^2 & x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & 1\end{vmatrix}$$

$$b_{\theta}^{\alpha}=\frac{2h_{\theta}^{\alpha}}{k_{\theta}^{\alpha}}\begin{vmatrix}{x_{1\theta}^{\alpha}}^2 & -x_{1\theta}^{\alpha}y_{1\theta}^{\alpha} & x_{1\theta}^{\alpha} & y_{1\theta}^{\alpha} & 1\\ {x_{2\theta}^{\alpha}}^2 & -x_{2\theta}^{\alpha}y_{2\theta}^{\alpha} & x_{2\theta}^{\alpha} & y_{2\theta}^{\alpha} & 1\\ {x_{3\theta}^{\alpha}}^2 & -x_{3\theta}^{\alpha}y_{3\theta}^{\alpha} & x_{3\theta}^{\alpha} & y_{3\theta}^{\alpha} & 1\\ {x_{4\theta}^{\alpha}}^2 & -x_{4\theta}^{\alpha}y_{4\theta}^{\alpha} & x_{4\theta}^{\alpha} & y_{4\theta}^{\alpha} & 1\\ {x_{5\theta}^{\alpha}}^2 & -x_{5\theta}^{\alpha}y_{1\theta}^{\alpha} & x_{5\theta}^{\alpha} & y_{5\theta}^{\alpha} & 1\end{vmatrix}$$

$$g_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & -x_{1\theta}^\alpha y_{1\theta}^\alpha & y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & -x_{2\theta}^\alpha y_{2\theta}^\alpha & y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & -x_{3\theta}^\alpha y_{3\theta}^\alpha & y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & -x_{4\theta}^\alpha y_{4\theta}^\alpha & y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & -x_{5\theta}^\alpha y_{5\theta}^\alpha & y_{5\theta}^\alpha & 1 \end{vmatrix}$$

$$f_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & -x_{1\theta}^\alpha y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & -x_{2\theta}^\alpha y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & -x_{3\theta}^\alpha y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & -x_{4\theta}^\alpha y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & -x_{5\theta}^\alpha y_{5\theta}^\alpha & 1 \end{vmatrix}$$

$$c_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & y_{1\theta}^\alpha & -x_{1\theta}^\alpha y_{1\theta}^\alpha \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & y_{2\theta}^\alpha & -x_{2\theta}^\alpha y_{2\theta}^\alpha \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & y_{3\theta}^\alpha & -x_{3\theta}^\alpha y_{3\theta}^\alpha \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & y_{4\theta}^\alpha & -x_{4\theta}^\alpha y_{4\theta}^\alpha \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & y_{5\theta}^\alpha & -x_{5\theta}^\alpha y_{5\theta}^\alpha \end{vmatrix}$$

and

$$k_\theta^\alpha = \begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & y_{5\theta}^\alpha & 1 \end{vmatrix}.$$

These determinants are composed by writing column

$$\begin{bmatrix} -x_{1\theta}^\alpha & y_{1\theta}^\alpha \\ -x_{2\theta}^\alpha & y_{2\theta}^\alpha \\ -x_{3\theta}^\alpha & y_{3\theta}^\alpha \\ -x_{4\theta}^\alpha & y_{4\theta}^\alpha \\ -x_{5\theta}^\alpha & y_{5\theta}^\alpha \end{bmatrix}$$

instead of columns in determinant

$$\begin{vmatrix} {x_{1\theta}^\alpha}^2 & {y_{1\theta}^\alpha}^2 & x_{1\theta}^\alpha & y_{1\theta}^\alpha & 1 \\ {x_{2\theta}^\alpha}^2 & {y_{2\theta}^\alpha}^2 & x_{2\theta}^\alpha & y_{2\theta}^\alpha & 1 \\ {x_{3\theta}^\alpha}^2 & {y_{3\theta}^\alpha}^2 & x_{3\theta}^\alpha & y_{3\theta}^\alpha & 1 \\ {x_{4\theta}^\alpha}^2 & {y_{4\theta}^\alpha}^2 & x_{4\theta}^\alpha & y_{4\theta}^\alpha & 1 \\ {x_{5\theta}^\alpha}^2 & {y_{5\theta}^\alpha}^2 & x_{5\theta}^\alpha & y_{5\theta}^\alpha & 1 \end{vmatrix}$$

