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ON 3-DIMENSIONAL α-PARA KENMOTSU MANIFOLDS

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Abstract. The aim of the present paper is to study 3-dimensional α-para Kenmotsu manifolds. First we consider 3-dimensional Ricci semisymmetric α-para Kenmotsu manifolds and obtain some equivalent conditions. Next we study cyclic parallel Ricci tensor in 3-dimensional α-para Kenmotsu manifolds. Moreover, we investigate η-parallel Ricci tensor in 3-dimensional α-para Kenmotsu manifolds. Continuing our study, we consider locally ϕ-symmetric 3-dimensional α-para Kenmotsu manifolds. Next, we study gradient Ricci solitons in 3-dimensional α-para Kenmotsu manifolds. Finally, we give an example of a 3-dimensional α-para Kenmotsu manifold which verifies some results.

1. Introduction

In 1972, Chen and Yano [8] introduced the notion of quasi-constant curvature as a conformally flat manifold with the curvature tensor $'R$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned} 'R(X,Y,Z,W) &= p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad +q[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &\quad +g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \end{aligned} \tag{1.1}$$

where $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type $(1,3)$, p, q are scalar functions and T is a non zero 1-form defined by

$$g(X,\rho) = T(X). \tag{1.2}$$

It can be easily seen that if the curvature tensor $'R$ is of the form (1.1) then the manifold is of conformally flat. Vranceanu [28] defined the notion of almost constant curvature by the same expression (1.1). Later Mocanu [22] proved that the manifold introduced by Chen and Yano and the manifold introduced by Vranceanu are the same. Hence a Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $'R$ satisfies the relation (1.1). In particular $q = 0$, the

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manifold is said to be a manifold of constant curvature. Noticing, a manifold is of quasi-constant curvature is a generalization of a manifold of constant curvature. A manifold of quasi-constant curvature have been studied by several authors such as Adati and Wang [3], Adati [1], Wang [29], De and Ghosh [11] and many others.

A Riemannian manifold M is called locally symmetric if its curvature tensor R is parallel, that is, $\nabla R = 0$, where ∇ is the Levi-Civita connection. The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by $R(X,Y) \cdot R = 0$, where $R(X,Y)$ acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabó in [26]. The classification results of Szabó were presented in the book [6]. Also in [19], Kowalski classified 3-dimensional Riemannian spaces satisfying $R(X,Y) \cdot R = 0$.
A Riemannian manifold is said to be Ricci semisymmetric if $R(X,Y) \cdot S = 0$ where S denotes the Ricci tensor of type $(0,2)$. A general classification of these manifolds has been worked out recently by Mirzoyan [21].

A Ricci soliton [7] is a generalization of an Einstein metric. In a manifold M a Ricci soliton is a triplet (g,V,λ), with g, a pseudo Riemannian metric, V a vector field (called the potential vector field) and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.3}$$

where $\pounds_V$ is the Lie derivative with respect to V and S is the Ricci tensor of type $(0,2)$. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive, respectively. The compact Ricci solitons are the fixed points of the Ricci flow

$$\frac{\partial}{\partial t} g = -2S$$

projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings. If the complete vector field V is the gradient of a potential function $-f$, then g is said to be a gradient Ricci soliton and equation (1.3) takes the form

$$\nabla\nabla f = S + \lambda g, \tag{1.4}$$

where ∇ denotes the Riemannian connection.

