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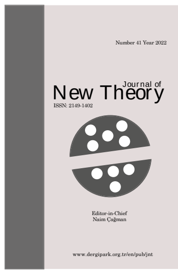


# Bivariate Variations of Fibonacci and Narayana Sequences and Universal Codes

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**Abstract** − In this study, we worked on the third-order bivariate variant of the Fibonacci universal code and the second-order bivariate variant of the Narayana universal code, depending on two negative integer variables $u$ and $v$. We then showed in tables these codes for $1 \leq k \leq 100$, $u = -1, -2, \dots, -20$, and $v = -2, -3, \dots, -21$ ($u$ and $v$ are consecutive, $v < u$). Moreover, we obtained some significant results from these tables. Furthermore, we compared the use of these codes in cryptography. Finally, we obtained the third-order bivariate variant of Fibonacci codes is more valuable than the second-order bivariate variant of Narayana codes.

***Keywords*** − *Fibonacci sequence, Narayana sequence, cryptography, variant Fibonacci code, variant Narayana code*

***Mathematics Subject Classification (2020)*** – 11B37, 14G50

## 1. Introduction

Number sequences have been popular with scientists for centuries. The most popular of these sequences is the Fibonacci sequence. The Fibonacci sequence, $\{F_k\}_0^\infty$, is a series of numbers, starting with the integers 0 and 1, in which the value of any element is computed by taking the summation of the two consecutive numbers. Here, for $k \geq 2$, $F_k = F_{k-1} + F_{k-2}$ [1]. In 1202, this sequence was introduced by Fibonacci. The Fibonacci numbers in the range of $-8 \leq k \leq 8$ are as follows:

$$-21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21$$

A sequence that based on the Fibonacci sequence and is a generalization of the Fibonacci sequence is the Tribonacci sequence. The Tribonacci sequence, $\{T_k\}_0^\infty$, is a series of numbers, starting with the integers 0, 1, and 1, in which the value of any element is computed by taking the summation of the preceding three terms [2]. If so, for $k \geq 3$, $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ [1]. In 1914, this number series was presented by Agronomof. The Tribonacci numbers in the range of $-8 \leq k \leq 8$ are as follows:

$$4, 1, -3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, 24, 44$$

Another sequence that based on a similar problem with Fibonacci numbers is the Narayana sequence. The Narayana sequence $\{N_k\}_0^\infty$, is a series of numbers, starting with the integers 1, 1, and 1, in which the value of any element is computed by taking the summation of the previous term and term two places before. Then, for $k \geq 3$, $N_{k+1} = N_k + N_{k-2}$ [1].

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In 1356, this number series was presented by Narayana in the treatise named Gaṇita Kaumudi. The Narayana numbers in the range of $-8 \leq k \leq 8$ are as follows:

$$-2, 1, 1, -1, 0, 1, 0, 0, 1, 1, 1, 2, 3, 4, 6, 9, 13$$

There are also different variants of the Fibonacci sequence. The author [3] defined a variation of the Fibonacci sequence that is the more general of Gopala Hemachandra (GH) sequence [4]:

$$\{u, v, u+v, u+2v, 2u+3v, 3u+5v\}$$

where $u, v$ are integers and a second order variant Fibonacci sequence, as the Gopala Hemachandra sequence above such that $v = 1 - u$.

In addition, different universal codes have been defined in the literature. Codes that are obtained by using number sequences are involving to them. The Fibonacci universal code, GH code (or variant Fibonacci code) and the Narayana code are examples of these can be given [3-9]. The most used of these codes is the Fibonacci code [8]. Fibonacci code is a universal code that encodes positive integers with binary representations according to Zeckendorf's Theorem [9]. To find the Fibonacci code for any positive integer, the following algorithm must hold.

One Fibonacci representation or code can be obtained for each positive integer $C$ with a binary string of length $t$, $g_1 g_2 \dots g_{t-1} g_t$, such that $C = \sum_{i=1}^{t} g_i F_i^{(m)}$. The representation is unique if and only if it is used the following algorithm: When it is given the integer $C$, it is selected the largest Fibonacci number $F_k^{(2)}$ equal to $C$ or smaller to $C$; after that, it is continued repeating with $C - F_k^{(2)}$ [8]. Finally, to find the Fibonacci code for any positive integer, it is added $1-$bit to the Fibonacci representation of the positive integer. For example, $31 = 2 + 8 + 21 = F_2^{(2)} + F_5^{(2)} + F_7^{(2)}$. Hence, its Fibonacci representation is 0100101 and its Fibonacci code is 01001011.

According to the above algorithm, there is no contiguous 1-bit in the binary representation. If we apply this algorithm to higher orders, it is seen that the same operations are carried out. But in the higher orders, it is added 1, $m - 1$ bits to the $m^{th}$ order Fibonacci representation of $k$ to build the $m^{th}$ order Fibonacci code. Therefore, there are no successive of $m$ bits [8].

Narayana and GH universal code are two generalizations of Fibonacci universal code. Thus, these codes also encode positive integers with binary representations according to above algorithm. That is, these universal codes and the variants of these codes can be obtained by using the same rule used to generate the standard Fibonacci code [3].

Moreover, cryptographic applications can be made using these codes or representations. There are many works on these codes in literature (see for example [5-10]).

This paper research the second-order bivariate variant of Narayana codes and the third-order bivariate GH codes and the properties of these codes. Furthermore, it also gives some results regarding the use of these codes in cryptography.

## 2. Preliminaries

In this section, the basic definitions and theorems that used in this study will be given.

**Definition 2.1.** [8] The $m^{th}$ order Fibonacci numbers, represented by $F_k^{(m)}$, are described with iteration relation as follows: $F_k^{(m)} = F_{k-1}^{(m)} + F_{k-2}^{(m)} + \dots + F_{k-m}^{(m)}$, for $k > 0$, and the boundary conditions $F_0^{(m)} = 1$ and $F_l^{(m)} = 0 \; (-m < l < 0)$ .

Thus, in 2007, a second-order variation of the Fibonacci code was presented for $m = 2$ as follows:

**Definition 2.2.** [3] The second-order variant Fibonacci sequences, $GH_u^{(2)}(k)$ is described with the sequences $\{u, v, u+v, u+2v, 2u+3v, 3u+5v\}$ where $v = 1-u$, that is , $GH_u^{(2)}(1) = u$; $GH_u^{(2)}(2) = 1-u$; and for $k \geq 3$, $GH_u^{(2)}(k) = GH_u^{(2)}(k-1) + GH_u^{(2)}(k-2)$.

Afterwards, in 2010, the authors [9] studied on the second-order variation Fibonacci code and obtained some results about the second-order variation Fibonacci codes of some positive integers. Then, in 2015, a third-order variation of the Fibonacci code was presented for $m = 3$ as follows:

**Definition 2.3.** [10] The third-order variant Fibonacci sequences, $GH_u^{(3)}(k)$ is described with the sequences $\{u, v, u+v, 2\,u+2v, 3u+4v, 6u+7v\}$ where $v = 1-u$, that is , $GH_u^{(3)}(1) = u$, $GH_u^{(3)}(2) = 1-u$, $GH_u^{(3)}(3) = 1$, and for $k \geq 4$, $GH_u^{(3)}(k) = GH_u^{(3)}(k-1) + GH_u^{(3)}(k-2) + GH_u^{(3)}(k-3)$.

In this situation, the second and the third order variant Fibonacci (or GH) codes can be defined above definitions. And we know that these universal codes can be obtained by using the same rule used to generate the standard Fibonacci code [3]. Thus, it is obvious that as the value of $u$ changes, a different sequence is obtained. Daykin [11] proved that only the Fibonacci sequence forms a unique Fibonacci code for all positive integers. Moreover, some integers have many GH codes, while others have no GH code. For instance, if $VF_{-5}^{(2)}(k) = \{-5,6,1,7,8,15,23,38, \dots\}$, then there is no GH code for $k = 20$. Similarly, if $VF_{-2}^{(2)}(k) = \{-2,3,1,4,5,9,14,23, \dots\}$, then there are two GH codes for $k = 13$. Because

$$13 = 4 + 9 = VF_{-2}^{(2)}(4) + VF_{-2}^{(2)}(6) = -2 + 1 + 14 = VF_{-2}^{(2)}(1) + VF_{-2}^{(2)}(3) + VF_{-2}^{(2)}(7)$$

these codes are 0001011 and 10100011. Furthermore, if $VF_{-11}^{(3)}(k) = \{-11,12,1,2,15,18,35,68, \dots\}$, then there is no GH code for $k = 11$ [10], and if $VF_{-5}^{(3)}(k) = \{-5,6,1,2,9,12,23,44, \dots\}$, then there are two GH codes for integer $k = 9$. Because $9 = VF_{-5}^{(3)}(5) = -5 + 2 + 12 = VF_{-5}^{(3)}(1) + VF_{-5}^{(3)}(4) + VF_{-5}^{(3)}(6)$ these codes are 0000111 and 10010111. Moreover, in 2016, it was obtained and proved the following theorem by Basu et al [12]. According to following theorem, it is obtained that the relationship between the second-order variant Fibonacci sequence and the Fibonacci sequence.

**Theorem 2.4.** [12] Let $VF_u(k)$ is the second-order variant Fibonacci sequence and $F(k)$ is the Fibonacci sequence. In this case, for $k \geq 1$, $VF_u(k) = F(k-2) - uF(k-4)$ where $k$ is an integer.

**Definition 2.5.** [5] A second-order variant of the Narayana sequence, $VN_u(k)$, is described the sequence $\{u, 3-u, 1-u, 1, 4-u, 5-2u, 6-2u, 10-3u, \dots\}$ with initial conditions $VN_u(0) = u$; $VN_u(1) = 3-u$; $VN_u(2) = 1-u$; and for $k \geq 3$, $VN_u(k) = VN_u(k-1) + VN_u(k-3)$.

Similarly, the second order variant Narayana codes can be defined above definition. Furthermore, we know that these universal codes can be obtained by using the same algorithm used to generate the standard Fibonacci code [3]. For instance, there is no variant Narayana code for 2 for $VN_{-3}(k) = \{-3,6\ 4,1,7,11,12,19\}$. Moreover, we obtain that there are two variant Narayana codes of 12 for $VN_{-1}(k) = \{-1,4,2,1,5,7,8,13,20, \dots\}$. Because

$$12 = 4 + 8 = VN_{-1}(2) + VN_{-1}(7) = -1 + 13 = VN_{-5}^{(3)}(1) + VN_{-5}^{(3)}(8)$$

these variant Narayana codes of 12 are 01000011 and 100000011.