Let A, B, C, F, G and K be the determinant values used to find the values of $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha, g_\theta^\alpha$ and k_θ^α respectively. So,

$$a_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}.A$$

$$b_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}.B$$

$$c_\theta^\alpha = \frac{2h_\theta^\alpha}{k_\theta^\alpha}.C$$

$$f_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}.F$$

$$g_\theta^\alpha = \frac{h_\theta^\alpha}{k_\theta^\alpha}.G$$

$$k_\theta^\alpha = K \tag{3}$$

are obtained.

The fuzzy ellipse $\overline{FE}$ that passes through $\bar{E}_i$ ‘s, $i = 1,2,3,4,5$ is the union of all possible ellipse E_θ^α ‘s that lies between $Q_{1\theta}^\alpha$ and $Q_{5\theta}^\alpha$ ‘s.

That

$$\overline{FE} = \bigvee_{\alpha\in[0,1]} \bigcup_{\theta\in[0,2\pi]} \left\{ \begin{matrix} (x,y) = a_\theta^\alpha x^2 + 2h_\theta^\alpha xy + b_\theta^\alpha y^2 \\ +2g_\theta^\alpha x + 2f_\theta^\alpha y + c_\theta^\alpha = 0 \end{matrix} \right\}$$

Now we find the membership degree of the point (1,0.5) on the fuzzy ellipse $\overline{FE}$. First, we adjust the set of ellipses E_θ^α ‘s which the point (1,0.5) lies.

Let replace point (1,0.5) in Equation (2). We need to identify the passible values of α. Then, we get the equation below;

$$a_\theta^\alpha.(1)^2 + 2.h_\theta^\alpha.(1).(0.5) + b_\theta^\alpha.(0.5)^2 + 2.g_\theta^\alpha.(1) + 2.f_\theta^\alpha.(0.5) + c_\theta^\alpha = 0$$

which simplifies to

$$a_\theta^\alpha + h_\theta^\alpha + 0.25\, b_\theta^\alpha + 2g_\theta^\alpha + f_\theta^\alpha + c_\theta^\alpha = 0 \tag{4}$$

Now let’s examine the angular values that L lines make with the x-axis.

First, we admit that $\theta_0 = 45°$. We calculate above determinant values for this angle and will find k_θ^α using the Maple program.

$$k_\theta^\alpha = K$$

$$K=\begin{vmatrix} [1+(1-\alpha)\cos 45°]^2 & [(1-\alpha)\sin 45°]^2 & 1+(1-\alpha)\cos 45° & (1-\alpha)\sin 45° & 1 \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & \left[\frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}} & \frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}} & 1 \\ \left[-\frac{1}{4}+(1-\alpha)\cos 45°\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 45°\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 45° & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 45° & 1 \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 45°}{\sqrt{1+3\sin^2 45°}} & -\sqrt{3}+\frac{(1-\alpha)\sin 45°}{\sqrt{1+3\sin^2 45°}} & 1 \\ \left[\frac{1}{3}+(1-\alpha)\cos 45°\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 45°\right]^2 & \left[\frac{1}{3}+(1-\alpha)\cos 45°\right] & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 45° & 1 \end{vmatrix}=0$$

But this angle is not suitable for calculating a_θ^α, $b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha$ and g_θ^α. Because, it makes equations (3) undefined.