In a compact manifold a Ricci soliton has constant curvature in dimension 2 (Hamilton [15]) and also in dimension 3 (Ivey [16]). On the other hand a Ricci soliton on a compact manifold is a gradient Ricci soliton [24]. Recently Yildiz et al [32] have studied Ricci solitons in 3-dimensional f-Kenmotsu manifolds. For details, we refer to Chow and Knopf [9], Bejan and Crasmareanu [4] and Derdzinski [12]. Motivated by the above studies in the present paper we investigate certain curvature conditions on 3-dimensional α-para Kenmotsu manifolds.
The paper is organized as follows: In section 2, we give a brief account on almost paracontact metric manifolds, normal almost paracontact metric manifolds and curvature properties of α-para Kenmotsu manifolds. Next in section 3 we consider 3-dimensional Ricci semisymmetric α-para Kenmotsu manifolds and obtain some equivalent conditions. Section 4 deals with cyclic parallel Ricci tensor in 3-dimensional α-para Kenmotsu manifolds and prove that a 3-dimensional α-para Kenmotsu manifold M has cyclic parallel Ricci tensor if and only if the scalar curvature $r = -6\alpha^2$. Section 5 is devoted to study η-parallel Ricci tensor in 3-dimensional α-para Kenmotsu manifolds and prove that a 3-dimensional α-para Kenmotsu manifold M has η-parallel Ricci tensor if and only if the scalar curvature $r = -2\alpha^2$. We discuss locally ϕ-symmetric 3-dimensional α-para Kenmotsu

manifolds in section 6. Next, we study gradient Ricci solitons in 3-dimensional α-para Kenmotsu manifolds. Finally, we give an example of a 3-dimensional α-para Kenmotsu manifold which verifies some results.

2. Preliminaries

2.1. **Almost paracontact metric manifolds.** A differential manifold M of dimension $2n+1$ is called an almost paracontact manifold ([20],[23]) equipped with the structure (ϕ, ξ, η) where ϕ is a tensor field of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = I - \eta \otimes \xi \text{ and } \eta(\xi) = 1. \tag{2.1}$$

From equation (2.1) it can easily deduced that

$$\phi\xi = 0,\ \eta(\phi X) = 0 \text{ and } \operatorname{rank}(\phi) = 2n. \tag{2.2}$$

If an almost paracontact manifold admits a pseudo-Riemannian metric g satisfying

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

where signature of g is $(n+1, n)$ for any vector fields $X, Y \in \chi(M)$, (where $\chi(M)$ is the set of all differential vector fields on M) then the manifold is called almost paracontact metric manifold.

An almost paracontact structure is said to be a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$ with the associated metric g [33]. For an almost paracontact metric manifold, there always exists a special kind of local pseudo orthonormal ϕ basis $\{X_i, \phi X_i, \xi\}$, X_i's and ξ are space-like vector fields and ϕX_i's are time-like. Thus, an almost paracontact metric manifold is an odd dimensional manifold.

2.2. **Normal almost paracontact metric manifolds.** An almost paracontact metric manifold is said to be normal if the induced almost paracomplex structure J on the product manifold $\mathrm{M}^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X + f\xi, \eta(X)\frac{d}{dt}) \tag{2.4}$$

is integrable where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $\mathrm{M}^{2n+1} \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ defined by $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ.

Proposition 2.1. [30]*For a 3-dimensional almost paracontact metric manifold M, the following conditions are mutually equivalent*

(a) *M is normal,*
(b) *there exist differential functions α, β on M such that*

$$(\nabla_X \phi)Y = \beta\{g(X, Y)\xi - \eta(Y)X\} + \alpha\{g(\phi X, Y)\xi - \eta(Y)\phi X\},$$

(c) *there exist differential functions α, β on M such that*

$$\nabla_X \xi = \alpha\{X - \eta(X)\xi\} + \beta\phi X,$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g and α, β are given by

$$2\alpha = \mathit{Trace}\{X \to \nabla_X \xi\}, 2\beta = \mathit{Trace}\{X \to \phi\nabla_X \xi\}.$$

Definition 2.1. A 3-dimensional normal almost paracontact metric manifold M is said to be

1. paracosymplectic if $\alpha = \beta = 0$ [10],
2. α-para Kenmotsu if α is a non-zero constant and $\beta = 0$ [31], in particular para Kenmotsu if $\alpha = 1$ [5],
3. quasi-para Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ [13],
4. β-para Sasakian if and only if $\alpha = 0$ and β is a non-zero constant, in particular para Sasakian if $\beta = -1$ [33].

