Turk. J. Math. Comput. Sci.
16(1)(2024) 70–77
© MatDer
DOI : 10.47000/tjmcs.1202704

Boundedness Character of the System of Recursive Difference Equations

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Received: 11-11-2022 • Accepted: 04-05-2023

Abstract. In this paper, we take into consideration the boundedness character of positive solutions of the difference system

$$x_n = \alpha + \prod_{i=1}^{k} y_{n-i}^{a_i},$$
$$y_n = \beta + \prod_{i=1}^{k} x_{n-i}^{b_i},$$

where $a_i, b_i \in \mathbb{R}$, $i = \overline{1,k}$, $a_k \neq 0$, $b_k \neq 0$ and α and β are nonnegative real numbers.

2010 AMS Classification: 39A10

Keywords: System of nonlinear difference equations ,bounded solutions, positive solutions.

1. Introduction

A number of problems in many branches of science like biology, economics, control theory, etc. can be modelled and solved by using discrete conceptions. In particular, difference operators are great mathematical tools for this aim. In this context, studying long-term behavior and boundedness character of difference operators is a crucial research area since this helps investigation of stability and periodicity of solutions of difference equations and systems [1–33]. In recent years, various authors have studied equations and systems with non-integer powers of their variables [10–12, 15, 19, 20, 23–27].

The boundedness character of positive solutions of the recursive sequence

$$x_n = \alpha + \prod_{j=1}^{k} x_{n-j}^{a_j}, \; n \in \mathbb{N},$$

where $\alpha > 0$, $a_j \in \mathbb{R}$, $j \in \{1, ..., k\}$ and $a_k \neq 0$ was considered in [24]. Based on the conceptions in the studies [24], we construct a recursive system of difference equations as follows:

$$\begin{aligned} x_n &= \alpha + \prod_{i=1}^{k} y_{n-i}^{a_i}, \\ y_n &= \beta + \prod_{i=1}^{k} x_{n-i}^{b_i}, \end{aligned} \tag{1.1}$$

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where $a_i, b_i \in \mathbb{R}$, $i = \overline{1,k}$, $a_k \neq 0$, $b_k \neq 0$, $\alpha > 0, \beta > 0$ and we expand the results to the system.

2. Boundedness Character of the System

We will investigate the boundedness character of the system (1.1) for three cases based on α and β, that is, $\alpha = \beta = 0$, $\alpha = 0$ or $\beta = 0$, and $\alpha \neq 0$ and $\beta \neq 0$.

Let us begin with the first case. If $\alpha = \beta = 0$, then (1.1) turns into the system

$$\begin{aligned} x_n &= \prod_{i=1}^{k} y_{n-i}^{a_i}, \\ y_n &= \prod_{i=1}^{k} x_{n-i}^{b_i}. \end{aligned} \tag{2.1}$$

From the system (2.1), we get

$$x_n = \prod_{i=1}^{k} \left(\prod_{j=1}^{k} x_{n-2j}^{b_j} \right)^{a_i}.$$

After a simple calculation, we obtain

$$x_n = \prod_{i=1}^{k} x_{n-2i}^{c_i} \tag{2.2}$$

for $c_i = b_i(a_1 + a_2 + ... + a_k)$, $\forall i = \overline{1,k}$. If we take the logarithm of both sides of the difference equation (2.2) and change the variables by $y_n = \ln x_n$, then we get

$$y_n - c_1 y_{n-2} - c_2 y_{n-4} - ... - c_k y_{n-2k} = 0, \; n \in \mathbb{N}, \tag{2.3}$$

which is a $2k$ degree of a linear difference equation with constant coefficients. Note that the associated characteristic polynomial for (2.3) is

$$P_{2k}(t) = t^{2k} - c_1 t^{2k-2} - c_2 t^{2k-4} - ... - c_k = 0. \tag{2.4}$$

As a matter of convenience, we will take $x = t^2$. So, (2.4) becomes

$$P_k(x) = x^k - c_1 x^{k-1} - c_2 x^{k-2} - ... - c_k = 0. \tag{2.5}$$

We'll investigate the boundedness character of the system (1.1) in terms of the roots of (2.5). In this paper, we assume that all characteristic roots of $P_k(x)$ belongs to the interval $(0, 1)$ which is equivalent to the the case that all characteristic roots of $P_{2k}(t)$ lies in the interval $(-1, 0) \cup (0, 1)$.