## 3. The Research Findings and Discussion

In this study, firstly, we obtained and demonstrated the following theorem, which is the relation between the third-order variant Fibonacci sequence and the Tribonacci sequence.

**Theorem 3.1.** Let $VF_u^{(3)}(k)$ is the third-order variant Fibonacci sequence and $T(k)$ is the Tribonacci sequence. In this case, for $k \geq 1$,

$$VF_u^{(3)}(k) = T(k-1) - uT(k-4)$$

where $k$ is an integer.

**Proof.** We obtain the following set from the third-order variant of the Fibonacci sequence.

$$VF_u^{(3)}(k) = \{u, 1-u, 1, 2, 4-u, 7-u, 13-2u, 24-4u, \dots\}$$

$$VF_u^{(3)}(1) = u = 0 - u(-1) = T(-1) - uT(-3)$$

$$VF_u^{(3)}(2) = 1 - u = 1 - u(1) = T(1) - uT(-2)$$

$$VF_u^{(3)}(3) = 1 = 1 - u(0) = T(2) - uT(-1)$$

The results are correct for $k = 1, k = 2$, and $k = 3$, as seen above. Suppose that the results are correct for $k = 1, 2 \dots, n$. In that case, $VF_u^{(3)}(n-2) = T(n-3) - uT(n-6)$ , $VF_u^{(3)}(n-1) = T(n-2) - uT(n-5)$, and $VF_u^{(3)}(n) = T(n-1) - uT(n-4)$. Since

$$\begin{aligned} VF_u^{(3)}(n+1) &= T(n-1) - uT(n-4) + T(n-2) \ - uT(n-5) + T(n-3) \ - uT(n-6) \\ &= T(n-1) + T(n-2) + T(n-3) - uT(n-4) + T(n-5) + T(n-6) \\ &= T(n) \ - uT(n-3) = T(n+1-1) \ - uT(n+1-4) \end{aligned}$$

this equation is true for $n+1$. Therefore, by the induction, $VF_u^{(3)}(k) = T(k-1) - uT(k-4), k \geq 1$.

## 3.1. The Third Order Bivariate Gopala Hemachandra Sequences and Codes

We will describe a new variation depending on two negative variables of the GH sequence for $m = 3$ as follows:

**Definition 3.1.1.** The third-order bivariate GH sequence $GH_{(u,v)}^{(3)}(k)$ is described the sequence $\{u, v, s, u+v+s, u+2v+2s, 2u+3v+4s, 4u+6v+7s, \dots\}$ where $s = 1-u-v$, with initial conditions $GH_{(u,v)}^{(3)}(1) = u$, $GH_{(u,v)}^{(3)}(2) = v$, $GH_{(u,v)}^{(3)}(3) = 1-u-v$ and for $k \geq 4$, $GH_{(u,v)}^{(3)}(k) = GH_{(u,v)}^{(3)}(k-1) + GH_{(u,v)}^{(3)}(k-2) + GH_{(u,v)}^{(3)}(k-3)$.

For instance, from the above definition for $u = -2, v = -3$, we have $\{-2, -3, 6, 1, 4, 11, 16, 31, \dots\}$. It is obviously seen that different sequences will be obtained for different values of $u$ and $v$. With these bivariate variants of GH sequences, we may compose a new universal source code, which we named the bivariate GH code or bivariate variant of Fibonacci universal code. To make, it can be used the same rule used to generate the standard Fibonacci code as in the second and the third order variant Fibonacci code. In addition, we know that GH sequences authorize having more than one Zeckendorf representation of any integer. Similarly, here, we have obtained the third-order bivariate variant of the GH sequences authorizing having more than one Zeckendorf representation of any integer. For example, for $GH_{(-2,-3)}^{(3)}(k) = \{-2, -3, 6, 1, 4, 11, 16, 31, \dots\}$,

$$17 = 6 + 11 = GH_{(-2,-3)}^{(3)}(3) + GH_{(-2,-3)}^{(3)}(6) = 1 + 16 = GH_{(-2,-3)}^{(3)}(4) + GH_{(-2,-3)}^{(3)}(7)$$

Therefore, the third order bivariate GH code of 17 would be both 00100111 and 000100111.

In this study, we investigated for which values $u$ and $v$, the third-order bivariate GH codes precisely exist or for which $k$ positive integers they do not exist. For instance, we obtained that each positive integer has at least one bivariate GH code for $GH^{(3)}_{(-2,-6)}(k) = \{-2, -6, 9, 1, 4, 14, 19, 37, \dots\}$. But there is no bivariate GH code of 5 for $GH^{(3)}_{(-6,-2)}(k) = \{-6, -2, 9, 1, 8, 18, 27, 53, \dots\}$.

Basu and Prasad [9] obtained the second-order GH codes $GH^{(2)}_u(k)$ or undetectable values of the positive integer $k$ for $k = 1, 2, \dots, 100$ and for $u = -2, \dots, -20$. Besides, Nallı and Özyılmaz [10] obtained the third-order GH codes $GH^{(3)}_u(k)$ or undetectable values of the positive integer $k$, for $k = 1, 2, \dots, 100$ and for $u = -2, \dots, -20$.

This section obtained the third-order bivariate GH codes $VF^{(3)}_{(u,v)}(k)$ or undetectable values (–) of the positive integer $k$ for $k = 1, 2, \dots, 100$ and for $u = -1, \dots, -20$ and $v = -2, \dots, -21$ ($u$ and $v$ are consecutive, $v < u$) Tables 1 and 2. From Tables 1 and 2, we got the following results for the third-order bivariate GH codes.

*i.* For the positive integers $k = 1,2,3$, the third-order bivariate GH code $VF^{(3)}_{(u,v)}(k)$ exactly exists, for $u = -1, -2, \dots, -20$ and $v = -2, -3, \dots, -21$ ($u$ and $v$ are consecutive, $v < u$).

*ii.* For $1 \leq k \leq 100$, there are at most $j$ consecutive undetectable (--) values in the third-order bivariate GH code in $VF^{(3)}_{(-3+j,-(4+j))}(k)$ column in which $1 \leq j \leq 17$.

*iii.* For $1 \leq k \leq 100$, as long as $j$ raises, the detectable of GH code is reduced in $VF^{(3)}_{(-3+j,-(4+j))}(k)$ column in which $1 \leq j \leq 17$.

**Table 1.** The third order bivariate GH codes ($u$ and $v$ are consecutive and $v < u$)

| $k$ | $(u=-1)$ $(v=-2)$ | $(u=-2)$ $(v=-3)$ | $(u=-3)$ $(v=-4)$ | $(u=-4)$ $(v=-5)$ | $(u=5)$ $(v=-6)$ | $(u=-6)$ $(v=-7)$ | $(u=-7)$ $(v=-8)$ | $(u=-8)$ $(v=-9)$ | $(u=-9)$ $(v=-10)$ | $(u=-10)$ $(v=-11)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 |
| **2** | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 |
| **3** | 0000111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 |
| **4** | 00111 | 0000111 | 01111 | – | – | – | – | – | – | – |
| **5** | 001111 | 0001111 | 0000111 | 01111 | – | – | – | – | – | – |
| **6** | 01000111 | 00111 | 0001111 | 0000111 | 01111 | – | – | – | – | – |
| **7** | 10000111 | 0110111 | 11000111 | 0001111 | 0000111 | 01111 | – | – | – | – |
| **8** | 00000111 | 01000111 | 00111 | 11000111 | 0001111 | 0000111 | 01111 | – | – | – |
| **9** | 00010111 | 10000111 | 001111 | 11010111 | 11000111 | 0001111 | 0000111 | 01111 | – | – |
| **10** | 010000111 | 10010111 | 01000111 | 00111 | 11010111 | 11000111 | 0001111 | 0000111 | 01111 | – |
| **11** | 100000111 | 00000111 | 10000111 | 001111 | – | 11010111 | 11000111 | 0001111 | 0000111 | 01111 |
| **12** | 000000111 | 00010111 | 10010111 | 01000111 | 00111 | – | 11010111 | 11000111 | 0001111 | 0000111 |
| **13** | 000100111 | 010000111 | 0010111 | 10000111 | 001111 | – | – | 11010111 | 11000111 | 0001111 |
| **14** | 100010111 | 100000111 | 00000111 | 10010111 | 01000111 | 00111 | – | – | 11010111 | 11000111 |
| **15** | 000010111 | 100100111 | 00010111 | 110000111 | 10000111 | 001111 | – | – | – | 11010111 |
| **16** | 001000111 | 000000111 | 010000111 | 0010111 | 10010111 | 01000111 | 00111 | – | – | – |
| **17** | 001100111 | 000100111 | 100000111 | 00000111 | 110000111 | 10000111 | 001111 | – | – | – |
| **18** | 101010111 | 100010111 | 100100111 | 00010111 | 110100111 | 10010111 | 01000111 | 00111 | – | – |
| **19** | 100001111 | 100110111 | 00001111 | 010000111 | 0010111 | 110000111 | 10000111 | 001111 | – | – |
| **20** | 000001111 | 000010111 | 000000111 | 100000111 | 00000111 | 110100111 | 10010111 | 01000111 | 00111 | – |
| **21** | 0100000111 | 000110111 | 000100111 | 100100111 | 00010111 | – | 110000111 | 10000111 | 001111 | – |
| **22** | 0101000111 | 001000111 | 100010111 | 01100111 | 010000111 | 0010111 | 110100111 | 10010111 | 01000111 | 00111 |
| **23** | 0000000111 | 001100111 | 100110111 | 00001111 | 100000111 | 00000111 | – | 110000111 | 10000111 | 001111 |
| **24** | 0001000111 | 101010111 | 011000111 | 000000111 | 100100111 | 00010111 | – | 110100111 | 10010111 | 01000111 |
| **25** | 1000100111 | 100001111 | 000010111 | 000100111 | 110110111 | 010000111 | 0010111 | – | 110000111 | 10000111 |

