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GENERALIZED DUAL-VARIABLE BERNSTEIN POLYNOMIALS

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Abstract. Bernstein polynomials are used in computer graphics for Computer Aided Geometric Design (CAGD). In this paper, we introduce the concept of the generalized dual-variable Bernstein polynomials and give its some properties. In particular, we investigate the limit and derivation equations of the dual-variable Bernstein polynomials.

1. Introduction and Preliminaries

Bernstein polynomials were initiated by Sergei Natanovich Bernstein in 1910. The Bernstein polynomials of degree n which are represented by the formula $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ for $i = 1, 2, \ldots, n$ and here $\binom{n}{i}, i = 1, 2, \ldots, n$ are the binomial coefficients. The Bernstein polynomials are useful mathematical tools for CAGD as they are simply defined, differentiated and also calculated quickly on computer systems. Hence one of the most appropriate way of defining curves on CAGD is the Bernstein polynomials. One of these kinds curves are Bezier curves defined by P. Bezier at Renault Company in 1959. The Bezier curves are introduced by the formula $b^n(t) = \sum_{i=0}^{n-r} b_i^r(t) B_i^{n-r}(t)$, where the polynomials $B_i^{n-r}(t)$ are Bernstein polynomial and the vectors $b_i^r(t)$ are the control points. The coefficients of the Bernstein polynomials are satisfied the condition $\sum\limits_{i=0}^{n} B_i^n(t) \equiv 1$. Some special case of Bernstein polynomials are $B_i^n(t) \equiv 0$ if $i < 0$ or $i > n$ and $B_i^n(t) \equiv 1$ if $i = n = 0$. Also the recursion $B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$ is satisfied, [1]. The further theoretical preliminaries of the Bernstein polynomials can be found in the references [2-7].

In the other hand, we will give some properties of dual numbers and dual-variable functions. Dual numbers were first introduced by W.K.Clifford in 1873. The number $z = (x, y) = x + \varepsilon y$ is called a dual number associated with the real unit 1 and the dual unit ε where $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. Therefore, the dual numbers are elements of the set

$$\mathbb{D} = \mathbb{R}[\varepsilon] = \left\{ z = x + \varepsilon y \mid (x, y) \in \mathbb{R}^2, \varepsilon^2 = 0,\ \varepsilon \neq 0 \right\}$$

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generated by 1 and ε^2. The addition, multiplication with a scalar and multiplication of the dual numbers are defined by

$$\begin{aligned} z_1 + z_2 &= (x_1 + \varepsilon y_1) + (x_2 + \varepsilon y_2) = x_1 + x_2 + \varepsilon (y_1 + y_2) \\ \lambda z_1 &= \lambda (x_1 + \varepsilon y_1) = \lambda x_1 + \varepsilon \lambda y_1 \\ z_1 . z_2 &= (x_1 + \varepsilon y_1) . (x_2 + \varepsilon y_2) = x_1 . x_2 + \varepsilon (x_1 y_2 + y_1 x_2) \end{aligned}$$

Since $(\mathbb{D}, +, .)$ is commutative, associative and distributive over addition, then the dual number set becomes an algebra. But the dual number algebra is not a field. Because the zero division element $\varepsilon y, y \in \mathbb{R}$ has no an inverse in the algebra $\mathbb{D}$. If $z = x + \varepsilon y$ is a dual number, x and y are called the real and dual parts of of z, respectively. Similar with complex numbers, the conguate $\bar{z}$ of the dual number $z = x + \varepsilon y$ is defined by $\bar{z} = x - \varepsilon y$. Then $z.\bar{z} = x^2$. Furthermore the division of dual numbers is

$$\frac{z_1}{z_2} = \frac{z_1 . \overline{z_2}}{z_2 . \overline{z_2}} = \frac{x_1 x_2 + \varepsilon (x_1 y_2 - x_2 y_1)}{{x_2}^2}$$