Let us introduce the following result, which tell us the solutions of (1.1) are bounded from below.

Theorem 2.1. *Let $k \in \mathbb{N} \setminus \{1\}$, and that a sequence of positive numbers $(x_n)_{n \geqslant -2k}$ satisfies the following difference inequality*

$$\prod_{j=1}^{k} x_{n-2j}^{c_j} \leq x_n, \; n \in \mathbb{N}_0, \tag{2.6}$$

where $c_j \in \mathbb{R}$, $j = \overline{1,k}$, $c_k \neq 0$, and all the zeros of polynomial (2.5) belong to the interval (0,1). Then, there is a positive number $m_{1,k}$ such that

$$x_n \geqslant m_{1,k} \qquad for\; n \geqslant -2k. \tag{2.7}$$

Proof. Assume that λ_j, $j = \overline{1,k}$ are the roots of polynomial (2.5). It is known from linear algebra that coefficients of the polynomial $P_k(x)$ can be found in terms of the basic symetric polynomials of λ_j for $j = \overline{1,k}$. That is,

$$c_j = (-1)^{j-1} \sigma_j(\lambda_1, \lambda_2, ..., \lambda_k), \; j = \overline{1,k}. \tag{2.8}$$

Let us begin the proof with the case $k = 2$. Then, from (2.8) we get

$$c_1 = \lambda_1 + \lambda_2 \text{ and } c_2 = -\lambda_1 \lambda_2. \tag{2.9}$$

Using (2.9) in (2.6), we have

$$x_{n-2}^{\lambda_1+\lambda_2} x_{n-4}^{-\lambda_1\lambda_2} \leq x_n, \quad n \in \mathbb{N}_0.$$

Since $(x_n)_{n\geqslant -4}$ is a positive sequence from the assumption, the following can be written

$$\left(\frac{x_{n-2}}{x_{n-4}^{\lambda_1}}\right)^{\lambda_2} \leq \frac{x_n}{x_{n-2}^{\lambda_1}}, \quad n \in \mathbb{N}_0. \tag{2.10}$$

Let us use the change of variables

$$y_n = \frac{x_n}{x_{n-2}^{\lambda_1}}. \tag{2.11}$$

So, from (2.10) and (2.11), we have

$$0 < y_{n-2}^{\lambda_2} \leq y_n, \quad n \in \mathbb{N}_0. \tag{2.12}$$

If the use of (2.12) is iterated, we obtain that there exists a constant d_0 such that $d_0 \leq y_n$, $n \in \mathbb{N}_0$, that is,

$$d_0 x_{n-2}^{\lambda_1} \leq x_n, \quad n \in \mathbb{N}_0. \tag{2.13}$$

Since $\lambda_1 \in (0,1)$, we can iterate the use of (2.13) and obtain that there exists an $m_{1,2} > 0$ such that $m_{1,2} \leq x_n$, $n \in \mathbb{N}_0$. So, the proof of the theorem is completed for $k = 2$.

From induction method, assume that (2.6) is satisfied for $k-1$. Let us represent this as every sequence of positive numbers $(z_n)_{n\geqslant -2k+2}$ satisfies the inequality

$$\prod_{j=1}^{k-1} z_{n-2j}^{f_j} \leq z_n, \quad n \in \mathbb{N}_0, \tag{2.14}$$

where $f_j \in \mathbb{R}$, $j = \overline{1, k-1}$, $f_{k-1} \neq 0$, and all the zeros of the polynomial

$$P_1(x) = x^{k-1} - f_1 x^{k-2} - \dots - f_{k-1},$$

lie in $(0,1)$.

From (2.8), we can write inequality (2.6) as

$$\prod_{j=1}^{k} x_{n-2j}^{(-1)^{j-1}\sigma_j(\lambda_1,\lambda_2,\dots,\lambda_k)} \leq x_n, \quad \text{for } n \in \mathbb{N}_0. \tag{2.15}$$

Since

$$P_k(\lambda) = (\lambda - \lambda_1)\left(\lambda^{k-1} - \sum_{i=1}^{k-1}(-1)^{i-1}\widetilde{\sigma}_i(\lambda_2,\dots,\lambda_k)\right),$$

for every $\lambda \in \mathbb{R}$, we have the following

$$1 - \sum_{j=1}^{k}(-1)^{j-1}\sigma_j(\lambda_1,\lambda_2,\dots,\lambda_k) = (1-\lambda_1)\left(1 - \sum_{i=1}^{k-1}(-1)^{i-1}\widetilde{\sigma}_i(\lambda_2,\dots,\lambda_k)\right), \tag{2.16}$$

where $\widetilde{\sigma}_i(s_1,\dots,s_{k-1})$, $i = \overline{1,k-1}$, are basic symetric polynomials of degree $k-1$ of variables $s_1, s_2, \dots, s_{k-1}$.