**Table 1.** Continued

| $k$ | $(u=-1)$ $(v=-2)$ | $(u=-2)$ $(v=-3)$ | $(u=-3)$ $(v=-4)$ | $(u=-4)$ $(v=-5)$ | $(u=5)$ $(v=-6)$ | $(u=-6)$ $(v=-7)$ | $(u=-7)$ $(v=-8)$ | $(u=-8)$ $(v=-9)$ | $(u=-9)$ $(v=-10)$ | $(u=-10)$ $(v=-11)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **26** | 0000100111 | 100101111 | 000110111 | 100010111 | 01100111 | 100000111 | 00000111 | – | 110100111 | 10010111 |
| **27** | 0010000111 | 000001111 | 00101111 | 100110111 | 00001111 | 100100111 | 00010111 | – | – | 110000111 |
| **28** | 0011000111 | 0100000111 | 001000111 | 00110111 | 000000111 | 110110111 | 010000111 | 0010111 | – | 110100111 |
| **29** | 0100010111 | 0101000111 | 001100111 | 011000111 | 000100111 | – | 100000111 | 00000111 | – | – |
| **30** | 0101010111 | 1001000111 | 101010111 | 000010111 | 100010111 | 01100111 | 100100111 | 00010111 | – | – |
| **31** | 0000010111 | 0101000111 | 100001111 | 000110111 | 100110111 | 00001111 | 110110111 | 010000111 | 0010111 | – |
| **32** | 0001010111 | 0001000111 | 100101111 | 110001111 | 00100111 | 000000111 | – | 100000111 | 00000111 | – |
| **33** | 0100001111 | 1000100111 | 001010111 | 00101111 | 00110111 | 000100111 | – | 100100111 | 00010111 | – |
| **34** | 0101001111 | 1001100111 | 000001111 | 001000111 | 011000111 | 100010111 | 01100111 | 110110111 | 010000111 | 0010111 |
| **35** | 0000001111 | 0000100111 | 0100000111 | 001100111 | 000010111 | 100110111 | 00001111 | – | 100000111 | 00000111 |
| **36** | 0001001111 | 0001100111 | 0101000111 | 101010111 | 000110111 | – | 000000111 | – | 100100111 | 00010111 |
| **37** | 1000101111 | 0010000111 | 1001000111 | 100001111 | 110001111 | 00100111 | 000100111 | – | 110110111 | 010000111 |
| **38** | 0000101111 | 0011000111 | 011001111 | 100101111 | 110101111 | 00110111 | 100010111 | 01100111 | – | 100000111 |
| **39** | 0010001111 | 0100010111 | 0101000111 | 1101000111 | 00101111 | 011000111 | 100110111 | 00001111 | – | 100100111 |
| **40** | 0011001111 | 1000010111 | 0001000111 | 001010111 | 001000111 | 000010111 | – | 000000111 | – | 110110111 |
| **41** | 01000000111 | 1001010111 | 1000100111 | 000001111 | 001100111 | 000110111 | – | 000100111 | – | – |
| **42** | 01010000111 | 0000010111 | 1001100111 | 0100000111 | 101010111 | 110001111 | 00100111 | 100010111 | 01100111 | – |
| **43** | 00000000111 | 0001010111 | 0110000111 | 0101000111 | 100001111 | 110101111 | 00110111 | 100110111 | 00001111 | – |
| **44** | 00010000111 | 0100001111 | 0000100111 | 1001000111 | 100101111 | – | 011000111 | – | 000000111 | – |
| **45** | 10001000111 | 1000001111 | 0001100111 | – | 1101000111 | 00101111 | 000010111 | 000010111 | 000100111 | – |
| **46** | 00001000111 | 1001001111 | 1100010111 | 011001111 | – | 001000111 | 000110111 | – | 100010111 | 01100111 |
| **47** | 00100000111 | 0000001111 | 0010000111 | 0101000111 | 001010111 | 001100111 | 110001111 | 00100111 | 100110111 | 00001111 |
| **48** | 00110000111 | 0001001111 | 0011000111 | 0001000111 | 000001111 | 101010111 | 110101111 | 00110111 | – | 000000111 |
| **49** | 01000100111 | 1000101111 | 0100010111 | 1000100111 | 0100000111 | 100001111 | – | 011000111 | – | 000100111 |
| **50** | 10000100111 | 1001101111 | 1000010111 | – | 0101000111 | 100101111 | – | 000010111 | 000010111 | 100010111 |
| **51** | 00000100111 | 0000101111 | 1001010111 | 001001111 | 1001000111 | 1101000111 | 00101111 | 000110111 | – | 100110111 |
| **52** | 00010100111 | 0001101111 | 1100001111 | 0110000111 | 1101100111 | – | 001000111 | 110001111 | 00100111 | – |
| **53** | 10001100111 | 11000000111 | 0000010111 | 0000100111 | – | – | 001100111 | 110101111 | 00110111 | – |
| **54** | 00001100111 | 0110101111 | 0001010111 | 0001100111 | 011001111 | 001010111 | 101010111 | – | 011000111 | – |
| **55** | 00000010111 | 01000000111 | 1000110111 | 1100010111 | 0101000111 | 000001111 | 100001111 | – | 000010111 | 000010111 |
| **56** | 00010010111 | 10000000111 | 1000001111 | 1101010111 | 0001000111 | 0100000111 | 100101111 | – | 000110111 | |
| **57** | 10001010111 | 10010000111 | 1001001111 | 0010000111 | 1000100111 | 0101000111 | 1101000111 | 00101111 | 110001111 | 00100111 |
| **58** | 00001010111 | 00000000111 | 0000110111 | 0011000111 | 1001100111 | 1001000111 | – | 001000111 | 110101111 | 00110111 |
| 59 | 00100010111 | 00010000111 | 0000001111 | 0100010111 | – | 1101100111 | – | 001100111 | – | 011000111 |
| 60 | 00110010111 | 10001000111 | 0001001111 | 1000010111 | 001001111 | – | – | 101010111 | – | 000010111 |
| 61 | 01000110111 | 01100000111 | 1000101111 | 1001010111 | 0110000111 | – | 001010111 | 100001111 | – | 000110111 |
| 62 | 10000110111 | 00001000111 | 1001101111 | 1100001111 | 0000100111 | 011001111 | 000001111 | 100101111 | – | 110001111 |
| 63 | 00000110111 | 00011000111 | 0110001111 | 0010100111 | 0001100111 | 0101000111 | 0100000111 | 1101000111 | 00101111 | 110101111 |
| 64 | 01000001111 | 00100000111 | 0000101111 | 0000010111 | 1100010111 | 0001000111 | 0101000111 | – | 001000111 | – |
| 65 | 10000001111 | 00110000111 | 0001101111 | 0001010111 | 1101010111 | 1000100111 | 1001000111 | – | 001100111 | – |
| 66 | 00000001111 | 10101000111 | 11000000111 | 1000110111 | – | – | 1101100111 | – | 101010111 | – |
| 67 | 00010001111 | 10000100111 | 11010000111 | 1000001111 | 0010000111 | – | – | – | 100001111 | – |
| 68 | 10001001111 | 10010100111 | 0110101111 | 1001001111 | 0011000111 | – | – | 001010111 | 100101111 | – |
| 69 | 00001001111 | 00000100111 | 01000000111 | 0110010111 | 0100010111 | 001001111 | – | 000001111 | 1101000111 | 00101111 |
| 70 | 00100001111 | 00010100111 | 10000000111 | 0000110111 | 1000010111 | 0110000111 | 011001111 | 0100000111 | 1001100111 | 001000111 |
| 71 | 00110001111 | 10001100111 | 10010000111 | 0000001111 | 1001010111 | 0000100111 | 0101000111 | 0101000111 | – | 001100111 |
| 72 | 01000101111 | 10000010111 | 11011000111 | 0001001111 | 1100001111 | 0001100111 | 0001000111 | 1001000111 | – | 101010111 |

**Table 1.** Continued

| $k$ | $(u=-1)$ $(v=-2)$ | $(u=-2)$ $(v=-3)$ | $(u=-3)$ $(v=-4)$ | $(u=-4)$ $(v=-5)$ | $(u=5)$ $(v=-6)$ | $(u=-6)$ $(v=-7)$ | $(u=-7)$ $(v=-8)$ | $(u=-8)$ $(v=-9)$ | $(u=-9)$ $(v=-10)$ | $(u=-10)$ $(v=-11)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 73 | 01010101111 | 00001100111 | 00000000111 | 1000101111 | 1101001111 | 1100010111 | 1000100111 | 1101100111 | – | 100001111 |
| 74 | 000001001111 | 00000010111 | 00010000111 | 1001101111 | 0010100111 | 1101010111 | 1001100111 | – | – | 100101111 |
| 75 | 110000000111 | 00010010111 | 10001000111 | 0011010111 | 0000010111 | – | – | – | 001010111 | 1101000111 |
| 76 | 010000000111 | 10001010111 | 10011000111 | 0110001111 | 0001010111 | – | – | – | 000001111 | – |
| 77 | 100000000111 | 01100010111 | 01100000111 | 0000101111 | 1000110111 | 0010000111 | – | – | 0100000111 | – |
| 78 | 000000000111 | 00001010111 | 00001000111 | 0001101111 | 1000001111 | 0011000111 | 001001111 | 011001111 | 0101000111 | – |
| 79 | 000100000111 | 00011010111 | 00011000111 | 11000000111 | 1001001111 | 0100010111 | 0110000111 | 0101000111 | 1001000111 | – |
| 80 | 100010000111 | 00100010111 | 11000100111 | 11010000111 | 1101101111 | 1000010111 | 0000100111 | 0001000111 | 1101100111 | – |
| 81 | 000010000111 | 00110010111 | 00100000111 | 0010001111 | 0110010111 | 1001010111 | 0001100111 | 1000100111 | – | – |
| 82 | 001000000111 | 01000110111 | 00110000111 | 0110101111 | 0000110111 | 1100001111 | 1100010111 | 1001100111 | – | 001010111 |
| 83 | 001100000111 | 10000110111 | 10101000111 | 01000000111 | 0000001111 | 1101001111 | 1101010111 | – | – | 000001111 |
| 84 | 010001000111 | 11000001111 | 10000100111 | 10000000111 | 0001001111 | – | – | – | – | 0100000111 |
| 85 | 100001000111 | 00000110111 | 10010100111 | 10010000111 | 1000101111 | 0010100111 | – | – | – | 0101000111 |
| 86 | 000001000111 | 01000001111 | 00101000111 | 11011000111 | 1001101111 | 0000010111 | – | – | 011001111 | 1001000111 |
| 87 | 000101000111 | 10000001111 | 00000100111 | 0010101111 | 0010010111 | 0001010111 | 0010000111 | 001001111 | 0101000111 | 1101100111 |
| 88 | 010000100111 | 10010001111 | 00010100111 | 00000000111 | 0011010111 | 1000110111 | 0011000111 | 0110000111 | 0001000111 | – |
| 89 | 100000100111 | 00000001111 | 10001100111 | 00010000111 | 0110001111 | 1000001111 | 0100010111 | 0000100111 | 1000100111 | – |
| 90 | 000000100111 | 00010001111 | 10000010111 | 10001000111 | 0000101111 | 1001001111 | 1000010111 | 0001100111 | 1001100111 | – |
| 91 | 000100100111 | 10001001111 | 10010010111 | 10011000111 | 0001101111 | – | 1001010111 | 1100010111 | – | – |
| 92 | 100010100111 | 01100001111 | 00001100111 | – | 11000000111 | – | 1100001111 | 1101010111 | – | – |
| 93 | 000010100111 | 00001001111 | 00000010111 | 01100000111 | 11010000111 | 0110010111 | 1101001111 | – | – | – |
| 94 | 001000100111 | 00011001111 | 00010010111 | 00001000111 | 0010110111 | 0000110111 | – | – | – | 011001111 |
| 95 | 001100100111 | 00100001111 | 10001010111 | 00011000111 | 0010001111 | 0000001111 | – | – | – | 0101000111 |
| 96 | 010001100111 | 00110001111 | 10011010111 | 11000100111 | 0110101111 | 0001001111 | 0010100111 | – | 001001111 | 0001000111 |
| 97 | 100001100111 | 01000101111 | 01100010111 | 11010100111 | 01000000111 | 1000101111 | 0000010111 | 0010000111 | 0110000111 | 1000100111 |
| 98 | 000001100111 | 01010101111 | 00001010111 | 00100000111 | 01010000111 | 1001101111 | 0001010111 | 0011000111 | 0000100111 | 1001100111 |
| 99 | 000101100111 | 00101001111 | 00011010111 | 00110000111 | 10010000111 | – | 1000110111 | 0100010111 | 0001100111 | – |
| 100 | 100000010111 | 110000000111 | 11000110111 | 10101000111 | 11011000111 | 0010010111 | 1000001111 | 1000010111 | 1100010111 | – |