Then we put $\theta_0 = 30°$ in (3.3) and we calculate a_θ^α, $b_\theta^\alpha, g_\theta^\alpha, f_\theta^\alpha, c_\theta^\alpha$ and k_θ^α.

$$k_\theta^\alpha = K$$

$$K=\begin{vmatrix} [1+(1-\alpha)\cos 30°]^2 & [(1-\alpha)\sin 30°]^2 & 1+(1-\alpha)\cos 30° & (1-\alpha)\sin 30° & 1 \\ \left[\frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \left[\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & \frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ \left[-\frac{1}{4}+(1-\alpha)\cos 30°\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30°\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30° & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30° & 1 \\ \left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ \left[\frac{1}{3}+(1-\alpha)\cos 30°\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30°\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30° & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30° & 1 \end{vmatrix}$$

and

$$K = -0{,}02\alpha^3 + 0{,}37\alpha^2 + 0{,}69\alpha - 1{,}02$$

We continue to find the other determinant values.

For the value of A,

$$\begin{vmatrix} -[1+(1-\alpha)\cos 30°][(1-\alpha)\cos 30°] & [(1-\alpha)\sin 30°]^2 & 1+(1-\alpha)\cos 30° & (1-\alpha)\sin 30° & 1 \\ -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]\left[\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right] & \left[\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & \frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ -\left[-\frac{1}{4}+(1-\alpha)\cos 30°\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30°\right] & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30°\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30° & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30° & 1 \\ -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right] & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30°}{\sqrt{1+3\sin^2 30°}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30°}{\sqrt{1+3\sin^2 30°}} & 1 \\ -\left[\frac{1}{3}+(1-\alpha)\cos 30°\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30°\right] & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30°\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30° & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30° & 1 \end{vmatrix}$$

$$=0.011\alpha^3 - 0.477\alpha^2 + 3.221\alpha - 1.696$$

For the value of B,

$$\begin{vmatrix}
[1+(1-\alpha)\cos 30^\circ]^2 & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] & 1+(1-\alpha)\cos 30^\circ & (1-\alpha)\sin 30^\circ & 1 \\
\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & \frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & \frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\
\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] & -\frac{1}{4}+(1-\alpha)\cos 30^\circ & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ & 1 \\
\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & -\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\
\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right] & \frac{1}{3}+(1-\alpha)\cos 30^\circ & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ & 1
\end{vmatrix}$$

$$=0.033\alpha^3-0.255\alpha^2+0.230\alpha+0.717$$

For the value of G,

$$\begin{vmatrix}
[1+(1-\alpha)\cos 30^\circ]^2 & [(1-\alpha)\sin 30^\circ]^2 & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] & (1-\alpha)\sin 30^\circ & 1 \\
\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & \frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\
\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right]^2 & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ & 1 \\
\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & -\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & 1 \\
\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right]^2 & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right] & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ & 1
\end{vmatrix}$$

$$=-0.555\alpha^3+5.777\alpha^2-10.915\alpha+5.693$$

For the value of F,

$$\begin{vmatrix}
[1+(1-\alpha)\cos 30^\circ]^2 & [(1-\alpha)\sin 30^\circ]^2 & 1+(1-\alpha)\cos 30^\circ & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] & 1 \\
\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & 1 \\
\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30^\circ & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] & 1 \\
\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] & 1 \\
\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30^\circ & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right] & 1
\end{vmatrix}$$

$$=0.057\alpha^3+0.141\alpha^2+0.649\alpha-0.848$$

For the value of C,

$$\begin{vmatrix}
[1+(1-\alpha)\cos 30^\circ]^2 & [(1-\alpha)\sin 30^\circ]^2 & 1+(1-\alpha)\cos 30^\circ & (1-\alpha)\sin 30^\circ & -[1+(1-\alpha)\cos 30^\circ][(1-\alpha)\cos 30^\circ] \\
\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & \frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[\frac{1}{5}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] \\
\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right]^2 & -\frac{1}{4}+(1-\alpha)\cos 30^\circ & \frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ & -\left[-\frac{1}{4}+(1-\alpha)\cos 30^\circ\right]\left[\frac{\sqrt{15}}{2}+(1-\alpha)\sin 30^\circ\right] \\
\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & \left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]^2 & -\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}} & -\left[-\frac{1}{2}+\frac{(1-\alpha)\cos 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right]\left[-\sqrt{3}+\frac{(1-\alpha)\sin 30^\circ}{\sqrt{1+3\sin^2 30^\circ}}\right] \\
\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]^2 & \left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right]^2 & \frac{1}{3}+(1-\alpha)\cos 30^\circ & \frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ & -\left[\frac{1}{3}+(1-\alpha)\cos 30^\circ\right]\left[\frac{4\sqrt{2}}{3}+(1-\alpha)\sin 30^\circ\right]
\end{vmatrix}$$