If the inequality (2.15) is divided by $x_{n-2}^{\lambda_1}$, and the change of variables (2.11) is used for $n \geqslant -2k+2$, then (2.6) becomes the next one

$$\prod_{j=1}^{k-1} y_{n-2j}^{f_j} \leq y_n, \quad n \in \mathbb{N}_0, \tag{2.17}$$

from (2.16) with

$$f_j = (-1)^{j-1}\widetilde{\sigma}_i(\lambda_2,\dots,\lambda_k), \quad j = \overline{1,k-1}.$$

It is clear that, the sequence $(y_n)_{n\geqslant -2k+2}$ in (2.17) satisfies the assumption (2.14). Hence, from the induction hypothesis, we can say that the sequence $(y_n)_{n\geqslant -2k+2}$ is bounded from below. Therefore, there exists a positive number d_1 such that $d_1 \leq y_n$, $n \geqslant -2k+2$, which is equivalent to

$$d_1 x_{n-2}^{\lambda_1} \leq x_n, \quad n \geqslant -2k+2. \tag{2.18}$$

From iterating the use of (2.18), we find that there is a positive constant $m_{1,k}$ such that (2.7) holds. The proof is completed. □

Now, we will consider the case that one of α and β is zero and the other one is nonzero. Without lost of generality, we will assume that $\beta = 0$ and $\alpha \neq 0$. In this case, we can write the system (1.1) by following:

$$x_n = \alpha + \prod_{i=1}^{k} x_{n-2i}^{c_i}. \tag{2.19}$$

From (2.19), we have

$$\prod_{i=1}^{k} x_{n-2i}^{c_i} \leq x_n \leq \alpha + \prod_{i=1}^{k} x_{n-2i}^{c_i}.$$

Now, let us introduce the following lemma that will be used later to prove that the solutions of the system (1.1) is bounded from above.

Lemma 2.2. *Let $\lambda \in (0,1)$, $b, c > 0$, and $(x_n)_{n\in\mathbb{N}_0}$ is a sequence of positive numbers satisfying*

$$x_{n+2} \leq bx_n^{\lambda} + c, \; n \in \mathbb{N}_0. \tag{2.20}$$

Then, there is a positive number M_3 such that $x_n \leq M_3$, $n \in \mathbb{N}_0$.

Proof. Let $(y_n)_{n\in\mathbb{N}_0}$ be the solution of the following difference equation

$$y_{n+2} = by_n^{\lambda} + c, \; n \in \mathbb{N}_0, \tag{2.21}$$

such that $y_1 = x_1$ and $y_0 = x_0$. Then, from (2.20), (2.21) and by induction we obtain

$$x_n \leq y_n \text{ for } n \in \mathbb{N}_0. \tag{2.22}$$

We can rewrite (2.21) in the form of

$$z_{2n+m} = bz_{2(n-1)+m}^{\lambda} + c, \; n \in \mathbb{N}_1,$$

where $m = 0, 1$. The solutions of this equation are formally similar to the solutions of

$$w_{n+1} = bw_n^{\lambda} + c, \; n \in \mathbb{N}_0,$$

which the boundedness of sequence $(w_n)_{n\in\mathbb{N}_0}$ was proved essentially in [25]. From this result and (2.22), we obtain a positive number M_3 such that $x_n \leq M_3$, $n \in \mathbb{N}_0$. □

Theorem 2.3. *Let $k \in \mathbb{N}\setminus\{1\}$, and a sequence of positive numbers $(x_n)_{n\geqslant -2k}$ holds the following difference inequalities*

$$\prod_{j=1}^{k} x_{n-2j}^{c_j} \leq x_n \leq \alpha + \prod_{j=1}^{k} x_{n-2j}^{c_j}, \quad n \in \mathbb{N}_0, \tag{2.23}$$

where $\alpha > 0$, $c_j \in \mathbb{R}$, $j = \overline{1,k}$, $c_k \neq 0$, and all the zeros of the polynomial (2.5) are in the interval (0,1). Then, there exists two positive numbers $M_{1,k}$ and $M_{2,k}$ such that

$$M_{1,k} \leq x_n \leq M_{2,k} \textit{ for } n \geqslant -2k. \tag{2.24}$$

Proof. From Theorem 2.1 existence of such $M_{1,k}$ is clear. Hence, we need to show the existance of $M_{2,k} > 0$ for the second inequality in (2.24). Let λ_j, $j = \overline{1,k}$, be the roots of (2.5). Then, (2.8) holds.