2.3. **Curvature properties of 3-dimensional α-para Kenmotsu manifolds.** In a 3-dimensional α-para Kenmotsu manifold, it follows that [25]

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2\alpha^2)[g(Y,Z)X - g(X,Z)Y] \\ &\quad -(\frac{r}{2} + 3\alpha^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ &\quad +(\frac{r}{2} + 3\alpha^2)[\eta(X)Y - \eta(Y)X]\eta(Z), \end{aligned} \tag{2.5}$$

where r is the scalar curvature of the manifold and g, pseudo-metric.
Clearly, (2.5) is of the form (1.1) and hence the manifold is of quasi-constant curvature. In addition, if $r = -6\alpha^2$ then the manifold is of constant curvature.
Contracting the equation (2.5) with respect to Z, we have

$$S(X,Y) = (\frac{r}{2} + \alpha^2)g(X,Y) - (\frac{r}{2} + 3\alpha^2)\eta(X)\eta(Y). \tag{2.6}$$

Proposition 2.2. *A 3-dimensional α-para Kenmotsu manifold is of quasi-constant curvature.*

In a 3-dimensional α-para Kenmotsu manifold the following relations hold [25]:

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.7}$$

$$g(X,\xi) = \eta(X), \tag{2.8}$$

$$S(X,\xi) = -2\alpha^2\eta(X), \tag{2.9}$$

$$R(X,Y)\xi = -\alpha^2\{\eta(Y)X - \eta(X)Y\}, \tag{2.10}$$

$$(\nabla_X\eta)Y = \alpha\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{2.11}$$

$$(\nabla_X\phi)Y = \alpha\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.12}$$

$$\nabla_X\xi = \alpha\{X - \eta(X)\xi\}, \tag{2.13}$$

for all vector fields X, Y, Z and $W \in \chi(M)$.

3. Ricci semisymmetric α-para Kenmotsu manifolds

In this section we consider a 3-dimensional Ricci semisymmetric α-para Kenmotsu manifold. A Riemannian or pseudo-Riemannian manifold is said to be Ricci semisymmetric [2] if

$$R(X,Y)\cdot S=0, \tag{3.1}$$

where $R(X,Y)$ denotes the derivation in the tensor algebra at each point of the manifold. It follows from equation (3.1)

$$(R(X,Y)\cdot S)(U,V)=0. \tag{3.2}$$

This implies

$$S(R(X,Y)U,V)+S(U,R(X,Y)V)=0. \tag{3.3}$$

Using (2.5) and (2.9) we get

$$\begin{aligned} S(R(X,Y)U,V) &= (\frac{r}{2}+2\alpha^2)[g(Y,U)S(X,V)-g(X,U)S(Y,V)] \\ &\quad +2\alpha^2(\frac{r}{2}+3\alpha^2)[g(Y,U)\eta(X)-g(X,U)\eta(Y)]\eta(V) \\ &\quad +(\frac{r}{2}+3\alpha^2)[S(Y,V)\eta(X)-S(X,V)\eta(Y)]\eta(U). \end{aligned} \tag{3.4}$$

Interchanging U and V in (3.4) yields

$$\begin{aligned} S(R(X,Y)V,U) &= (\frac{r}{2}+2\alpha^2)[g(Y,V)S(X,U)-g(X,V)S(Y,U)] \\ &\quad +2\alpha^2(\frac{r}{2}+3\alpha^2)[g(Y,V)\eta(X)-g(X,V)\eta(Y)]\eta(U) \\ &\quad +(\frac{r}{2}+3\alpha^2)[S(Y,U)\eta(X)-S(X,U)\eta(Y)]\eta(V). \end{aligned} \tag{3.5}$$

With the help of the equations (3.4) and (3.5), we obtain from (3.3)

$$\begin{aligned} &(\frac{r}{2}+2\alpha^2)\{g(Y,U)S(X,V)-g(X,U)S(Y,V) \\ &+g(Y,V)S(X,U)-g(X,V)S(Y,U)\} \\ &+2\alpha^2(\frac{r}{2}+3\alpha^2)\{g(Y,U)\eta(X)\eta(V)-g(X,U)\eta(Y)\eta(V) \\ &+g(Y,V)\eta(X)\eta(U)-g(X,V)\eta(Y)\eta(U)\} \\ &+(\frac{r}{2}+3\alpha^2)\{S(Y,U)\eta(X)\eta(V)-S(X,U)\eta(Y)\eta(V) \\ &+S(Y,V)\eta(X)\eta(U)-S(X,V)\eta(Y)\eta(U)\}=0. \end{aligned} \tag{3.6}$$