where the condition $z_2 \neq \varepsilon y$ must be satisfied. Note that the division is possible for x_2. One can verify, using the multiplication and Binomial development, that $(x + \varepsilon y)^n = x^n + \varepsilon n x^{n-1} y$ in the references [11]. Let $f : \Omega \subset \mathbb{D} \to \mathbb{D}$ and $z = x + \varepsilon y \in \mathbb{D}$. The function $f(z)$ is called dual-variable functions. We say that the dual-variable function $f(z)$ is continuous at z_0, if $\lim_{z \to z_0} f(z) = f(z_0)$. If the dual-variable function f is continuous at every point of Ω, the dual-variable function becomes continuous in $\Omega \subset \mathbb{D}$. The function f is said to be differentiable at $z_0 = x_0 + \varepsilon y_0$, if the limit $\frac{df}{dz}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists, then $\frac{df}{dz}(z_0)$ is called the derivative of f at the point $z_0 = x_0 + \varepsilon y_0$. Also the dual-variable function can be written in terms of its real and dual parts as $f(z) = \varphi(x, y) + \varepsilon \psi(x, y)$. Therefore, the derivation of the dual-variable function can be written as

$$\frac{df(z)}{dz} = \frac{d\varphi(x, y)}{dx} + \varepsilon \frac{d\psi(x, y)}{dx},$$

see in the reference [11]. More details about the dual numbers and the dual-variable functions can be found in the refrences [8-11]. The dual numbers have lots of applications in many field of fundamental sciences; robotic, computer science, physic, aerospace. Furthermore, the dual variable functions give us a chance for defining the dual curves which are commonly studied in dual differential geometry. Hence, the function $f(z)$ describes a dual curves.

In our literature research, we realized that the dual-variable Bernstein polynomials had not been defined until now. So, in our present paper we introduce the dual variable Bernstein polynomials and we study the differentiability properties of the dual-variable Bernstein polynomials.

2. Main Result

2.1. Generalized Dual-variable Bernstein Polynomials.

Definition 2.1. Let $z = x + \varepsilon y \in \mathbb{D}$ *and* $\varepsilon \neq 0$ $\varepsilon^2 = 0$, $x, y \in \mathbb{R}$. The dual-variable generalized Bernstein polynomials of degree n are defined by

$$B_i{}^n(z) = \binom{n}{i} z^i . (1 - z)^{n-i},$$

where the binomial coefficients are $\begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{i!(n-i)!}$ for $0 \leq i \leq n$ and otherwise binomial coefficients are zero.

Definition 2.2. For $z = x + \varepsilon y \in \mathbb{D},\ \varepsilon \neq 0, \varepsilon^2 = 0,\ \ x, y \in \mathbb{R}$, the multiplication of two dual-variable generalized Bernstein basis is

$$B_i^n(z) .B_k^m(z) = \frac{\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} m \\ k \end{pmatrix}}{\begin{pmatrix} n+m \\ i+k \end{pmatrix}} B_{i+k}^{n+m}(z).$$

Definition 2.3. Let $z \in \mathbb{D}$. The limit of the dual-variable Bernstein polynomial at the point $z_0 \in \mathbb{D}$ is defined by

$$\lim_{z \to z_0} B_i{}^n(z) = \lim_{z \to z_0} \begin{pmatrix} n \\ i \end{pmatrix} z^i.(1-z)^{n-i} = \begin{pmatrix} n \\ i \end{pmatrix} {z_0}^i.(1-z_0)^{n-i}.$$

Definition 2.4. The dual-variable generalized Bernstein polynomial is continuous if the condition $\lim\limits_{z \to z_0} B_i{}^n(z) = B_i{}^n(z_0)$ is satisfied.

Definition 2.5. The dual-variable Bernstein polynomial can be differentiable, if the dual-variable Bernstein polynomial has the limit and it is continuous, then

$$\frac{d}{dz} B_i^n(z) = \lim_{z \to z_0} \frac{B_i^n(z) - B_i^n(z_0)}{z - z_0}.$$