Let $k = 2$. Then, from (2.9), we can rewrite (2.23) as in the following form

$$x_{n-2}^{\lambda_1+\lambda_2} x_{n-4}^{-\lambda_1\lambda_2} \leq x_n \leq \alpha + x_{n-2}^{\lambda_1+\lambda_2} x_{n-4}^{-\lambda_1\lambda_2}, \quad n \in \mathbb{N}_0.$$

After dividing both sides by $x_{n-2}^{\lambda_1}$ in the second part of above inequality, we get

$$\frac{x_n}{x_{n-2}^{\lambda_1}} \leq \frac{\alpha}{x_{n-2}^{\lambda_1}} + \left(\frac{x_{n-2}}{x_{n-4}^{\lambda_1}}\right)^{\lambda_2}, \quad n \in \mathbb{N}_0. \tag{2.25}$$

If we use the change of variables (2.11) in (2.25), we obtain

$$y_n \leq y_{n-2}^{\lambda_2} + \frac{\alpha}{M_{1,2}^{\lambda_1}}, \quad n \in \mathbb{N}_0. \tag{2.26}$$

See that Lemma 2.2 is applicable in (2.26) since $\lambda_2 \in (0,1)$. Hence, we find that there exists a positive constant d_2 such that $y_n \le d_2$, $n \geqslant -2$, that is,

$$x_n \le d_2 x_{n-2}^{\lambda_1}, \quad n \geqslant -2.$$

By iterating the above inequality we can find that there exists a positive constant $M_{2,2}$ such that $x_n \le M_{2,2}$ for $n \geqslant -4$. Therefore, Theorem 2.3 satisfies for $k = 2$.

In able to prove the general case, we assume that (2.23) holds for $k-1$ from induction method. That is, every sequence of positive numbers $(z_n)_{n \geqslant -2k+2}$ satisfies the next inequalities

$$\prod_{j=1}^{k-1} z_{n-2j}^{f_j} \le z_n \le \widetilde{\alpha} + \prod_{j=1}^{k-1} z_{n-2j}^{f_j}, \tag{2.27}$$

where $\widetilde{\alpha} > 0$, $f_j \in \mathbb{R}$, $j = \overline{1, k-1}$, $f_{k-1} \neq 0$, and where the zeros of the polynomial

$$P_1(x) = x^{k-1} - f_1 x^{k-2} - ... - f_{k-1},$$

lies in the interval $(0,1)$.

By using (2.8), we can rewrite the inequalities in (2.23) in the following form

$$\prod_{j=1}^{k} x_{n-2j}^{(-1)^{j-1}\sigma_j(\lambda_1,\lambda_2,\ldots,\lambda_k)} \le x_n \le \alpha + \prod_{j=1}^{k} x_{n-2j}^{(-1)^{j-1}\sigma_j(\lambda_1,\lambda_2,\ldots,\lambda_k)}, \quad n \in \mathbb{N}_0. \tag{2.28}$$

Now, we will divide the inequalities in (2.28) by $x_{n-2}^{\lambda_1}$, and use the change of variables (2.11) for $n \geqslant -2k+2$. So, we get

$$\prod_{j=1}^{k-1} y_{n-2j}^{f_j} \le y_n \le \frac{\alpha}{x_{n-2}^{\lambda_1}} + \prod_{j=1}^{k-1} y_{n-2j}^{f_j}, \quad n \in \mathbb{N}_0, \tag{2.29}$$

for $f_j = (-1)^{j-1}\widetilde{\sigma}_i(\lambda_2, ..., \lambda_k)$, $\quad j = \overline{1, k-1}$. If we use the first inequality in (2.24) for (2.29), we obtain

$$\prod_{j=1}^{k-1} y_{n-2j}^{f_j} \le y_n \le \frac{\alpha}{M_{1,k}^{\lambda_1}} + \prod_{j=1}^{k-1} y_{n-2j}^{f_j}, \quad n \in \mathbb{N}_0. \tag{2.30}$$