**Table 2.** The third order bivariate GH codes ($u$ and $v$ are consecutive and $v < u$)

| $k$ | $(u=-11)$ $(v=-12)$ | $(u=-12)$ $(v=-13)$ | $(u=-13)$ $(v=-14)$ | $(u=-14)$ $(v=-15)$ | $(u=15)$ $(v=-16)$ | $(u=-16)$ $(v=-17)$ | $(u=-17)$ $(v=-18)$ | $(u=-18)$ $(v=-19)$ | $(u=-19)$ $(v=-20)$ | $(u=-20)$ $(v=21)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 | 000111 |
| **2** | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 | 1000111 |
| **3** | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 | 1001111 |
| **4** | – | – | – | – | – | – | – | – | – | – |
| **5** | – | – | – | – | – | – | – | – | – | – |
| **6** | – | – | – | – | – | – | – | – | – | – |
| **7** | – | – | – | – | – | – | – | – | – | – |
| **8** | – | – | – | – | – | – | – | – | – | – |
| **9** | – | – | – | – | – | – | – | – | – | – |
| **10** | – | – | – | – | – | – | – | – | – | – |
| **11** | – | – | – | – | – | – | – | – | – | – |
| **12** | 01111 | – | – | – | – | – | – | – | – | – |
| **13** | 0000111 | 01111 | – | – | – | – | – | – | – | – |
| **14** | 0001111 | 0000111 | 01111 | – | – | – | – | – | – | – |

**Table 2.** Continued

| $k$ | $(u=-11)$ $(v=-12)$ | $(u=-12)$ $(v=-13)$ | $(u=-13)$ $(v=-14)$ | $(u=-14)$ $(v=-15)$ | $(u=15)$ $(v=-16)$ | $(u=-16)$ $(v=-17)$ | $(u=-17)$ $(v=-18)$ | $(u=-18)$ $(v=-19)$ | $(u=-19)$ $(v=-20)$ | $(u=-20)$ $(v=21)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **15** | 11000111 | 0001111 | 0000111 | 01111 | – | – | – | – | – | – |
| **16** | 11010111 | 11000111 | 0001111 | 0000111 | 01111 | – | – | – | – | – |
| **17** | – | 11010111 | 11000111 | 0001111 | 0000111 | 01111 | – | – | – | – |
| **18** | – | – | 11010111 | 11000111 | 0001111 | 0000111 | 01111 | – | – | – |
| **19** | – | – | – | 11010111 | 11000111 | 0001111 | 0000111 | 01111 | – | – |
| **20** | – | – | – | – | 11010111 | 11000111 | 0001111 | 0000111 | 01111 | – |
| **21** | – | – | – | – | – | 11010111 | 11000111 | 0001111 | 0000111 | 01111 |
| **22** | – | – | – | – | – | – | 11010111 | 11000111 | 0001111 | 0000111 |
| **23** | – | – | – | – | – | – | – | 11010111 | 11000111 | 0001111 |
| **24** | 00111 | – | – | – | – | – | – | – | 11010111 | 11000111 |
| **25** | 001111 | – | – | – | – | – | – | – | – | 11010111 |
| **26** | 01000111 | 00111 | – | – | – | – | – | – | – | – |
| **27** | 10000111 | 001111 | – | – | – | – | – | – | – | – |
| **28** | 10010111 | 01000111 | 00111 | – | – | – | – | – | – | – |
| **29** | 110000111 | 10000111 | 001111 | – | – | – | – | – | – | – |
| **30** | 110100111 | 10010111 | 01000111 | 00111 | – | – | – | – | – | – |
| **31** | – | 110000111 | 10000111 | 001111 | – | – | – | – | – | – |
| **32** | – | 110100111 | 10010111 | 01000111 | 00111 | – | – | – | – | – |
| **33** | – | – | 110000111 | 10000111 | 001111 | – | – | – | – | – |
| **34** | – | – | 110100111 | 10010111 | 01000111 | 00111 | – | – | – | – |
| **35** | – | – | – | 110000111 | 10000111 | 001111 | – | – | – | – |
| **36** | – | – | – | 110100111 | 10010111 | 01000111 | 00111 | – | – | – |
| **37** | 0010111 | – | – | – | 110000111 | 10000111 | 001111 | – | – | – |
| **38** | 00000111 | – | – | – | 110100111 | 10010111 | 01000111 | 00111 | – | – |
| **39** | 00010111 | – | – | – | – | 110000111 | 10000111 | 001111 | – | – |
| **40** | 010000111 | 0010111 | – | – | – | 110100111 | 10010111 | 01000111 | 00111 | – |
| **41** | 100000111 | 00000111 | – | – | – | – | 110000111 | 10000111 | 001111 | – |
| **42** | 100100111 | 00010111 | – | – | – | – | 110100111 | 10010111 | 01000111 | 00111 |
| **43** | 110110111 | 010000111 | 0010111 | – | – | – | – | 110000111 | 10000111 | 001111 |
| **44** | – | 100000111 | 00000111 | – | – | – | – | 110100111 | 10010111 | 01000111 |
| **45** | – | 100100111 | 00010111 | – | – | – | – | – | 110000111 | 10000111 |
| **46** | – | 110110111 | 010000111 | 0010111 | – | – | – | – | 110100111 | 10010111 |
| **47** | – | – | 100000111 | 00000111 | – | – | – | – | – | 110000111 |
| **48** | – | – | 100100111 | 00010111 | – | – | – | – | – | 110100111 |
| **49** | – | – | 110110111 | 010000111 | 0010111 | – | – | – | – | – |
| **50** | 01100111 | – | – | 100000111 | 00000111 | – | – | – | – | – |
| **51** | 00001111 | – | – | 100100111 | 00010111 | – | – | – | – | – |
| **52** | 000000111 | – | – | 110110111 | 010000111 | 0010111 | – | – | – | – |
| **53** | 000100111 | – | – | – | 100000111 | 00000111 | – | – | – | – |
| **54** | 100010111 | 01100111 | – | – | 100100111 | 00010111 | – | – | – | – |
| **55** | 100110111 | 00001111 | – | – | 110110111 | 010000111 | 0010111 | – | – | – |
| **56** | – | 000000111 | – | – | – | 100000111 | 00000111 | – | – | – |
| **57** | – | 000100111 | – | – | – | 100100111 | 00010111 | – | – | – |
| **58** | – | 100010111 | 01100111 | – | – | 110110111 | 010000111 | 0010111 | – | – |
| **59** | – | 100110111 | 00001111 | – | – | – | 100000111 | 00000111 | – | – |
| **60** | – | – | 000000111 | – | – | – | 100100111 | 00010111 | – | – |
| **61** | – | – | 000100111 | – | – | – | 110110111 | 010000111 | 0010111 | – |

**Table 2.** Continued

| $k$ | $(u=-11)$ $(v=-12)$ | $(u=-12)$ $(v=-13)$ | $(u=-13)$ $(v=-14)$ | $(u=-14)$ $(v=-15)$ | $(u=15)$ $(v=-16)$ | $(u=-16)$ $(v=-17)$ | $(u=-17)$ $(v=-18)$ | $(u=-18)$ $(v=-19)$ | $(u=-19)$ $(v=-20)$ | $(u=-20)$ $(v=21)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| **62** | 00100111 | – | 100010111 | 01100111 | – | – | – | 100000111 | 00000111 | – |
| **63** | 00110111 | – | 100110111 | 00001111 | – | – | – | 100100111 | 00010111 | – |
| **64** | 011000111 | – | – | 000000111 | – | – | – | 110110111 | 010000111 | 0010111 |
| **65** | 000010111 | – | – | 000100111 | – | – | – | – | 100000111 | 00000111 |
| **66** | 000110111 | – | – | 100010111 | 01100111 | – | – | – | 100100111 | 00010111 |
| **67** | 110001111 | 00100111 | – | 100110111 | 00001111 | – | – | – | 110110111 | 010000111 |
| **68** | 110101111 | 00110111 | – | – | 000000111 | – | – | – | – | 100000111 |
| **69** | – | 011000111 | – | – | 000100111 | – | – | – | – | 100100111 |
| **70** | – | 000010111 | – | – | 100010111 | 01100111 | – | – | – | 110110111 |
| **71** | – | 000110111 | – | – | 100110111 | 00001111 | – | – | – | – |
| **72** | – | 110001111 | 00100111 | – | – | 000000111 | – | – | – | – |
| **73** | – | 110101111 | 00110111 | – | – | 000100111 | – | – | – | – |
| **74** | – | – | 011000111 | – | – | 100010111 | 01100111 | – | – | – |
| **75** | 00101111 | – | 000010111 | – | – | 100110111 | 00001111 | – | – | – |
| **76** | 001000111 | – | 000110111 | – | – | – | 000000111 | – | – | – |
| **77** | 011010111 | – | 110001111 | 00100111 | – | – | 000100111 | – | – | – |
| **78** | 101010111 | – | 110101111 | 00110111 | – | – | 100010111 | 01100111 | – | – |
| **79** | 100001111 | – | – | 011000111 | – | – | 100110111 | 00001111 | – | – |
| **80** | 100101111 | – | – | 000010111 | – | – | – | 000000111 | – | – |
| **81** | 1101000111 | 00101111 | – | 000110111 | – | – | – | 000100111 | – | – |
| **82** | – | 001000111 | – | 110001111 | 00100111 | – | – | 100010111 | 01100111 | – |
| **83** | – | 001100111 | – | 110101111 | 00110111 | – | – | 100110111 | 00001111 | – |
| **84** | – | 101010111 | – | – | 011000111 | – | – | – | 000000111 | – |
| **85** | – | 100001111 | – | – | 000010111 | – | – | – | 000100111 | – |
| **86** | – | 100101111 | – | – | 000110111 | – | – | – | 100010111 | 01100111 |
| **87** | – | 1101000111 | 00101111 | – | 110001111 | 00100111 | – | – | 100110111 | 00001111 |
| **88** | – | – | 001000111 | – | 110101111 | 00110111 | – | – | – | 000000111 |
| **89** | 001010111 | – | 001100111 | – | – | 011000111 | – | – | – | 000100111 |
| **90** | 000001111 | – | 101010111 | – | – | 000010111 | – | – | – | 100010111 |
| **91** | 000101111 | – | 100001111 | – | – | 000110111 | – | – | – | 100110111 |
| **92** | 0101000111 | – | 100101111 | – | – | 110001111 | 00100111 | – | – | – |
| **93** | 1001000111 | – | 1101000111 | 00101111 | – | 110101111 | 00110111 | – | – | – |
| **94** | 1101100111 | – | – | 001000111 | – | – | 011000111 | – | – | – |
| **95** | – | – | – | 001100111 | – | – | 000010111 | – | – | – |
| **96** | – | 001010111 | – | 101010111 | – | – | 000110111 | – | – | – |
| **97** | – | 000001111 | – | 100001111 | – | – | 110001111 | 00100111 | – | – |
| 98 | – | 0100000111 | – | 100101111 | – | – | 110101111 | 00110111 | – | – |
| 99 | – | 1000000111 | – | 1101000111 | 00101111 | – | – | 011000111 | – | – |
| 100 | – | 1001000111 | – | – | 001000111 | – | – | 000010111 | – | – |