Lemma 2.1. *Let $z = x + \varepsilon y \in \mathbb{D}$, $c \in \mathbb{R}$, and the Bernstein polinomials $B_i{}^n(z)$, $B_j{}^m(z)$ and $B_k{}^p(z)$ be differentiable. The dual-variable Bernstein polynomials are satisfy following properties:*

$$1)\frac{d}{dz}\left[B_i{}^n(z) + cB_j{}^m(z)\right] = \frac{d}{dz} B_i{}^n(z) + c.\frac{d}{dz} B_j{}^m(z)$$
$$2)\frac{d}{dz}\left[B_i{}^n(z) .B_j{}^m(z)\right] = \frac{d}{dz} B_i{}^n(z) .B_j{}^m(z) + B_i{}^n(z) .\frac{d}{dz} B_j{}^m(z)$$
$$3)\frac{d}{dz}\left[\frac{B_i{}^n(z)}{B_j{}^m(z)}\right] = \frac{\frac{d}{dz} B_i{}^n(z) .B_j{}^m(z) - B_i{}^n(z) .\frac{d}{dz} B_j{}^m(z)}{\left(B_j{}^m(z)\right)^2}, \quad B_j{}^m(z) \neq 0$$
$$4)\frac{d}{dz}\left[B_k{}^p(z)\, o B_j{}^m(z)\right] = \frac{dB_k{}^p(z)}{dz}\left(B_j{}^m(z)\right) \frac{dB_j{}^m(z)}{dz}.$$

Proof: From the properties of derivation of the dual-variable functions in [11], above equations are obviously satisfied.

Theorem 2.1. *The first order derivative of the dual-variable generalized Bernstein polynomial at the point $z = x + \varepsilon y \in \mathbb{D}$ is obtained with*

$$\frac{d}{dz} B_i{}^n(z) = n.\left(B_{i-1}{}^{n-1}(z) - B_i{}^{n-1}(z)\right).$$

Proof: From the derivation of the Definition (2.1), we get

$$\begin{aligned}
\frac{d}{dz}B_i{}^n(z) &= \frac{d}{dz}\left(\begin{pmatrix} n \\ i \end{pmatrix} z^i.(1-z)^{n-i}\right) \\
&= \frac{d}{dz}\left(\frac{n!}{i!\,(n-i)!}z^i.(1-z)^{n-i}\right) \\
&= \frac{n!}{i!\,(n-i)!}.\frac{dz^i}{dz}.(1-z)^{n-i} + \frac{n!}{i!\,(n-i)!}.z^i.\frac{d(1-z)^{n-i}}{dz} \\
&= \frac{n.\,(n-1)!}{(i-1)!\,(n-i)!}.z^{i-1}.(1-z)^{n-i} - \frac{n.\,(n-1)!}{i!\,(n-i-1)!}.z^i.(1-z)^{n-i-1} \\
&= n.\left[\frac{(n-1)!}{(i-1)!\,(n-i)!}.z^{i-1}.(1-z)^{n-i} - \frac{(n-1)!}{i!\,(n-i-1)!}.z^i.(1-z)^{n-i-1}\right] \\
&= n.\left[B_{i-1}{}^{n-1}(z) - B_i{}^{n-1}(z)\right]
\end{aligned}$$

Theorem 2.2. *Let $z = x + \varepsilon y \in \mathbb{D}$. The second order derivatives of $B_i{}^n(z) = \begin{pmatrix} n \\ i \end{pmatrix} z^i.(1-z)^{n-i}$ is calculated as following:*

$$\frac{d^2}{dz^2}B_i{}^n(z) = n.\,(n-1)\,.\left[B_{i-2}{}^{n-2}(z) - 2B_{i-1}{}^{n-2}(z) + B_i{}^{n-2}(z)\right]$$

Proof: Taking the derivative of the equation $\frac{d}{dz}B_i{}^n(z) = n.\left(B_{i-1}{}^{n-1}(z) - B_i{}^{n-1}(z)\right)$, we can get the second order derivatives of the polynomial, i.e.