Using (2.6) it follows from (3.6)

$$\begin{aligned} &(\frac{r\alpha^2}{2}+3\alpha^4)\{g(Y,U)\eta(X)\eta(V)-g(X,U)\eta(Y)\eta(V) \\ &+g(Y,V)\eta(X)\eta(U)-g(X,V)\eta(Y)\eta(U)\}=0. \end{aligned} \tag{3.7}$$

Putting $X=U=\xi$ in (3.7) we have

$$(\frac{r\alpha^2}{2}+3\alpha^4)\{g(Y,V)-\eta(Y)\eta(V)\}=0. \tag{3.8}$$

By making contraction (3.8) and using the fact $\alpha\neq 0$ yields

$$r=-6\alpha^2. \tag{3.9}$$

Hence the manifold M is of constant curvature. Consequently the manifold M is an Einstein manifold. Indeed we have

$$S(X,Y) = -2\alpha^2 g(X,Y). \tag{3.10}$$

Conversely, if the manifold is an Einstein manifold then

$$R(X,Y)\cdot S = 0.$$

By the above discussions we have the following:

Theorem 3.1. *In a 3-dimensional α-para Kenmotsu manifold M, the following conditions are equivalent:*

(a) *M is Ricci semisymmetric, that is, $R(X,Y)\cdot S = 0$ for any X, Y,*
(b) *the scalar curvature $r = -6\alpha^2$,*
(c) *the manifold M is of constant curvature,*
(d) *M is an Einstein manifold.*

4. α-para Kenmotsu manifolds with cyclic parallel Ricci tensor

This section deals with α-para Kenmotsu manifolds with cyclic parallel Ricci tensor. We prove the following:

Theorem 4.1. *A 3-dimensional α-para Kenmotsu manifold M satisfies cyclic parallel Ricci tensor if and only if the scalar curvature $r = -6\alpha^2$.*

Proof. Suppose the manifold under consideration satisfies cyclic parallel Ricci tensor [14], then the Ricci tensor S satisfies

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0, \tag{4.1}$$

for all $X, Y, Z \in \chi(M)$. It is known [17] that Cartan hypersurfaces are manifolds with non-parallel Ricci tensor, satisfying (4.1). From (4.1), it follows that $r =$ constant. Then

$$\begin{aligned}&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)\\ &= -(\frac{r}{2} + 3\alpha^2)[(\nabla_X \eta)Y\eta(Z) + (\nabla_X \eta)Z\eta(Y) + (\nabla_Y \eta)X\eta(Z)\\ &+(\nabla_Y \eta)Z\eta(X) + (\nabla_Z \eta)X\eta(Y) + (\nabla_Z \eta)Y\eta(X)].\end{aligned} \tag{4.2}$$

From (4.1) and (4.2) we get

$$\begin{aligned}&(\frac{r}{2} + 3\alpha^2)[(\nabla_X \eta)Y\eta(Z) + (\nabla_X \eta)Z\eta(Y) + (\nabla_Y \eta)X\eta(Z)\\ &+(\nabla_Y \eta)Z\eta(X) + (\nabla_Z \eta)X\eta(Y) + (\nabla_Z \eta)Y\eta(X)] = 0.\end{aligned} \tag{4.3}$$

By (2.11) it follows from (4.3)

$$\begin{aligned}&(\frac{r}{2} + 3\alpha^2)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + g(Y,X)\eta(Z) + g(Y,Z)\eta(X)\\ &+g(Z,X)\eta(Y) + g(Z,Y)\eta(X) - 6\eta(X)\eta(Y)\eta(Z)] = 0.\end{aligned} \tag{4.4}$$