In this section, we have also obtained the following theorem, which is the relation between the third-order bivariate variant of the Fibonacci sequence and the Tribonacci sequence.

**Theorem 3.1.2.** Let $VF^{(3)}_{(u,v)}(k)$ is the third-order bivariate variant of the Fibonacci sequence and $T(k)$ is the Tribonacci sequence. In this case, for integers $k \geq 1$,

$$VF^{(3)}_{(u,v)}(k) = T(k-2) - u\big(T(k-4) + T(k-5)\big) - vT(k-5)$$

PROOF. We have the following set from the third-order bivariate variant of the Fibonacci sequence.

$$VF^{(3)}_{(u,v)}(k) = \{u, v, 1-u-v, 1, 2-u, 4-2u-v, 7-3u-v, 13-6u-2v, 24-11u-4v, \cdots\}$$

$$VF^{(3)}_{(u,v)}(1) = u = 0 - u\big(0+(-1)\big) - v.0 = T(-1) - u.\big(T(-3)+T(-4)\big) - vT(-4)$$

$$VF^{(3)}_{(u,v)}(2) = v = 0 - u\big((-1)+1\big) - v.(-1) = T(0) - u.\big(T(-2)+T(-3)\big) - vT(-3)$$

$$VF^{(3)}_{(u,v)}(3) = 1-u-v = 1 - u(0+1) - v.1 = T(1) - u.\big(T(-1)+T(-2)\big) - vT(-2)$$

The results are correct for $k = 1, k = 2$, and $k = 3$, as seen above. Suppose that the results are correct for $k = 1,2 \dots, n$. In that case,

$$VF^{(3)}_{(u,v)}(n-2) = T(n-4) - u\big(T(n-6)+T(n-7)\big) - vT(n-7)$$

$$VF^{(3)}_{(u,v)}(n-1) = T(n-3) - u\big(T(n-5)+T(n-6)\big) - vT(n-6)$$

$$VF^{(3)}_{(u,v)}(n) = T(n-2) - u\big(T(n-4)+T(n-5)\big) - vT(n-5)$$

Let us show that this equation is true for $n+1$.

$$\begin{aligned} VF^{(3)}_{(u,v)}(n+1) &= T(n-4) - u\big(T(n-6)+T(n-7)\big) - vT(n-7) + T(n-3) - u(T(n-5)+T(n-6)) \\ &\quad -vT(n-6) + T(n-2) - u(T(n-4)+T(n-5) - vT(n-5) \\ &= T(n-4) + T(n-3) + T(n-2) - u(T(n-4) + T(n-5) + T(n-5) \\ &\quad +T(n-6) + T(n-6) + T(n-7)) - v(T(n-7) + T(n-6) + T(n-5)) \\ &= T(n-1) - u\big(T(n-3)+T(n-4)\big) - vT(n-4) \\ &= T(n+1-2) - u\big(T(n+1-4)+T(n+1-5)\big) - vT(n+1-5) \end{aligned}$$

Therefore, by the induction, $VF^{(3)}_{(u,v)}(k) = T(k-2) - u\big(T(k-4)+T(k-5)\big) - vT(k-5), k \geq 1$.

### 3.2. The Second-Order Bivariate Variant of Narayana Sequences and Codes

This section describes a new variation depends on two negative variables of the Narayana sequence as follows:

**Definition 3.2.1.** The second-order bivariate variant of the Narayana sequence $VN^{(2)}_{(u,v)}(k)$ is described the sequence $\{u, v, s, u+s, v+u+s, v+u+2s, v+2u+3s, \dots\}$ where $s = 1-u-v$, with initial conditions $VN^{(2)}_{(u,v)}(1) = u$; $VN^{(2)}_{(u,v)}(2) = v$; $VN^{(2)}_{(u,v)}(3) = 1-u-v$; and for $k \geq 4$, $VN^{(2)}_{(u,v)}(k) = VN^{(2)}_{(u,v)}(k-1) + VN^{(2)}_{(u,v)}(k-3)$.

Here, if $u = -2$ and $v = -3$, then the second-order bivariate variant of the Narayana sequence is $\{-2, -3, 6, 4, 1, 7, 11, 12, 19, \dots\}$. It is obviously seen that different sequences will be obtained for different values of $u$ and $v$.

With these bivariate variant of Narayana sequences, we may compose a new universal source code, which we have named the bivariate variant of the Narayana code. To make, it can be used the same rule used to generate the standard Fibonacci code as in the second and the third order bivariate variant of Fibonacci code.

Moreover, as in the variations of Fibonacci sequences, the bivariate variant of Narayana sequences has more than one Zeckendorf representation of any integer, too. For example, the second order bivariate variant of

Narayana code of 17 would be 1000000011 or 00100011 for $VN^{(2)}_{(-2,-3)}(k) = \{-2,-3,6,4,1,7,11,12,19,\dots\}$. Because

$$17 = -2 + 19 = VN^{(2)}_{(-2,-3)}(1) + VN^{(2)}_{(-2,-3)}(9) = 6 + 11 = VN^{(2)}_{(-2,-3)}(3) + VN^{(2)}_{(-2,-3)}(7)$$

This study has also investigated for which values $u$ and $v$ the second-order bivariate variant of Narayana codes precisely exist or for which $k$ positive integers they don't exist. For instance, we have obtained that there is no bivariate variant of the Narayana code of 2 for $VN^{(2)}_{(-1,-3)}(k) = \{-1,-3,5,4,1,6,10,11,17\}$.

Das and Sinha have obtained the second-order variant of Narayana codes $VN^{(2)}_{u}(k)$ or undetectable values of the positive integer $k$ for $k = 1,2,\dots,50$ and for $u = -1,-2,\dots,-20$ in [13]. This section has obtained the second-order bivariate variant of Narayana codes $VN^{(2)}_{(u,v)}(k)$ or undetectable values (–) of the positive integer $k$ for $k = 1,2,\dots,100$ and for $u = -1,-2,\dots,-20$ and $v = -2,-3,\dots,-21$ ($u$ and $v$ are consecutive, $v < u$) Tables 3 and 4. From Tables 3 and 4, we got the following results for the second-order bivariate variant of Narayana codes.

*i.* For the positive integers $k = 1$, the second-order bivariate variant of the Narayana code $VN^{(2)}_{(u,v)}(k)$ exactly exists for $u = -1,-2,\dots,-20$ and $v = -2,-3,\dots,-21$ ($u$ and $v$ are consecutive, $v < u$).

*ii.* For $1 \le k \le 100$, there are at most $j$ consecutive undetectable (--) values in the second-order bivariate variant of Narayana code in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \le j \le 19$.

*iii.* For $1 \le k \le 100$, as long as $j$ raises, the detectable of Narayana code is reduced in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \le j \le 19$.

**Table 3.** The second order bivariate variant of Narayana codes ($u$ and $v$ are consecutive and $v < u$)

| $k$ | $(u=-1)$ $(v=-2)$ | $(u=-2)$ $(v=-3)$ | $(u=-3)$ $(v=-4)$ | $(u=-4)$ $(v=-5)$ | $(u=5)$ $(v=-6)$ | $(u=-6)$ $(v=-7)$ | $(u=-7)$ $(v=-8)$ | $(u=-8)$ $(v=-9)$ | $(u=-9)$ $(v=-10)$ | $(u=-10)$ $(v=-11)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 |
| 2 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 |
| 3 | 00011 | – | – | – | – | – | – | – | – | – |
| 4 | 0011 | 00011 | – | – | – | – | – | – | – | – |
| 5 | 0000011 | 1000011 | 00011 | – | – | – | – | – | – | – |
| 6 | 01000011 | 0011 | 1000011 | 00011 | – | – | – | – | – | – |
| 7 | 10000011 | 0000011 | – | 1000011 | 00011 | – | – | – | – | – |
| 8 | 00000011 | 01000011 | 0011 | – | 1000011 | 00011 | – | – | – | – |
| 9 | 000000011 | 100100011 | 0000011 | – | – | 1000011 | 00011 | – | – | – |
| 10 | 000010011 | 100000011 | 01000011 | 0011 | – | – | 1000011 | 00011 | – | – |
| 11 | 100100011 | 00000011 | 100100011 | 0000011 | – | – | – | 1000011 | 00011 | – |
| 12 | 0100000011 | 000000011 | 100000011 | 01000011 | 0011 | – | – | – | 1000011 | 00011 |
| 13 | 1000000011 | 000010011 | 100010011 | 1001011 | 0000011 | – | – | – | – | 1000011 |
| 14 | 0000000011 | 100100011 | 00000011 | 100000011 | 01000011 | 0011 | – | – | – | – |
| 15 | 0000100011 | 00010011 | 000000011 | 100010011 | 1001011 | 0000011 | – | – | – | – |
| 16 | 1001000011 | 0100000011 | 000010011 | – | 100000011 | 01000011 | 0011 | – | – | – |
| 17 | 0001000011 | 1000000011 | 0010011 | 00000011 | 100010011 | 1001011 | 0000011 | – | – | – |
| 18 | 0010000011 | 1000100011 | – | 000000011 | – | 100000011 | 01000011 | 0011 | – | – |
| 19 | 0000010011 | 0000000011 | 00010011 | 000010011 | – | 100010011 | 1001011 | 0000011 | – | – |
| 20 | 01000000011 | 0000100011 | 0100000011 | 100100011 | 00000011 | – | 100000011 | 01000011 | 0011 | – |
| 21 | 10000000011 | 1001000011 | 1000000011 | 0010011 | 000000011 | – | 100010011 | 1001011 | 0000011 | – |
| 22 | 00000000011 | – | 1000100011 | – | 000010011 | – | – | 100000011 | 01000011 | 0011 |
| 23 | 00001000011 | 0001000011 | 001000011 | 00010011 | 100100011 | 00000011 | – | 100010011 | 1001011 | 0000011 |