$$\begin{aligned}
\frac{d}{dz}\left[\frac{d}{dz}B_i{}^n(z)\right] &= \frac{d}{dz}\left[n.\left(B_{i-1}{}^{n-1}(z) - B_i{}^{n-1}(z)\right)\right] \\
&= n.\frac{d}{dz}B_{i-1}{}^{n-1}(z) - n.\frac{d}{dz}B_i{}^{n-1}(z) \\
&= n.\left\{(n-1)\left[B_{i-2}{}^{n-2}(z) - B_{i-1}{}^{n-2}(z)\right]\right\} \\
&\quad -n.\left\{(n-1)\left[B_{i-1}{}^{n-2}(z) - B_i{}^{n-2}(z)\right]\right\} \\
&= n.\,(n-1)\left[B_{i-2}{}^{n-2}(z) - B_{i-1}{}^{n-2}(z) - B_{i-1}{}^{n-2}(z) + B_i{}^{n-2}(z)\right] \\
&= n.\,(n-1)\left[B_{i-2}{}^{n-2}(z) - 2.B_{i-1}{}^{n-2}(z) + B_i{}^{n-2}(z)\right].
\end{aligned}$$

Theorem 2.3. *The third order derivative of $B_i{}^n(z) = \begin{pmatrix} n \\ i \end{pmatrix} z^i.(1-z)^{n-i}$ at the point $z \in \mathbb{D}$ is obtained*

$$\frac{d^3}{dz^3}B_i{}^n(z) = n.\,(n-1)\,.\,(n-2)\,[B_{i-3}{}^{n-3}(z) - 3.B_{i-2}{}^{n-3}(z) + 3.B_{i-1}{}^{n-3}(z) - B_i{}^{n-3}(z)].$$

Proof: Taking the derivative of $\frac{d^2}{dz^2}B_i{}^n(z)$, we can obtain the third-derivative equation as following

$$\begin{aligned}
&\frac{d^3}{dz^3}B_i{}^n(z)=\frac{d}{dz}\left[\frac{d^2}{dz^2}B_i{}^n(z)\right]\\
=\ &\frac{d}{dz}\left[n.(n-1).\left[B_{i-2}{}^{n-2}(z)-B_{i-1}{}^{n-2}(z)-B_{i-1}{}^{n-2}(z)+B_i{}^{n-2}(z)\right]\right]\\
=\ &n.(n-1).\left[\frac{d}{dz}B_{i-2}{}^{n-2}(z)-\frac{d}{dz}B_{i-1}{}^{n-2}(z)-\frac{d}{dz}B_{i-1}{}^{n-2}(z)+\frac{d}{dz}B_i{}^{n-2}(z)\right]\\
=\ &n.(n-1).(n-2)\left[\begin{array}{l}B_{i-3}{}^{n-3}(z)-B_{i-2}{}^{n-3}(z)-\left(B_{i-2}{}^{n-3}(z)-B_{i-1}{}^{n-3}(z)\right)\\-\left(B_{i-2}{}^{n-3}(z)-B_{i-1}{}^{n-3}(z)\right)+\left(B_{i-1}{}^{n-3}(z)-B_i{}^{n-3}(z)\right)\end{array}\right]\\
=\ &n.(n-1).(n-2)\left[\begin{array}{l}B_{i-3}{}^{n-3}(z)-B_{i-2}{}^{n-3}(z)-B_{i-2}{}^{n-3}(z)+B_{i-1}{}^{n-3}(z)\\-B_{i-2}{}^{n-3}(z)+B_{i-1}{}^{n-3}(z)+B_{i-1}{}^{n-3}(z)-B_i{}^{n-3}(z)\end{array}\right]\\
=\ &n.(n-1).(n-2)\left[B_{i-3}{}^{n-3}(z)-3.B_{i-2}{}^{n-3}(z)+3.B_{i-1}{}^{n-3}(z)-B_i{}^{n-3}(z)\right]
\end{aligned}$$

Theorem 2.4. *The r th. order derivative of the generated dual-variable Bernstein polynomial at the point $z=x+\varepsilon y\in\mathbb{D}$ is calculated by the formula*

$$\frac{d^r}{dz^r}B_i{}^n(z)=\frac{n!}{(n-r)!}\left[\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\begin{pmatrix}r\\k\end{pmatrix}B_{i-k}{}^{n-r}(z)\right].$$