Let $\{e_i\}(1 \le i \le 3)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Y = e_i$ in (4.4) and taking summation over i, we get

$$(\frac{r}{2} + 3\alpha^2)\eta(Z) = 0. \tag{4.5}$$

Since $\eta(Z) \neq 0$, thus we have from (4.5)

$$r = -6\alpha^2. \tag{4.6}$$

On the other hand if $r = -6\alpha^2$, then from the equation (4.2) it follows that (4.1) holds. Thus our theorem is proved. □

Again if $r = -6\alpha^2$, then the manifold M is of constant curvature and consequently Einstein. Hence from Theorem 3.1 we arrive to the following assertion:

Corollary 4.1. *If a 3-dimensional α-para Kenmotsu manifold M satisfies cyclic parallel Ricci tensor, then the manifold is Ricci semisymmetric.*

5. α-para Kenmotsu manifolds with η-parallel Ricci tensor

This section is devoted to study α-para Kenmotsu manifolds with η-parallel Ricci tensor.

Definition 5.1. The Ricci tensor S of an α-para Kenmotsu manifold M is called η-parallel if it satisfies the condition

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \text{ for all } X, Y, Z \in \chi(M). \tag{5.1}$$

The notion of η-parallel Ricci tensor for Sasakian manifolds was introduced by Kon [18].
We have from (2.6) and (2.3)

$$S(\phi Y, \phi Z) = -(\frac{r}{2} + \alpha^2)\{g(Y, Z) - \eta(Y)\eta(Z)\}. \tag{5.2}$$

Now differentiating (5.2) covariently along the vector field X we obtain

$$\begin{aligned}(\nabla_X S)(\phi Y, \phi Z) &= -\frac{dr(X)}{2}\{g(Y, Z) - \eta(Y)\eta(Z)\} \\ &\quad +(\frac{r}{2} + \alpha^2)[(\nabla_X \eta)Y\eta(Z) + (\nabla_X \eta)Z\eta(Y)].\end{aligned} \tag{5.3}$$

Using (2.11), the above equation implies

$$\begin{aligned}(\nabla_X S)(\phi Y, \phi Z) &= -\frac{dr(X)}{2}\{g(Y, Z) - \eta(Y)\eta(Z)\} \\ &\quad +\alpha(\frac{r}{2} + \alpha^2)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) \\ &\quad -2\eta(X)\eta(Y)\eta(Z)].\end{aligned} \tag{5.4}$$

Suppose the manifold under consideration satisfies the condition (5.1). Therefore

$$\begin{aligned}&-\frac{dr(X)}{2}\{g(Y, Z) - \eta(Y)\eta(Z)\} \\ &+\alpha(\frac{r}{2} + \alpha^2)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) \\ &-2\eta(X)\eta(Y)\eta(Z)] = 0.\end{aligned} \tag{5.5}$$

Let $\{e_i\}(1 \leq i \leq 3)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (5.5) and taking summation over i, we get

$$dr(X) = 0. \tag{5.6}$$

Consequently

$$r = \text{constant}. \tag{5.7}$$

From (5.5) and (5.6) yields

$$\alpha(\frac{r}{2}+\alpha^2)[g(X,Y)\eta(Z)+g(X,Z)\eta(Y)-2\eta(X)\eta(Y)\eta(Z)]=0. \tag{5.8}$$

By making suitable contraction of (5.8), we obtain

$$r=-2\alpha^2. \tag{5.9}$$

Conversely, if we take $r=-2\alpha^2$, then the equation (5.1) holds.
This leads to the following:

Theorem 5.1. *A 3-dimensional α-para Kenmotsu manifold M has η-parallel Ricci tensor if and only if the scalar curvature $r=-2\alpha^2$.*

6. Locally ϕ-symmetric α-para Kenmotsu manifolds

In this section we discuss locally ϕ-symmetric α-para Kenmotsu manifolds.

Definition 6.1. A 3-dimensional α-para Kenmotsu manifold is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X,Y)Z=0,$$

for all vector fields X,Y,Z,W orthogonal to ξ.