$$=-0.194\alpha^4+2.264\alpha^3-5.842\alpha^2+6.232\alpha-3.518$$

Let substitute these values in the equation (3) and we obtain the coefficients $a_\theta^\alpha, b_\theta^\alpha, c_\theta^\alpha, f_\theta^\alpha, g_\theta^\alpha$ and c_θ^α. If we substitude these coefficients in the conic equation (4) and simplify the equation with h_θ^α ($h_\theta^\alpha \neq 0$), we obtain the following equation;

$$-0{,}389\alpha^4+3{,}475\alpha^3-0{,}692\alpha^2-1{,}706\alpha-0{,}555=0 \qquad (5)$$

By solving the equation (5), the real values of α;

$$0.98, 0.85$$

As alpha represents the grade of membership, it must be [0,1].

So the appropriate alpha real number is $\theta = 0.98$ as it is supremum. Thus for $\theta_0 = 30°$, the set S_{θ_0} of all passible value of α is the set $\{0.98\}$.

We vary θ_0 across $[0,2\pi]$ and keep an identifying the value of S_{θ_0}. Finally, we get supremum of all S_{θ_0}'s. One can easily verify that the supremum value for the considered point is 0,98 which is attained for the value of $\theta_0 = 30°$. Eventually, we note that the conic passing through to the same points for $\theta_0 = 30°$ and $\alpha = 0{,}98$. As a result, we find that the conic which passing through to the same-points are;

$(1.01,0.01) \in \overline{E_1}, (0.21,1.96) \in \overline{E_2}, (-0.23,1.94) \in \overline{E_3}, (-0.48,-1.72) \in \overline{E_4},$ and $(0.35,1.89) \in \overline{E_5}$

Then we obtain the conic equation (6) which passing through to the same -points:

$$0.96x^2 - 0.04xy + 0.27y^2 + 0.05x - 0.03y - 1.03 = 0 \qquad (6)$$

This conic equation contains the point $(1, 0.5)$. And, we have

$$\mu\big((1, 0.5)|\overline{FE}\big) = 0.98$$

alpha cut of fuzzy ellipse.

Now we give an example of a fuzzy ellipse whose core ellipse is decentralized.

Example 3.2. Let $\overline{E_1} = (1,4), \overline{E_2} = \left(0, \frac{2\sqrt{35}}{3}\right), \overline{E_3} = \left(-2, 2\sqrt{3}\right), \overline{E_4} = \left(-3, -\frac{4\sqrt{5}}{3}\right)$ and $\overline{E_5} = \left(4, -2\sqrt{3}\right)$ are fuzzy poinys. We get the fuzzy ellipse that passes through these points.

The core ellipse equation is that passes through these points;

$$\left\{(x,y): \frac{(x-1)^2}{36} + \frac{y^2}{16} = 1\right\}$$

Now let take the membership functions of these five ellipses as circle, ellipse, circle, circle and ellipse respectively;

$$\{(x,y): (x-1)^2 + (y-4)^2 \leq 1\}$$

$$\left\{(x,y): x^2 + 9\left(y - \frac{2\sqrt{35}}{3}\right)^2 \leq 1\right\}$$

$$\left\{(x,y): (x+2)^2 + \left(y + 2\sqrt{3}\right)^2 \leq 1\right\}$$

$$\left\{(x,y): (x+3)^2 + \left(y + \frac{4\sqrt{5}}{3}\right)^2 \leq 1\right\}$$