Proof: Using the induction method, we can obtain the r th. order derivative of the polinomial $B_i^n(z)$. For $r=1$, the first derivative of dual variable Bernstein polinomial is $\frac{d}{dz}B_i^n(z)=n\left(B_{i-1}^{n-1}(z)-B_i^{n-1}(z)\right)$. Suppose that $r.th$ derivation of $B_i^n(z)$

$$\frac{d^r}{dz^r}B_i{}^n(z)=\frac{n!}{(n-r)!}\left[\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\begin{pmatrix}r\\k\end{pmatrix}B_{i-k}{}^{n-r}(z)\right]$$

holds. Then, we will prove that the $r+1$ th. order derivative of the polinomial $\frac{d^{r+1}}{dz^{r+1}}B_i{}^n(z)$ is equal to

$$\frac{n!}{(n-r-1)!}\left[\sum_{k=\max(0,i+r+1-n)}^{\min(i,r+1)}(-1)^{k+r+1}\begin{pmatrix}r+1\\k\end{pmatrix}B_{i-k}{}^{n-r-1}(z)\right].$$

Now for $\frac{d^{r+1}}{dz^{r+1}}{B_i}^n(z)$, we will take the derivation of $\left[\frac{d^r}{dz^r}{B_i}^n(z)\right]$. Then, we obtain

$$\begin{aligned}
&\frac{d^{r+1}}{dz^{r+1}}{B_i}^n(z) = \frac{d}{dz}\left[\frac{d^r}{dz^r}{B_i}^n(z)\right] \\
= \;& \frac{d}{dz}\left[\frac{n!}{(n-r)!}\left(\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\binom{r}{k}{B_{i-k}}^{n-r}(z)\right)\right] \\
= \;& \frac{n!}{(n-r)!}\left(\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\binom{r}{k}\frac{d{B_{i-k}}^{n-r}(z)}{dz}\right) \\
= \;& \frac{n!}{(n-r)!}\left(\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\binom{r}{k}\left({B_{i-k}}^{n-r-1}(z)-{B_{i-k}}^{n-r-1}(z)\right)\right) \\
= \;& \frac{n!\,(n-r)}{(n-r)!}\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\binom{r}{k}\left({B_{i-k}}^{n-r-1}(z)\right) \\
& -\frac{n!\,(n-r)}{(n-r-1)!}\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\binom{r}{k}\left({B_{i-k}}^{n-r-1}(z)\right) \\
= \;& \frac{n!}{(n-r-1)!}\sum_{k=\max(0,i+r+1-n)}^{\min(i,r+1)}(-1)^{k+r+1}\binom{r}{k-1}\left({B_{i-k}}^{n-r-1}(z)\right) \\
& +\frac{n!\,(n-r)}{(n-r-1)!}\sum_{k=\max(0,i+r+1-n)}^{\min(i,r)}(-1)^{k+r}\binom{r}{k}\left({B_{i-k}}^{n-r-1}(z)\right) \\
= \;& \frac{n!}{(n-r-1)!}\sum_{k=\max(0,i+r+1-n)}^{\min(i,r+1)}(-1)^{k+r+1}\left(\binom{r}{k-1}+\binom{r}{k}\right)B_{i-k}^{n-r-1}(z) \\
= \;& \frac{n!}{(n-r-1)!}\sum_{k=\max(0,i+r+1-n)}^{\min(i,r+1)}(-1)^{k+r+1}\binom{r+1}{k}B_{i-k}^{n-r-1}(z)
\end{aligned}$$

which completes the induction and proves the theorem.

Theorem 2.5. *Let $z = x + \varepsilon y \in \mathbb{D}$. Dual-variable Bernstein polynomials for can be also written as*

$$(2.1)\qquad {B_i}^n(z) = (1-z)\,.{B_i}^{n-1}(z) + z.{B_{i-1}}^{n-1}(z)\,.$$

Proof: Using the definition of generated dual variable Bernstein polynomial, the Bernstein polynomial can written also

$$\begin{aligned}
{B_i}^n(z) &= \binom{n}{i} z^i.(1-z)^{n-i} \\
&= \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right].z^i.(1-z)^{n-i} \\
&= \binom{n-1}{i}.z^i.(1-z)^{n-i} + \binom{n-1}{i-1}.z^i.(1-z)^{n-i} \\
&= (1-z).\binom{n-1}{i}.z^i.(1-z)^{n-i-1} + z.\binom{n-1}{i-1}.z^{i-1}.(1-z)^{n-1-(i-1)} \\
&= (1-z)\,{B_i}^{n-1}(z) + z.{B_{i-1}}^{n-1}(z),
\end{aligned}$$

where $z = x + \varepsilon y \in \mathbb{D}$ and $\varepsilon \neq 0, , \varepsilon^2 = 0$.