**Table 3.** Continued

| $k$ | $(u=-1)$ $(v=-2)$ | $(u=-2)$ $(v=-3)$ | $(u=-3)$ $(v=-4)$ | $(u=-4)$ $(v=-5)$ | $(u=5)$ $(v=-6)$ | $(u=-6)$ $(v=-7)$ | $(u=-7)$ $(v=-8)$ | $(u=-8)$ $(v=-9)$ | $(u=-9)$ $(v=-10)$ | $(u=-10)$ $(v=-11)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 24 | 10010000011 | 1000010011 | 0000000011 | 0100000011 | – | 000000011 | – | – | 100000011 | 01000011 |
| 25 | 00010000011 | 0010000011 | 0000100011 | 1000000011 | 0010011 | 000010011 | – | – | 100010011 | 1001011 |
| 26 | 00100000011 | 0000010011 | 1001000011 | 1000100011 | – | 100100011 | 00000011 | – | – | 100000011 |
| 27 | 00000100011 | 01000000011 | – | 00100011 | 00010011 | – | 000000011 | – | – | 100010011 |
| 28 | 01000010011 | 10000000011 | – | 001000011 | 0100000011 | – | 000010011 | – | – | – |
| 29 | 010000000011 | 10001000011 | 0001000011 | 0000000011 | 1000000011 | 0010011 | 100100011 | 00000011 | – | – |
| 30 | 100000000011 | 00000000011 | 1000010011 | 0000100011 | 1000100011 | – | – | 000000011 | – | – |
| 31 | 000000000011 | 00001000011 | – | 1001000011 | – | 00010011 | – | 000010011 | – | – |
| 32 | 000010000011 | 10010000011 | 0010000011 | – | 00100011 | 0100000011 | – | 100100011 | 00000011 | – |
| 33 | 100100000011 | – | 0000010011 | – | 00101011 | 1000000011 | 0010011 | – | 000000011 | – |
| 34 | 000100000011 | 00010000011 | 01000000011 | – | 0000000011 | 1000100011 | – | – | 000010011 | – |
| 35 | 001000000011 | 10000100011 | 10000000011 | 0001000011 | 0000100011 | – | 00010011 | – | 100100011 | 00000011 |
| 36 | 000001000011 | 00100000011 | 10001000011 | 1000010011 | 1001000011 | – | 0100000011 | – | – | 000000011 |
| 37 | 010000100011 | 00000100011 | – | – | – | 00100011 | 1000000011 | 0010011 | – | 000010011 |
| 38 | 100000100011 | 01000010011 | 00000000011 | – | – | 001000011 | 1000100011 | – | – | 100100011 |
| 39 | 000000100011 | 010000010011 | 00001000011 | 0010000011 | – | 0000000011 | – | 00010011 | – | – |
| 40 | 000000010011 | 100000000011 | 10010000011 | 0010100011 | – | 0000100011 | – | 0100000011 | – | – |
| 41 | 000010010011 | 100010000011 | – | 01000000011 | 0001000011 | 1001000011 | – | 1000000011 | 0010011 | – |
| 42 | 100100010011 | 000000000011 | – | 10000000011 | 1000010011 | – | 00100011 | 1000100011 | – | – |
| 43 | 0100000000011 | 000010000011 | 00010000011 | 10001000011 | – | – | 001000011 | – | 00010011 | – |
| 44 | 1000000000011 | 100100000011 | 10000100011 | – | – | – | 0000000011 | – | 0100000011 | – |
| 45 | 0000000000011 | 00010010011 | – | – | – | – | 0000100011 | – | 1000000011 | 0010011 |
| 46 | 0000100000011 | 000100000011 | 00100000011 | 00000000011 | 0010000011 | – | 1001000011 | – | 1000100011 | – |
| 47 | 1001000000011 | 100001000011 | 00000100011 | 00001000011 | 0010100011 | 0001000011 | – | 00100011 | – | 00010011 |
| 48 | 0001000000011 | 001000000011 | 01000010011 | 10010000011 | 01000000011 | 1000010011 | – | 00101011 | – | 0100000011 |
| 49 | 0010000000011 | 000001000011 | 01001010011 | – | 10000000011 | – | – | 0000000011 | – | 1000000011 |
| 50 | 0000010000011 | 010000100011 | 10001010011 | – | 10001000011 | – | – | 0000100011 | – | 1000100011 |
| 51 | 0100001000011 | 100000100011 | 10001001011 | – | – | – | – | 1001000011 | – | – |
| 52 | 1000001000011 | 100000010011 | 00000010011 | 00010000011 | – | – | – | – | 00100011 | – |
| 53 | 0000001000011 | 000000100011 | 000000000011 | 10000100011 | – | 0010000011 | 0001000011 | – | 001000011 | – |
| 54 | 0000000100011 | 000000010011 | 000010000011 | – | 00000000011 | 0010100011 | 1000010011 | – | 0000000011 | – |
| 55 | 0000100100011 | 000010010011 | 100100000011 | – | 00001000011 | 01000000011 | – | – | 0000100011 | – |
| 56 | 1001000100011 | 100100010011 | – | 00100000011 | 10010000011 | 10000000011 | – | – | 1001000011 | – |
| 57 | 0100000010011 | 000100100011 | – | 00000100011 | – | 10001000011 | – | – | – | 00100011 |
| 58 | 1000000010011 | 0100000000011 | 000100000011 | 01000010011 | – | – | – | – | – | 001000011 |
| 59 | 0000000010011 | 1000000000011 | 100001000011 | 010000000011 | – | – | – | 0001000011 | – | 0000000011 |
| 60 | 0000100010011 | 1000100000011 | – | 100000000011 | – | – | 0010000011 | 1000010011 | – | 0000100011 |
| 61 | 1001000010011 | 0000000000011 | 001000000011 | 100010000011 | 00010000011 | – | 0010100011 | – | – | 1001000011 |
| 62 | 0001000010011 | 0000100000011 | 000001000011 | – | 10000100011 | 00000000011 | 01000000011 | – | – | – |
| 63 | 0010000010011 | 1001000000011 | 010000100011 | 00000010011 | – | 00001000011 | 10000000011 | – | – | – |
| 64 | 0000010010011 | – | 100000100011 | 000000000011 | – | 10010000011 | 10001000011 | – | – | – |
| 65 | 01000000000011 | 0001000000011 | 100000010011 | 000010000011 | – | – | – | – | 0001000011 | – |
| 66 | 10000000000011 | 1000010000011 | 100010010011 | 100100000011 | 00100000011 | – | – | – | 1000010011 | – |
| 67 | 00000000000011 | 0010000000011 | 000000100011 | – | 00000100011 | – | – | 0010000011 | – | – |
| 68 | 00001000000011 | 0000010000011 | 000000010011 | – | 01000010011 | – | – | 0010100011 | – | – |
| 69 | 10010000000011 | 0100001000011 | 000010010011 | 00010010011 | 010000000011 | – | – | 01000000011 | – | – |
| 70 | 00010000000011 | 1000001000011 | 100100010011 | 000100000011 | 100000000011 | 00010000011 | 00000000011 | 10000000011 | – | – |

**Table 3.** Continued

| $k$ | $(u=-1)$ $(v=-2)$ | $(u=-2)$ $(v=-3)$ | $(u=-3)$ $(v=-4)$ | $(u=-4)$ $(v=-5)$ | $(u=5)$ $(v=-6)$ | $(u=-6)$ $(v=-7)$ | $(u=-7)$ $(v=-8)$ | $(u=-8)$ $(v=-9)$ | $(u=-9)$ $(v=-10)$ | $(u=-10)$ $(v=-11)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 71 | 00100000000011 | 1000000100011 | – | 100001000011 | 100010000011 | 10000100011 | 00001000011 | 10001000011 | – | 0001000011 |
| 72 | 00000100000011 | 0000001000011 | 000100100011 | – | – | – | 10010000011 | – | – | 1000010011 |
| 73 | 01000010000011 | 0000000100011 | 0100000000011 | – | – | – | – | – | – | – |
| 74 | 10000010000011 | 0000100100011 | 1000000000011 | 001000000011 | 00000010011 | – | – | – | 0010000011 | – |
| 75 | 00000010000011 | 1001000100011 | 1000100000011 | 000001000011 | 000000000011 | – | – | – | 0010100011 | – |
| 76 | 00000001000011 | 0001001000011 | – | 010000100011 | 000010000011 | 00100000011 | – | – | 01000000011 | – |
| 77 | 00001001000011 | 0001000100011 | 0000000000011 | 100000100011 | 100100000011 | 00000100011 | – | – | 10000000011 | – |
| 78 | 10010001000011 | 1000000010011 | 0000100000011 | 100000010011 | – | 01000010011 | – | 00000000011 | 10001000011 | – |
| 79 | 01000000100011 | 1000100010011 | 1001000000011 | 100010010011 | 00100100011 | 010000000011 | 00010000011 | 00001000011 | – | – |
| 80 | 10000000100011 | 0000000010011 | – | – | – | 100000000011 | 10000100011 | 10010000011 | – | – |
| 81 | 00000000100011 | 0000100010011 | – | 000000100011 | 00010010011 | 100010000011 | – | – | – | 0010000011 |
| 82 | 00001000100011 | 1001000010011 | 0001000000011 | 000000010011 | 000100000011 | – | – | – | – | 0010100011 |
| 83 | 10010000100011 | – | 1000010000011 | 000010010011 | – | – | – | – | – | 01000000011 |
| 84 | 00010000100011 | 0001000010011 | – | 100100010011 | – | – | – | – | – | 10000000011 |
| 85 | 00100000100011 | 1000010010011 | 0010000000011 | 001001000011 | – | 00000010011 | – | – | – | 10001000011 |
| 86 | 00000100100011 | 0010000010011 | 0000010000011 | – | – | 000000000011 | 00100000011 | – | 00000000011 | – |
| 87 | 01000000010011 | 0000010010011 | 0100001000011 | 000100100011 | 001000000011 | 000010000011 | 00000100011 | – | 00001000011 | – |
| 88 | 10000000010011 | 01000000000011 | 1000001000011 | 0100000000011 | 000001000011 | 100100000011 | 01000010011 | 00010000011 | 10010000011 | – |
| 89 | 00000000010011 | 10000000000011 | 1000000100011 | 1000000000011 | 010000100011 | – | 010000000011 | 10000100011 | – | – |
| 90 | 00001000010011 | 10001000000011 | 1000100100011 | 1000100000011 | 100000100011 | – | 100000000011 | – | – | – |
| 91 | 10010000010011 | 00000000000011 | 0000001000011 | – | 100010100011 | – | 100010000011 | – | – | – |
| 92 | 00010000010011 | 00001000000011 | 0000000100011 | – | 100010010011 | – | – | – | – | – |
| 93 | 00100000010011 | 10010000000011 | 0000100100011 | 0000000000011 | – | – | – | – | – | – |
| 94 | 00000100010011 | – | 1001000100011 | 0000100000011 | – | 000100000011 | – | – | – | 00000000011 |
| 95 | 01000010010011 | 00010000000011 | – | 1001000000011 | 000000100011 | 100001000011 | – | – | 0010010011 | 00001000011 |
| 96 | 010000000000011 | 10000100000011 | 0001001000011 | – | 000000010011 | – | 00000010011 | 00100000011 | – | 10010000011 |
| 97 | 100000000000011 | 00100000000011 | 0100000010011 | – | 000010010011 | – | 000000000011 | 00000100011 | 00010000011 | – |
| 98 | 000000000000011 | 00000100000011 | 0100100010011 | – | – | – | 000010000011 | 01000010011 | 10000100011 | – |
| 99 | 000010000000011 | 01000010000011 | 1000100010011 | 0001000000011 | – | – | 100100000011 | 010000000011 | – | – |
| 100 | 100100000000011 | 10000010000011 | 0010101000011 | 1000010000011 | 001001000011 | 001000000011 | – | 100000000011 | – | – |