This notation was introduced for Sasakian manifolds by Takahashi [27]. Differentiating (2.5) covariently along the vector field W we obtain

$$\begin{aligned}(\nabla_W R)(X,Y)Z \;=\;& \frac{dr(W)}{2}\{g(Y,Z)X-g(X,Z)Y-g(Y,Z)\eta(X)\xi\\ &+g(X,Z)\eta(Y)\xi+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X\}\\ &-(\frac{r}{2}+3\alpha^2)\{g(Y,Z)\eta(X)\nabla_W\xi-g(X,Z)\eta(Y)\nabla_W\xi\\ &+g(Y,Z)(\nabla_W\eta)(X)\xi-g(X,Z)(\nabla_W\eta)(Y)\xi\\ &-(\nabla_W\eta)X\eta(Z)Y-(\nabla_W\eta)Z\eta(X)Y\\ &+(\nabla_W\eta)Y\eta(Z)X+(\nabla_W\eta)Z\eta(Y)X\}.\end{aligned} \tag{6.1}$$

Let us suppose that the manifold M under consideration is locally ϕ-symmetric. With the help of (2.11), (2.2) and (6.1) we obtain

$$\phi^2(\nabla_W R)(X,Y)Z=\frac{dr(W)}{2}\{g(Y,Z)\phi^2X-g(X,Z)\phi^2Y\}. \tag{6.2}$$

Using (2.1) it follows that

$$\frac{dr(W)}{2}\{g(Y,Z)X-g(X,Z)Y\}=0. \tag{6.3}$$

Taking inner product of (6.3) with ξ, we have

$$\frac{dr(W)}{2}\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}=0. \tag{6.4}$$

Contracting Y,Z in (6.4) yields

$$\frac{dr(W)}{2}=0,$$

that is,

$$r=\text{constant}.$$

Conversely, if r =constant, then from (6.2) it follows that the manifold is locally ϕ-symmetric. Hence we can state the following:

Theorem 6.1. *A 3-dimensional α-para Kenmotsu manifold is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant.*

7. Gradient Ricci solitons

In [4], Bejan and Crasmareanu have been studied Ricci solitons in 3-dimensional normal paracontact geometry and obtain the necessary conditions for their existence, moreover in the case of α-para Kenmotsu manifolds the Ricci tensor S satisfies the condition $S = -(2\alpha^2 + \alpha)g + \alpha\eta \otimes \eta$. Let us suppose that a 3-dimensional α-para Kenmotsu manifold admits a Ricci soliton defined by (1.3). It is well known that $\nabla g = 0$. Since λ is constant, thus $\nabla\lambda g = 0$. Therefore $\pounds_V g + 2S$ is parallel. Here we recall a lemma proved by Srivastava and Srivastava in [25]:

Lemma 7.1. *A parallel symmetric $(0, 2)$ tensor field in a 3-dimensional non-para-cosymplectic α-para Kenmotsu manifold is a constant multiple of the associated metric tensor.*

By Lemma 7.1 we can say that $\pounds_V g + 2S$ is a constant multiple of pseudo-metric tensor g. Thus $\pounds_V g + 2S = kg$, where k is constant. Hence $\pounds_V g + 2S + 2\lambda g$ reduces to $(k + 2\lambda)g$. By (1.3) we have $\lambda = -\frac{k}{2}$. This leads to the following:

Proposition 7.1. *In a 3-dimensional non-para-cosymplectic α-para Kenmotsu manifold, the Ricci soliton (g, V, λ) is shrinking or expanding according as k is positive or negative.*

Let M be a 3-dimensional non-para-cosymplectic α-para Kenmotsu manifold with constant structure function α and g a gradient Ricci soliton. Then the equation (1.4) can be put in of the form

$$\nabla_Y Df = QY + \lambda Y \tag{7.1}$$

for all vector fields $Y \in \chi(M)$, where D denotes the gradient operator of pseudo-metric g.
From (7.1) it follows that