$$\left\{(x,y): 9(x-4)^2 + \left(y+2\sqrt{3}\right)^2 \leq 1\right\}$$

The vertices of the membership function are $(1,4)$, $\left(0,\frac{2\sqrt{35}}{3}\right)$, $\left(-2,2\sqrt{3}\right)$, $\left(-3,-\frac{4\sqrt{5}}{3}\right)$ and $\left(4,-2\sqrt{3}\right)$ respectively. Now let write five same-points on $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{E}_4$ and $\bar{E}_5$ whose membership degrees are alpha

$$Q_{1\theta}^{\alpha}: (x_{1\theta}^{\alpha}, y_{1\theta}^{\alpha}) = (1 + (1-\alpha).\cos\theta\, , 4 + (1-\alpha).\sin\theta)$$

$$Q_{2\theta}^{\alpha}: (x_{2\theta}^{\alpha}, y_{2\theta}^{\alpha}) = \left((1-\alpha).\frac{\cos\theta}{\sqrt{1+8\sin^2\theta}}, \frac{2\sqrt{35}}{3} + (1-\alpha).\frac{\sin\theta}{\sqrt{1+8\sin^2\theta}}\right)$$

$$Q_{3\theta}^{\alpha}: (x_{3\theta}^{\alpha}, y_{3\theta}^{\alpha}) = \left(-2 + (1-\alpha).\cos\theta\, , 2\sqrt{3} + (1-\alpha).\sin\theta\right)$$

$$Q_{4\theta}^{\alpha}: (x_{4\theta}^{\alpha}, y_{4\theta}^{\alpha}) = \left(-3 + (1-\alpha).\cos\theta\, , -\frac{4\sqrt{5}}{3} + (1-\alpha).\sin\theta\right)$$

$$Q_{5\theta}^{\alpha}: (x_{5\theta}^{\alpha}, y_{5\theta}^{\alpha}) = \left(4 + (1-\alpha).\frac{\cos\theta}{\sqrt{1+8\cos^2\theta}}, -2\sqrt{3} + (1-\alpha).\frac{\sin\theta}{\sqrt{1+8\cos^2\theta}}\right)$$

The ellipse E_{θ}^{α} that passes through $Q_{1\theta}^{\alpha}, Q_{2\theta}^{\alpha}, Q_{3\theta}^{\alpha}, Q_{4\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$ can be determinant by the equation below again as in the previous example;

$$a_{\theta}^{\alpha}x^2 + 2h_{\theta}^{\alpha}xy + b_{\theta}^{\alpha}y^2 + 2g_{\theta}^{\alpha}x + 2\, f_{\theta}^{\alpha}y + c_{\theta}^{\alpha} = 0$$

with ${h_{\theta}^{\alpha}}^2 < a_{\theta}^{\alpha}.b_{\theta}^{\alpha}$.

We gave on the previous example how to find a_{θ}^{α}, b_{θ}^{α}, g_{θ}^{α}, f_{θ}^{α} and c_{θ}^{α}. In this example we apply the same. The fuzzy ellipse $\overline{FE}_{1\ldots5}$ that passes through $\bar{E}_i$'s $i = 1,2,3,4,5$ is the union of all possible ellipse E_{θ}^{α} 's that lies between $Q_{1\theta}^{\alpha}$ and $Q_{5\theta}^{\alpha}$'s. That is,

$$\overline{FE}_{1\ldots5} = \bigvee_{\alpha\in[0,1]} \bigcup_{\theta\in[0,2\pi]} \left\{ \begin{matrix} (x,y) = a_{\theta}^{\alpha}x^2 + \ 2h_{\theta}^{\alpha}xy + \ b_{\theta}^{\alpha}y^2 \\ +2g_{\theta}^{\alpha}x + 2f_{\theta}^{\alpha}y + c_{\theta}^{\alpha} = 0 \end{matrix} \right\}$$

Now we find the membership degree of the point $(1,4.1)$ on the fuzzy ellipse $\overline{FE}$. We adjust the set of ellipses E_{θ}^{α} 's on which the point $(1,4.1)$ lies.