Equation (2.30) shows that the sequence $(y_n)_{n \geqslant -2k+2}$ holds our assumption in (2.27) with $\widetilde{\alpha} = \frac{\alpha}{M_{1,k}^{\lambda_1}}$. Therefore, we have that the sequence $(y_n)_{n \geqslant -2k+2}$ is bounded from above. That is, there exists a positive constant $M_{2,k} > 0$ such that the second part of the (2.24) is satisfied. So, the proof is finished by the induction hypothesis. □

Now, we will consider the case both α and β are different from zero. Note that there is a number B which is large enough, such that

$$\prod_{i=1}^{k}\left(\beta + \prod_{j=1}^{k} x_{n-2j}^{b_j}\right)^{a_i} \le \prod_{i=1}^{k}\left(B \prod_{j=1}^{k} x_{n-2j}^{b_j}\right)^{a_i}, n \in \mathbb{N}_0.$$

Without lost of generality, in this case the system (1.1) can be rewritten as

$$\begin{aligned} x_n &= \alpha + \prod_{i=1}^{k}\left(\beta + \prod_{j=1}^{k} x_{n-2j}^{b_j}\right)^{a_i} \\ &\le \alpha + \prod_{i=1}^{k}\left(B \prod_{j=1}^{k} x_{n-2j}^{b_j}\right)^{a_i} \\ &= \alpha + B^{a_1+a_2+\ldots+a_k} \prod_{j=1}^{k} x_{n-2j}^{c_j}. \end{aligned}$$

Hence, we have $\prod\limits_{j=1}^{k} x_{n-2j}^{c_j} \le x_n \le \alpha + B_k \prod\limits_{j=1}^{k} x_{n-2j}^{c_j}$, for $n \in \mathbb{N}_0$ and $B_k = B^{a_1+a_2+\ldots+a_k}$. Since $a_j \in \mathbb{R}$ for $j = \overline{1,k}$, we can find a positive constant P such that $B_k \le P$, $\forall k \in \mathbb{N} \setminus \{1\}$. We will use the constant P in the next theorem.

Theorem 2.4. *Let $k \in \mathbb{N} \setminus \{1\}$, and a sequence of positive numbers $(x_n)_{n \geqslant -2k}$ holds the following difference inequalities*

$$\prod_{j=1}^{k} x_{n-2j}^{c_j} \leq x_n \leq \alpha + P \prod_{j=1}^{k} x_{n-2j}^{c_j}, \quad n \in \mathbb{N}_0,$$

where $\alpha > 0$, $c_j \in \mathbb{R}$, $j = \overline{1,k}$, $c_k \neq 0$, and all the zeros of the polynomial (2.5) are in the interval (0,1). Then, there exists two positive numbers $M_{1,k}$ and $M_{2,k}$ such that

$$M_{1,k} \leq x_n \leq M_{2,k} \text{ for } n \geqslant -2k.$$

Proof. Proof of Theorem 2.4 can be showed in a similar way to the proof of Theorem 2.3. □

Throughout the paper, we investigated the boundedness character of the system (1.1) in terms of the sequence (x_n). Note that the calculations become in same manner for the sequence (y_n) except change of variables.

3. Applications

Exercise 3.1. Let initial conditions $x_{-1} = 0.2$, $x_{-2} = 2$, $y_{-1} = 0.4$, $y_{-2} = 4$ and parameters $a_1 = 0.4$, $a_2 = 0.6$, $b_1 = 0.6$, $b_2 = -0.8$ in the system (1.1) with $\alpha = \beta = 0$. The solution $(x_n, y_n)_{n \geq -2}$ is given by Figure 3.1. The Figure 3.1 corrects the Theorem 2.1.

Exercise 3.2. Let initial conditions $x_{-1} = 2$, $x_{-2} = 0.1$, , $x_{-3} = 4$, $y_{-1} = 0.4$, $y_{-2} = 5$, $y_{-3} = 4$ and parameters $a_1 = 0.1$, $a_2 = 0.4$, $a_3 = 0.5$, $b_1 = 1.4$, $b_2 = -0.56$, $b_3 = 0.064$ in the system (1.1) with $\alpha = 2, \beta = 0$. The solution $(x_n, y_n)_{n \geq -3}$ of the system (1.1) is given by Figure 3.2. The Figure 3.2 corrects the Theorem 2.3.