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{7.2}$$

Using (2.6) in (7.2) we have

$$\begin{aligned} R(X, Y)Df &= \frac{dr(X)}{2}[Y - \eta(Y)\xi] - \frac{dr(Y)}{2}[X - \eta(X)\xi] \\ &\quad -\alpha(\frac{r}{2} + 3\alpha^2)[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{7.3}$$

Substituting $X = \xi$ in (7.3) yields

$$R(\xi, Y)Df = \{\frac{dr(\xi)}{2} + \alpha(\frac{r}{2} + 3\alpha^2)\}\{Y - \eta(Y)\xi\}. \tag{7.4}$$

By (2.5) it follows that

$$g(R(\xi, Y)Z, \xi) = -\alpha^2\{g(Y, Z) - \eta(Y)\eta(Z)\}. \tag{7.5}$$

With the help of (7.4) and (7.5) we obtain

$$g((\nabla_\xi Q)Y - (\nabla_Y Q)\xi, \xi) = 0. \tag{7.6}$$

This implies

$$g(Y, Df) = \xi f \eta(Y),$$

that is,

$$Df = (\xi f)\xi. \tag{7.7}$$

Again using (2.13) and (7.7) we obtain

$$g(\nabla_Y Df, X) = Y(\xi f)\eta(X) + \alpha(\xi f)\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{7.8}$$

From (7.1) and (7.8) we get

$$S(X,Y) + \lambda g(X,Y) = Y(\xi f)\eta(X) + \alpha(\xi f)\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{7.9}$$

Putting $X = \xi$ and using (2.6) we have

$$Y(\xi f) = (\lambda - 2\alpha^2)\eta(Y). \tag{7.10}$$

Therefore we have from (7.10) and (7.8)

$$\nabla_Y Df = -2\alpha^2\eta(Y)\xi + \lambda\eta(Y)\xi + \alpha(\xi f)Y - \alpha(\xi f)\eta(Y)\xi. \tag{7.11}$$

On the other hand (7.3) implies

$$g(R(X,Y)Df, \xi) = 0. \tag{7.12}$$

With the help of (7.11), (2.3) and (2.11) we obtain

$$\begin{aligned} \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df &= \alpha(\xi f)[\nabla_X Y - \nabla_Y X] \\ &\quad -\alpha^2(\xi f)[X\eta(Y) - Y\eta(X)]. \end{aligned} \tag{7.13}$$

Again we have from (7.11)

$$\nabla_{[X,Y]} Df = (\lambda - 2\alpha^2 - \alpha(\xi f))\eta([X,Y])\xi + \alpha(\xi f)[\nabla_X Y - \nabla_Y X]. \tag{7.14}$$

From (7.13) and (7.14) we get

$$\begin{aligned} R(X,Y)Df &= -\alpha^2(\xi f)[X\eta(Y) - Y\eta(X)] \\ &\quad -\{\lambda - 2\alpha^2 - \alpha(\xi f)\}\eta([X,Y])\xi. \end{aligned} \tag{7.15}$$

Taking inner product of (7.15) with ξ yields

$$g(R(X,Y)Df, \xi) = -(\lambda - 2\alpha^2 - \alpha(\xi f))\eta([X,Y]). \tag{7.16}$$

From (7.12) and (7.16) we get

$$\lambda - 2\alpha^2 = \alpha(\xi f). \tag{7.17}$$

Again we have from (7.17) and (7.10)

$$Y(\xi f) = \alpha(\xi f)\eta(Y). \tag{7.18}$$

Using (7.17) we get from (7.11)

$$\nabla_Y Df = \alpha(\xi f)Y. \tag{7.19}$$

Making use of (7.18) and (7.17) we obtain from (7.9)

$$S(X,Y) = -2\alpha^2 g(X,Y).$$

Hence M is an Einstein manifold. Thus we can state the following:

Theorem 7.1. *If the pseudo-metric g of a 3-dimensional non-para-cosymplectic α-para Kenmotsu manifold is a gradient Ricci soliton, then the manifold is an Einstein manifold.*

Again it is well known that a 3-dimensional Einstein manifold is a manifold of constant curvature. Thus we have the following:

Corollary 7.1. *If the pseudo-metric g of a 3-dimensional non-para-cosymplectic α-para Kenmotsu manifold is a gradient Ricci soliton, then the manifold is a manifold of constant curvature.*

8. Example of a 3-dimensional α-para Kenmotsu manifold

Example 8.1. We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are standard coordinate of $\mathbb{R}^3$.
The vector fields

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M.
Let g be the pseudo-Riemannian metric defined by

$$g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0,$$
$$g(e_1,e_1) = -1,\ g(e_2,e_2) = g(e_3,e_3) = 1.$$

Let η be the 1-form defined by $\eta(X) = g(X,e_3)$ for any $X \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = e_2,\ \phi(e_2) = e_1,\ \phi(e_3) = 0$. Then we have

$$\eta(e_3) = g(e_3,e_3) = 1,\ \ \phi^2 Z = Z - \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = -g(Z,W) + \eta(Z)\eta(W)$$

for any $Z, W \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ,ξ,η,g) defines an almost paracontact metric structure on M. Further one can easily verified $[\phi,\phi](e_i,e_j) - 2d\eta(e_i,e_j) = 0, \quad 1 \le i < j \le 3$, which implies that the structure is normal. Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then we have

$$\begin{aligned}[e_1,e_3] = e_1e_3 - e_3e_1 \qquad &= (z\frac{\partial}{\partial x})(z\frac{\partial}{\partial z}) - (z\frac{\partial}{\partial z})(z\frac{\partial}{\partial x}) \\ &= z^2\frac{\partial^2}{\partial x\partial z} - z\frac{\partial}{\partial x} - z^2\frac{\partial^2}{\partial z\partial x} \\ &= -z\frac{\partial}{\partial x} = -e_1.\end{aligned}$$

Similarly

$$[e_1,e_2] = 0,\ [e_2,e_3] = -e_2,$$
$$[e_1,e_1] = [e_2,e_2] = [e_3,e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned}2g(\nabla_X Y, Z) \;\; = \;\; & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]).\end{aligned} \tag{8.1}$$

Using (8.1) we have,

$$2g(\nabla_{e_1} e_3, e_3) = 0, \tag{8.2}$$

$$2g(\nabla_{e_1} e_3, e_2) = 0, \tag{8.3}$$

$$2g(\nabla_{e_1} e_3, e_1) = -2g(e_1,e_1). \tag{8.4}$$

From (8.2), (8.3) and (8.4) we get

$$2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X), \tag{8.5}$$

for any $X \in \chi(M)$.
Therefore

$$\nabla_{e_1} e_3 = -e_1.$$

Further we get from (8.1)

$$\begin{aligned} &\nabla_{e_1} e_1 = -e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = -e_1,\\ &\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = e_3,\ \nabla_{e_2} e_3 = -e_2,\\ &\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0. \end{aligned} \tag{8.6}$$

It is easily verify that the condition (2.12) holds.
It is well known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{8.7}$$

With the previous results and using (8.7) we get the following:

$$\begin{aligned} &R(e_1,e_2)e_1 = -e_2,\ R(e_1,e_2)e_2 = -e_1,\ R(e_1,e_2)e_3 = 0,\\ &R(e_2,e_3)e_1 = 0,\ R(e_2,e_3)e_2 = e_3,\ R(e_2,e_3)e_3 = -e_2,\\ &R(e_1,e_3)e_1 = -e_3,\ R(e_1,e_3)e_2 = 0,\ R(e_1,e_3)e_3 = -e_1. \end{aligned}$$

Using the above expressions of the curvature tensor we get

$$S(e_1,e_1) = 2,\ S(e_2,e_2) = S(e_3,e_3) = 0.$$

In a pseudo-Riemannian manifold the scalar curvature

$$r = \sum_{i=1}^{3} g(e_i,e_i) S(e_i,e_i) = -2.$$

Here $\alpha = -1$. Thus the manifold under consideration is an α-para Kenmotsu manifold which verifies Theorem 5.1 and Theorem 6.1.

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