**Table 4.** The second order bivariate variant of Narayana codes ($u$ and $v$ are consecutive and $v < u$)

| $k$ | $(u=-11)$ $(v=-12)$ | $(u=-12)$ $(v=-13)$ | $(u=-13)$ $(v=-14)$ | $(u=-14)$ $(v=-15)$ | $(u=15)$ $(v=-16)$ | $(u=-16)$ $(v=-17)$ | $(u=-17)$ $(v=-18)$ | $(u=-18)$ $(v=-19)$ | $(u=-19)$ $(v=-20)$ | $(u=-20)$ $(v=21)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 | 000011 |
| 2 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 | 10011 |
| 3 | – | – | – | – | – | – | – | – | – | – |
| 4 | – | – | – | – | – | – | – | – | – | – |
| 5 | – | – | – | – | – | – | – | – | – | – |
| 6 | – | – | – | – | – | – | – | – | – | – |
| 7 | – | – | – | – | – | – | – | – | – | – |
| 8 | – | – | – | – | – | – | – | – | – | – |
| 9 | – | – | – | – | – | – | – | – | – | – |
| 10 | – | – | – | – | – | – | – | – | – | – |
| 11 | – | – | – | – | – | – | – | – | – | – |

**Table 4.** Continued

| $k$ | $(u=-11)$ $(v=-12)$ | $(u=-12)$ $(v=-13)$ | $(u=-13)$ $(v=-14)$ | $(u=-14)$ $(v=-15)$ | $(u=15)$ $(v=-16)$ | $(u=-16)$ $(v=-17)$ | $(u=-17)$ $(v=-18)$ | $(u=-18)$ $(v=-19)$ | $(u=-19)$ $(v=-20)$ | $(u=-20)$ $(v=21)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 12 | – | – | – | – | – | – | – | – | – | – |
| 13 | 00011 | – | – | – | – | – | – | – | – | – |
| 14 | 1000011 | 00011 | – | – | – | – | – | – | – | – |
| 15 | – | 1000011 | 00011 | – | – | – | – | – | – | – |
| 16 | – | – | 1000011 | 00011 | – | – | – | – | – | – |
| 17 | – | – | – | 1000011 | 00011 | – | – | – | – | – |
| 18 | – | – | – | – | 1000011 | 00011 | – | – | – | – |
| 19 | – | – | – | – | – | 1000011 | 00011 | – | – | – |
| 20 | – | – | – | – | – | – | 1000011 | 00011 | – | – |
| 21 | – | – | – | – | – | – | – | 1000011 | 00011 | – |
| 22 | – | – | – | – | – | – | – | – | 1000011 | 00011 |
| 23 | – | – | – | – | – | – | – | – | – | 1000011 |
| 24 | 0011 | – | – | – | – | – | – | – | – | – |
| 25 | 0000011 | – | – | – | – | – | – | – | – | – |
| 26 | 01000011 | 0011 | – | – | – | – | – | – | – | – |
| 27 | 10000011 | 0000011 | – | – | – | – | – | – | – | – |
| 28 | 100000011 | 01000011 | 0011 | – | – | – | – | – | – | – |
| 29 | 100010011 | 10000011 | 0000011 | – | – | – | – | – | – | – |
| 30 | – | 100000011 | 01000011 | 0011 | – | – | – | – | – | – |
| 31 | – | 100010011 | 10000011 | 0000011 | – | – | – | – | – | – |
| 32 | – | – | 100000011 | 01000011 | 0011 | – | – | – | – | – |
| 33 | – | – | 100010011 | 10000011 | 0000011 | – | – | – | – | – |
| 34 | – | – | – | 100000011 | 01000011 | 0011 | – | – | – | – |
| 35 | – | – | – | 100010011 | 10000011 | 0000011 | – | – | – | – |
| 36 | – | – | – | – | 100000011 | 01000011 | 0011 | – | – | – |
| 37 | – | – | – | – | 100010011 | 10000011 | 0000011 | – | – | – |
| 38 | 00000011 | – | – | – | – | 100000011 | 01000011 | 0011 | – | – |
| 39 | 000000011 | – | – | – | – | 100010011 | 10000011 | 0000011 | – | – |
| 40 | 000010011 | – | – | – | – | – | 100000011 | 01000011 | 0011 | – |
| 41 | 100100011 | 00000011 | – | – | – | – | 100010011 | 10000011 | 0000011 | – |
| 42 | | 000000011 | – | – | – | – | – | 100000011 | 01000011 | 0011 |
| 43 | – | 000010011 | – | – | – | – | – | 100010011 | 10000011 | 0000011 |
| 44 | – | 100100011 | 00000011 | – | – | – | – | – | 100000011 | 01000011 |
| 45 | – | – | 000000011 | – | – | – | – | – | 100010011 | 10000011 |
| 46 | – | – | 000010011 | – | – | – | – | – | – | 100000011 |
| 47 | – | – | 100100011 | 00000011 | – | – | – | – | – | 100010011 |
| 48 | – | – | – | 000000011 | – | – | – | – | – | – |
| 49 | 0010011 | – | – | 000010011 | – | – | – | – | – | – |
| 50 | – | – | – | 100100011 | 00000011 | – | – | – | – | – |
| 51 | 0100000011 | – | – | – | 000010011 | – | – | – | – | – |
| 52 | 1000000011 | 0010011 | – | – | 100100011 | 00000011 | – | – | – | – |
| 53 | 1000100011 | – | – | – | – | 000000011 | – | – | – | – |
| 54 | – | 00010011 | – | – | – | 000010011 | – | – | – | – |
| 55 | – | 0100000011 | – | – | – | 100100011 | 00000011 | – | – | – |
| 56 | – | 1000000011 | 0010011 | – | – | – | 000000011 | – | – | – |
| 57 | – | 1000100011 | – | – | – | – | 000010011 | – | – | – |
| 58 | – | – | 00010011 | – | – | – | 100100011 | 00000011 | – | – |

**Table 4.** Continued

| $k$ | $(u=-11)$ $(v=-12)$ | $(u=-12)$ $(v=-13)$ | $(u=-13)$ $(v=-14)$ | $(u=-14)$ $(v=-15)$ | $(u=15)$ $(v=-16)$ | $(u=-16)$ $(v=-17)$ | $(u=-17)$ $(v=-18)$ | $(u=-18)$ $(v=-19)$ | $(u=-19)$ $(v=-20)$ | $(u=-20)$ $(v=21)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 59 | – | – | 0100000011 | – | – | – | – | 000000011 | – | – |
| 60 | – | – | 1000000011 | 0010011 | – | – | – | 000010011 | – | – |
| 61 | 00100011 | – | 1000100011 | – | – | – | – | 100100011 | 00000011 | – |
| 62 | 001000011 | – | – | 00010011 | – | – | – | – | 000000011 | – |
| 63 | 0000000011 | – | – | 0100000011 | – | – | – | – | 000010011 | – |
| 64 | 0000100011 | – | – | 1000000011 | 0010011 | – | – | – | 100100011 | 00000011 |
| 65 | 1001000011 | – | – | 1000100011 | – | – | – | – | – | 000000011 |
| 66 | – | 00100011 | – | – | 00010011 | – | – | – | – | 000010011 |
| 67 | – | 001000011 | – | – | 0100000011 | – | – | – | – | 100100011 |
| 68 | – | 0000000011 | – | – | 1000000011 | 0010011 | – | – | – | – |
| 69 | – | 0000100011 | – | – | 1000100011 | – | – | – | – | – |
| 70 | – | 1001000011 | – | – | – | 00010011 | – | – | – | – |
| 71 | – | – | 00100011 | – | – | 0100000011 | – | – | – | – |
| 72 | – | – | 001000011 | – | – | 1000000011 | 0010011 | – | – | – |
| 73 | – | – | 0000000011 | – | – | 1000100011 | – | – | – | – |
| 74 | – | – | 0000100011 | – | – | – | 00010011 | – | – | – |
| 75 | – | – | 1001000011 | – | – | – | 0100000011 | – | – | – |
| 76 | – | – | 1001000011 | – | – | – | 0100000011 | – | – | – |
| 77 | 0001000011 | – | – | 00100011 | – | – | 1000000011 | 0010011 | – | – |
| 78 | – | – | – | 001000011 | – | – | 1000100011 | – | – | – |
| 79 | – | – | – | 0000000011 | – | – | – | 00010011 | – | – |
| 80 | – | – | – | 0000100011 | – | – | – | 0100000011 | – | – |
| 81 | – | – | – | 1001000011 | – | – | – | 1000000011 | 0010011 | – |
| 82 | – | – | – | – | 00100011 | – | – | 1000100011 | – | – |
| 83 | – | 0001000011 | – | – | 001000011 | – | – | – | 00010011 | – |
| 84 | – | – | – | – | 0000000011 | – | – | – | 0100000011 | – |
| 85 | – | – | – | – | 0000100011 | – | – | – | 1000000011 | 0010011 |
| 86 | – | – | – | – | 1001000011 | – | – | – | 1000100011 | – |
| 87 | – | – | – | – | – | 00100011 | – | – | – | 00010011 |
| 88 | 0010000011 | – | – | – | – | 001000011 | – | – | – | 0100000011 |
| 89 | 0000010011 | – | 0001000011 | – | – | 0000000011 | – | – | – | 1000000011 |
| 90 | 01000000011 | – | – | – | – | 0000100011 | – | – | – | 1000100011 |
| 91 | 10000000011 | – | – | – | – | 1001000011 | – | – | – | – |
| 92 | 10001000011 | – | – | – | – | – | 00100011 | – | – | – |
| 93 | – | – | – | – | – | – | 001000011 | – | – | – |
| 94 | – | – | – | – | – | – | 0000000011 | – | – | – |
| 95 | – | 0010000011 | – | 0001000011 | – | – | 0000100011 | – | – | – |
| 96 | – | 0000010011 | – | – | – | – | 1001000011 | – | – | – |