Theorem 2.6. *Let $z = x + \varepsilon y \in \mathbb{D}$, $\varepsilon \neq 0, \varepsilon^2 = 0$. The first derivation of dual-variable Bernstein polynomial can be also written as*

$$\begin{aligned}
\frac{d}{dz}{B_i}^n(z) &= (-1).{B_i}^{n-1}(z) + (1-z).\left[{B_{i-1}}^{n-2}(z) - {B_i}^{n-2}(z)\right] \\
&\quad + {B_{i-1}}^{n-1}(z) + z.\left[{B_{i-2}}^{n-2}(z) - {B_{i-1}}^{n-2}(z)\right].
\end{aligned}$$

Proof: By using the Theorem (2.5), we can obtain a different way of the derivative of the dual-variable Bernstein polynomial. If we take the first differentiation of the Equation (2.1), we obtain the following equations

$$\begin{aligned}
\frac{d}{dz}{B_i}^n(z) &= \frac{d}{dz}\left[(1-z).{B_i}^{n-1}(z) + z.{B_{i-1}}^{n-1}(z)\right] \\
&= \frac{d}{dz}\left[(1-z).{B_i}^{n-1}(z)\right] + \frac{d}{dz}\left[z.{B_{i-1}}^{n-1}(z)\right] \\
&= \frac{d}{dz}(1-z).{B_i}^{n-1}(z) + (1-z).\frac{d}{dz}{B_i}^{n-1}(z) \\
&\quad + \frac{dz}{dz}.{B_{i-1}}^{n-1}(z) + z.\frac{d}{dz}{B_{i-1}}^{n-1}(z) \\
&= (-1).{B_i}^{n-1}(z) + (1-z).\frac{d}{dz}{B_i}^{n-1}(z) \\
&\quad + {B_{i-1}}^{n-1}(z) + z.\frac{d}{dz}{B_{i-1}}^{n-1}(z)
\end{aligned}$$

Unifying $\frac{d}{dz}{B_i}^n(z) = n.\left[{B_{i-1}}^{n-1}(z) - {B_i}^{n-1}(z)\right]$ and $\frac{d}{dz}{B_i}^{n-1}(z) = n.\left[{B_{i-1}}^{n-2}(z) - {B_i}^{n-2}(z)\right]$, we calculate the first derivative

$$\begin{aligned}
\frac{d}{dz}{B_i}^n(z) = (-1).{B_i}^{n-1}(z) + (1-z).\left[{B_{i-1}}^{n-2}(z) - {B_i}^{n-2}(z)\right] \\
+ {B_{i-1}}^{n-1}(z) + z.\left[{B_{i-2}}^{n-2}(z) - {B_{i-1}}^{n-2}(z)\right].
\end{aligned}$$

Theorem 2.7. *Let $z = x + \varepsilon y \in \mathbb{D}$ and for $\varepsilon \neq 0, \varepsilon^2 = 0$. Then*

$$\sum_{i=0}^{n} {B_i}^n(z) = \sum_{i=0}^{n-1} {B_i}^{n-1}(z).$$

Proof: Subsituting the Theorem (2.5), we obtain the sum of Bernstein polynomials as following;

$$\begin{aligned}
\sum_{i=0}^{n} B_i{}^{n}(z) &= \sum_{i=0}^{n} \left[(1-z).B_i{}^{n-1}(z) + z.B_{i-1}{}^{n-1}(z)\right] \\
&= (1-z)\left[\sum_{i=0}^{n} B_i{}^{n-1}(z)\right] + z.\left[\sum_{i=0}^{n} B_{i-1}{}^{n-1}(z)\right] \\
&= (1-z)\left[\sum_{i=0}^{n} B_i{}^{n-1}(z) + B_n{}^{n-1}(z)\right] + z.\left[\sum_{i=0}^{n} B_{i-1}{}^{n-1}(z) + B_{-1}{}^{n}(z)\right] \\
&= (1-z).\sum_{i=0}^{n} B_i{}^{n-1}(z) + z.\sum_{i=0}^{n-1} B_i{}^{n-1}(z) \\
&= \sum_{i=0}^{n-1} B_i{}^{n-1}(z),
\end{aligned}$$

here $B_n{}^{n-1}(z) = B_{-1}{}^{n}(z) = 0$.