Let replace point $(1, 4.1)$ in equation (2) we need to identify the possible values of α. Then, we get the equation below:

$$a_{\theta}^{\alpha}(1)^2 + \ 2h_{\theta}^{\alpha}(1).(4.1) + \ b_{\theta}^{\alpha}(4.1)^2 + 2g_{\theta}^{\alpha}(1) + 2f_{\theta}^{\alpha}(0.5) + c_{\theta}^{\alpha} = 0$$

which simplifies to

$$a_{\theta}^{\alpha} + \ 8.2h_{\theta}^{\alpha} + \ 16.81b_{\theta}^{\alpha} + 2g_{\theta}^{\alpha} + 8.2f_{\theta}^{\alpha} + c_{\theta}^{\alpha} = 0 \qquad (7)$$

Then we put $\theta_0 = 45°$ in (7) and we calculate $a_\theta^\alpha, b_\theta^\alpha, g_\theta^\alpha,\ f_\theta^\alpha$ and c_θ^α.

We find them and replace in (7) then we obtain the following non-linear equation which determinants are found:

$$-31.18\alpha^4 - 1303.24\alpha^3 - 5634.93\alpha^2 - 6568.61\alpha + 10404.88 = 0$$

By solving this equation, real alpha values are found;

$$\alpha = 0.85, -2.78$$

But we get 0.85 from 0 to 1 from these real two values.

Thus, for $\theta_0 = 45°$, the set S_{θ_0} of all passible value of α is the set $\{0.85\}$.

We vary θ_0 across $[0,2\pi]$ and keep an identifying the value of s_{θ_0}. Finaly, we get supremum of all s_{θ_0}'s. One can easily verify that the supremum value fort he considered point is 0,85 which is attained for the value of $\theta_0 = 45°$. Eventually, we note that the conic passing through to the same points, for $\theta_0 = 45°$ and $\alpha = 0,85$.

As a result, we find that the conic which passing through to the same-points are;

$$(1.31, 4.31) \in \bar{E}_1, (0.13,4.08) \in \bar{E}_2, (-1.68,3.77) \in \bar{E}_3,$$

$$(-2.68, -2.67) \in \bar{E}_4, (4.13, -3.32) \in \bar{E}_5$$

Then we obtain the following conic equation (8) which passing through to the same-points:

$$-524.84x^2 + 74.07xy - 2027.74y^2 + 1902.47x + 907.46y + 28574.06 = 0 \quad (8)$$

This conic equation contains the point (1,4.1). And, we have

$$\mu\big((1,4.1)\big|\overline{FE}\big) = 0.85$$

alpha cut of fuzzy ellipse and membership degree of a core ellipse in a fuzzy ellipse.

4. Conclusion

The concept of fuzzy ellipse has been initiated and basic properties of fuzzy ellipse have been explained in this study in details. The membership degrees of fuzzy points have a specific role for the graph of ellipse. The needed explanations based on these roles were made on the drawn graphics. Equations of conics such as hyperbola and ellipse can be obtained by determining five points. Starting from these, we developed a method for obtaining the fuzzy ellipse equation in the study. But we can't do this with five random points. We used the points which called the same-points. In the study, we have presented a method by determining the necessary properties for selecting points. As seen in the figures, the fuzzy ellipses can be depicted with different curves. Depending on the degree of membership Fuzzy ellipse can be use kidney stones crushing machines, billiard games, aerospace engineering and laser technology etc. We have shown the applicability of the method in the examples. When we create the fuzzy ellipse equations in the 3rd section, it is seen that the necessary coefficients for the calculation of these equations will be made with high-dimensional determinants. In the examples given, the maple program was used to calculate the membership of the coefficients using the selected points and angles. Thus, it will be possible to find fuzzy ellipses at different point and angle selections. The determinants we presented in section 3 can be easily calculated with mathematical programs.

Fuzzy ellipse can be use kidney stones crushing machines, billiard games, aerospace engineering and laser technology etc. The method and applications we have shown in this study will be a guide for studies in these areas.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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