**Table 4.** Continued

| $k$ | $(u=-11)$ $(v=-12)$ | $(u=-12)$ $(v=-13)$ | $(u=-13)$ $(v=-14)$ | $(u=-14)$ $(v=-15)$ | $(u=15)$ $(v=-16)$ | $(u=-16)$ $(v=-17)$ | $(u=-17)$ $(v=-18)$ | $(u=-18)$ $(v=-19)$ | $(u=-19)$ $(v=-20)$ | $(u=-20)$ $(v=21)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 97 | – | 01000000011 | – | – | – | – | – | 00100011 | – | – |
| 98 | – | 10000000011 | – | – | – | – | – | 001000011 | – | – |
| 99 | – | 10001000011 | – | – | – | – | – | 0000000011 | – | – |
| 100 | – | – | – | – | – | – | – | 0000100011 | – | – |

Moreover, this section obtains the following theorem, which is the relation between the second-order bivariate variant of the Narayana sequence and the Narayana sequence.

**Theorem 3.2.3.** Let $VN_{(u,v)}^{(2)}(k)$ is the second-order bivariate variant of the Narayana sequence and $N(k)$ is the Narayana sequence. In this case, for $k \geq 1$,

$$VN_{(u,v)}^{(2)}(k) = N(k-3) - uN(k-6) - v\big(N(k-6) + N(k-7)\big)$$

where $k$ is an integer.

Proof. We have the following set from the second-order bivariant variant of the Narayana sequence.

$$VN_{(u,v)}^{(2)}(k) = \{u, v, 1-u-v, 1-v, 1, 2-u-v, 3-u-2v, 4-u-2v, 6-2u-3v, \ldots\}$$

$$VN_{(u,v)}^{(2)}(1) = u = 0 - u(-1) - v\big((-1)+1\big) = N(-2) - uN(-5) - v(N(-5) + N(-6))$$

$$VN_{(u,v)}^{(2)}(2) = v = 0 - u0 - v(0-1) = N(-1) - uN(-4) - v(N(-4) + N(-5))$$

The results are correct for $k=1, k=2$, and $k=3$, as seen above. Suppose that the results are correct for $k = 1,2 \ldots, n$. In that case,

$$VN_{(u,v)}^{(2)}(n-2) = N(n-5) - uN(n-8) - v(N(n-8) + N(n-9))$$

$$VN_{(u,v)}^{(2)}(n-1) = N(n-4) - uN(n-7) - v(N(n-7) + N(n-8))$$

$$VN_{(u,v)}^{(2)}(n) = N(n-3) - uN(n-6) - v(N(n-6) + N(n-7))$$

Since

$$\begin{aligned} VN_{(u,v)}^{(2)}(n+1) = {} & N(n-5) - uN(n-8) - v\big(N(n-8) + N(n-9)\big) + N(n-4) - uN(n-7) \\ & -v\big(N(n-7) + N(n-8)\big) + N(n-3) - uN(n-6) - v\big(N(n-6)\big) + N(n-7)) \\ = {} & N(n-5) + N(n-4) + N(n-3) - u\big(N(n-8) + N(n-7) + N(n-6)\big) \\ & -v\Big(\big(N(n-6) + N(n-7) + N(n-7)\big) + N(n-8) + N(n-8) + N(n-9)\Big) \\ = {} & N(n-2) - uN(n-5) - v\big(N(n-5) + N(n-6)\big) \\ = {} & N(n+1-3) - uN(n+1-6) - v\big(N(n+1-6) + N(n+1-7)\big). \end{aligned}$$

this equation is true for $n+1$. Therefore, by the induction, $VN_{(u,v)}^{(2)}(k) = N(k-3) - uN(k-6) - v\big(N(k-6) + N(k-7)\big), k \geq 1$.

### 3.3. Cryptographic Comparison of Bivariate Gopala Hemachandra and Variant Narayana Codes

Cryptography is used to make sense of incomprehensible messages [14]. What cryptography focuses on is privacy. Cryptography aims to provide secure communication between two people. The information is named plaintext [15].

Cryptography is divided into two according to a key structure. These are symmetric cryptography and asymmetric cryptography. DES and AES are examples of symmetric cryptography, while RSA and ElGamal are examples of asymmetric cryptography. Different cryptographic systems are also built, too. One of them is the system obtained using source coding. A few examples made with the system can be found in [10], [12]. Hence, new source codes of third and fourth order identified in this study can also be used in cryptographic applications. But columns with undetectable codes can't be used in cryptography. While for the third-order bivariate GH code, there are no undetectable codes in $GH^{(3)}_{(-1,-2)}(k)$, $GH^{(3)}_{(-2,-3)}(k)$, $GH^{(3)}_{(-3,-4)}(k)$ columns, for the second-order bivariate variant of the Narayana code, there are no undetectable codes in only $VN^{(2)}_{(-1,-2)}(k)$ column. In addition, for the positive integers $k = 1, 2, 3$, the third-order bivariate GH code $GH^{(3)}_{(u,v)}(k)$ precisely exists, for a bivariate variant of Narayana code, for the positive integers $k = 1$, the second order bivariate variant of Narayana code $VN^{(2)}_{(u,v)}(k)$ exactly exists. Similarly, for $1 \leq k \leq 100$, while there is at most $j$ consecutive undetectable (--) values, the third-order bivariate GH code in $GH^{(3)}_{(-(3+j),-(4+j))}(k)$ column in which $1 \leq j \leq 17$, there is at most $j$ consecutive undetectable $(--)$ values the second order bivariate variant of Narayana code in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \leq j \leq 19$. At that rate, we obtained that the third-order bivariate variant of the Fibonacci code is more valuable than the second-order bivariate variant of the Narayana code in terms of cryptography.

## 4. Conclusion

In this study, firstly, we examined Fibonacci, Tribonacci, and Narayana sequences and investigated the third-order variant of the Fibonacci sequence, and also obtained which is the relation between the third-order variant of the Fibonacci sequence $VF^{(3)}_u(k)$ and the Tribonacci sequence $T(k)$ for any integer $k \geq 1$ with a theorem. Then, we described a new variant of the Fibonacci sequence and a new variant of the Narayana sequence, depending on two negative integer variables $u$ and $v$. Moreover, we named these new sequences the third-order bivariate variant of the Fibonacci sequence and the second-order bivariate variant of the Narayana sequence, respectively. Then, we obtained a bivariate variant of the GH code and a bivariate variant of the Narayana universal code based on these bivariate variant sequences we described.

In addition, we obtained the relation between the third-order bivariate variant of the Fibonacci sequence $VF^{(3)}_{(u,v)}(k)$ and the Tribonacci sequence $T(k)$, and the second-order bivariate variant of the Narayana sequence $VN^{(2)}_{(u,v)}(k)$ and the Narayana sequence $N(k)$ for any integer $k \geq 1$.

Afterwards, we showed in tables $VF^{(3)}_{(u,v)}(k)$, $VN^{(2)}_{(u,v)}(k)$ we have defined for $1 \leq k \leq 100$ and $u = -1, -2, \dots, -20$ and $v = -2, -3, \dots, -21$ ($u$ and $v$ are consecutive, $v < u$). We got some important results from the tables for these bivariate variants of universal codes. For $k = 1, 2, 3$, the third-order bivariate GH code $GH^{(3)}_{(u,v)}(k)$ exactly exists. There are at most $j$ consecutive undetectable (-) values in the third-order bivariate GH code in $GH^{(3)}_{(-(3+j),-(4+j))}(k)$ column in which $1 \leq j \leq 17$ and for $1 \leq k \leq 100$. As long as $j$ raises, the detectable of GH code is reduced in $GH^{(3)}_{(-(3+j),-(4+j))}(k)$ column in which $1 \leq j \leq 17$ and for $1 \leq k \leq 100$. For $k = 1$, the second-order bivariate variant of the Narayana code $VN^{(2)}_{(u,v)}(k)$ exactly exists. For

$1 \leq k \leq 100$, there are at most $j$ consecutive undetectable (--) values in the second-order bivariate variant of Narayana code in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \leq j \leq 19$. And for $1 \leq k \leq 100$, as long as $j$ raises, the detectable of Narayana code is reduced in $VN^{(2)}_{(-(1+j),-(2+j))}(k)$ column in which $1 \leq j \leq 19$.

Finally, we compared the two universal codes defined here regarding the existence of these codes for positive integers $k$ and the use of these sequences in cryptography. And, according to the above results, in cryptographic applications, columns $GH^{(3)}_{(-1,-2)}(k)$, $GH^{(3)}_{(-2,-3)}(k)$, $GH^{(3)}_{(-3,-4)}(k)$ can be precisely used for a bivariate GH code for $m = 3$, the column $VN^{(2)}_{(-1,-2)}(k)$ can be precisely used for a bivariate variant of the Narayana code for $m = 2$. And hence, we acquired the third-order bivariate variant of the Fibonacci code is more valuable than the second-order bivariate variant of the Narayana code.

## Author Contributions

The author read and approved the last version of the manuscript.

## Conflict of Interest

The author declares no conflict of interest.

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