Corollary 2.1. *For the dual-variable Bernstein ploynomials, the following equations*

$$\sum_{i=0}^{n} B_i{}^{n}(z) = \sum_{i=0}^{n-1} B_i{}^{n-1}(z) = \sum_{i=0}^{n-2} B_i{}^{n-2}(z) = ... = \sum_{i=0}^{1} B_i{}^{1}(z) = 1$$

are satisfied.

2.2. **The Dual-variable Bernstein Polynomials with Real and Dual Parts.** Now we will denote the dual-variable Bernstein polinomials with real and dual parts.

Theorem 2.8. *From the properties of the dual numbers we can calculate another form of the dual-variable Bernstein polinomials as*

$$B_i{}^{n}(z) = B_i{}^{n}(x) + \varepsilon y \frac{d}{dx} B_i{}^{n}(x). \tag{2.2}$$

Proof: By using the Equation to the dual-variable Bernstein polynomials, the equations

$$\begin{aligned}
B_i{}^n(z) &= \binom{n}{i} z^i.(1-z)^{n-i} \\
&= \binom{n}{i} (x+\varepsilon y)^i.[(1-x)-\varepsilon y]^{n-i} \\
&= \binom{n}{i} \left(x^i + i.x^{i-1}\varepsilon y\right)\left[(1-x)^{n-i} - (n-i).(1-x)^{n-i-1}.y.\varepsilon\right] \\
&= \binom{n}{i} \left[x^i.(1-x)^{n-i} + \varepsilon\left\{i.x^{i-1}.y.(1-x)^{n-i} - (n-i).x^i.(1-x)^{n-i-1}.y\right\}\right] \\
&= \binom{n}{i} x^i.(1-x)^{n-i} + \varepsilon y\left[i.x^{i-1}.(1-x)^{n-i} - x^i.(n-i).(1-x)^{n-i-1}\right] \\
&= \binom{n}{i} x^i.(1-x)^{n-i} + \varepsilon y\left[\frac{d}{dx}\left(x^i\right).(1-x)^{n-i} + x^i.\frac{d}{dx}(1-x)^{n-i}\right] \\
&= B_i{}^n(x) + \varepsilon y.\frac{d}{dx}B_i{}^n(x)
\end{aligned}$$

are found.

Theorem 2.9. *If we use alternative method (2.2), the derivative of the Bernstein polynomial can be obtain following equation*

$$\begin{aligned}
\frac{d}{dz}\mathrm{B}_i{}^n(z) &= n.\left[B_{i-1}{}^{n-1}(x) - B_i{}^{n-1}(x)\right] \\
&\quad +\varepsilon y\left[n.(n-1)\left(B_{i-2}{}^{n-2}(x) - 2.B_{i-1}{}^{n-2}(x) + B_i{}^{n-2}(x)\right)\right].
\end{aligned}$$

Proof: From the Equation (2.2);

$$\begin{aligned}
\frac{d}{dx}B_i{}^n(x) &= n.\left(B_{i-1}{}^{n-1}(x) - B_i{}^{n-1}(x)\right) \\
\frac{d^2}{dx^2}B_i{}^n(x) &= n.(n-1)\left(B_{i-2}{}^{n-2}(x) - 2.B_{i-1}{}^{n-2}(x) + B_i{}^{n-2}(x)\right)
\end{aligned}$$