Exercise 3.3. Let initial conditions $x_{-1} = 0.2$, $x_{-2} = 0.3$, $x_{-3} = 0.4$, $x_{-4} = 5$, $y_{-1} = 1$, $y_{-2} = 3$, $y_{-3} = 0.5$, $y_{-4} = 0.6$ and parameters $a_1 = 0.4$, $a_2 = 0.3$, $a_3 = 0.2$, $a_4 = 0.1$, $b_1 = 1$, $b_2 = -0.35$, $b_3 = 0.05$, $b_4 = -0.0024$. The solution $(x_n, y_n)_{n \geq -4}$ of the system (1.1) with $k = 4, \alpha = 2$ and $\beta = 0$ is given by Figure 3.3. The Figure 3.3 corrects the Theorem 2.4.

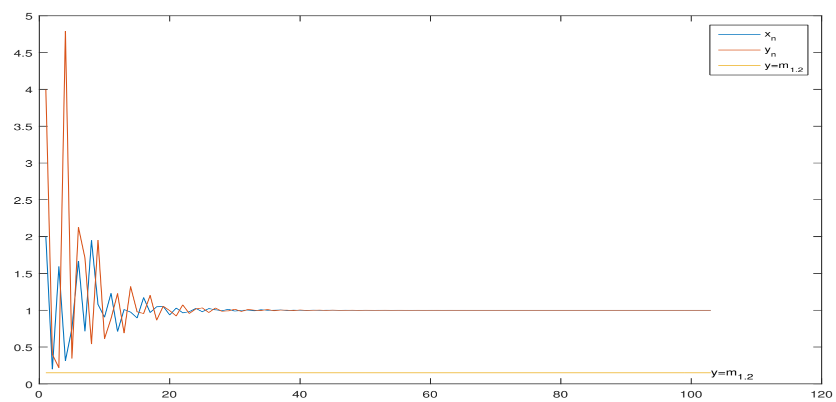

Figure 3.1.

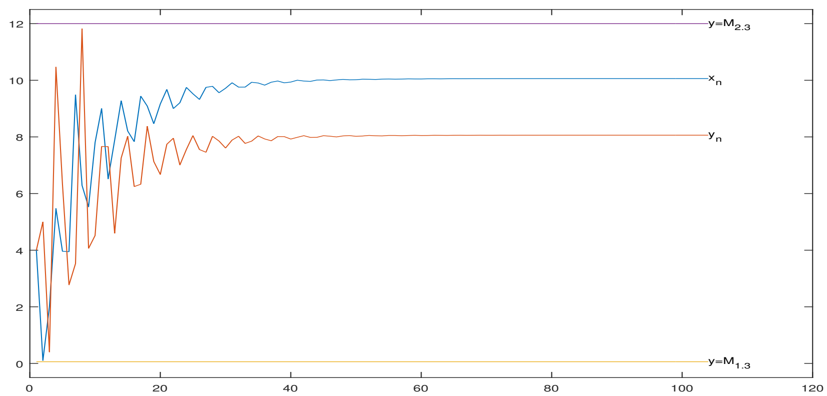

FIGURE 3.2.

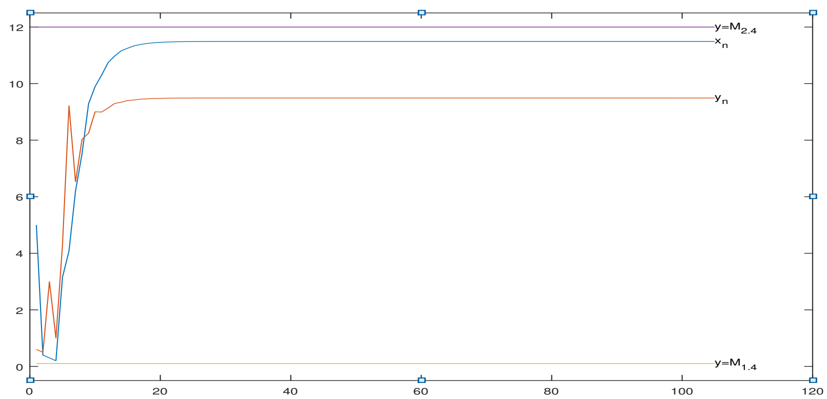

FIGURE 3.3.

AUTHORS CONTRIBUTION STATEMENT

The authors have read and agreed to the published version of the manuscript.

CONFLICT OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

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