Using the above equations, we get the first derivative of the polynomial by the following equation

$$\begin{aligned}
\frac{d}{dz}B_i{}^n(z) &= \frac{d}{dx}B_i{}^n(x) + \frac{d}{dx}\left[y\varepsilon\frac{d}{dx}B_i{}^n(x)\right] \\
&= \frac{d}{dx}B_i{}^n(x) + \varepsilon y\frac{d^2}{dx^2}B_i{}^n(x) \\
&= n.\left[B_{i-1}{}^{n-1}(x) - B_i{}^{n-1}(x)\right] \\
&\quad +\varepsilon y\left[n.(n-1)\left(B_{i-2}{}^{n-2}(x) - 2.B_{i-1}{}^{n-2}(x) + B_i{}^{n-2}(x)\right)\right]
\end{aligned}$$

Theorem 2.10. *If we use the alternative method, the derivative of the Bernstein polynomial formed by the Equation (2.2) can be obtain following equation*

$$\begin{aligned}\frac{d}{dz}\mathrm{B}_i{}^n(z) &= \frac{n!}{(n-1)!}\left[\sum_{k=\max(0,i+1-n)}^{\min(i,1)}(-1)^{k+1}\binom{1}{k}B_{i-k}{}^{n-1}(x)\right]\\ &\quad+\varepsilon y\left[\frac{n!}{(n-2)!}\left[\sum_{k=\max(0,i+2-n)}^{\min(i,2)}(-1)^{k+2}\binom{2}{k}B_{i-k}{}^{n-2}(x)\right]\right].\end{aligned}$$

Proof: The real and dual part of the $B_i{}^n(x)$ are

$$\begin{aligned}\frac{d}{dx}B_i{}^n(x) &= \frac{n!}{(n-1)!}\left[\sum_{k=\max(0,i+1-n)}^{\min(i,1)}(-1)^{k+1}\binom{1}{k}B_{i-k}{}^{n-1}(x)\right],\\ \frac{d^2}{dx^2}B_i{}^n(x) &= \frac{n!}{(n-2)!}\left[\sum_{k=\max(0,i+2-n)}^{\min(i,2)}(-1)^{k+2}\binom{2}{k}B_{i-k}{}^{n-2}(x)\right].\end{aligned}$$

Then substitution of above equations on the Eq.(2.2) we get

$$\begin{aligned}\frac{d}{dz}\mathrm{B}_i{}^n(z) &= \frac{n!}{(n-1)!}\left[\sum_{k=\max(0,i+1-n)}^{\min(i,1)}(-1)^{k+1}\binom{1}{k}B_{i-k}{}^{n-1}(x)\right]\\ &\quad+\varepsilon y\left[\frac{n!}{(n-2)!}\left[\sum_{k=\max(0,i+2-n)}^{\min(i,2)}(-1)^{k+2}\binom{2}{k}B_{i-k}{}^{n-2}(x)\right]\right].\end{aligned}$$

Theorem 2.11. *The second order derivative of $B_i{}^n(z)$ can be also written as*

$$\begin{aligned}\frac{d^2}{dz^2}\mathrm{B}_i{}^n(z) &= \frac{n!}{(n-2)!}\left[\sum_{k=\max(0,i+2-n)}^{\min(i,2)}(-1)^{k+2}\binom{2}{k}B_{i-k}{}^{n-2}(x)\right]\\ &\quad+\varepsilon y\left[\frac{n!}{(n-3)!}\left[\sum_{k=\max(0,i+3-n)}^{\min(i,3)}(-1)^{k+3}\binom{3}{k}B_{i-k}{}^{n-3}(x)\right]\right].\end{aligned}$$

Proof: The proof is obvious from Theorem (2.10).

Theorem 2.12. *The r^{th} order derivative of the dual-variable polynomial formed with $B_i{}^n(z) = B_i{}^n(x) + \varepsilon y\frac{dB_i{}^n(x)}{dx}$ is calculated by the formula*

$$\begin{aligned}\frac{d^r}{dz^r}\mathrm{B}_i{}^n(z) &= \frac{n!}{(n-r)!}\left[\sum_{k=\max(0,i+r-n)}^{\min(i,r)}(-1)^{k+r}\binom{r}{k}B_{i-k}{}^{n-r}(x)\right]\\ &+\varepsilon y\left[\frac{n!}{(n-(r+1))!}\left[\sum_{k=\max(0,i+r+1-n)}^{\min(i,r+1)}(-1)^{k+r+1}\binom{r+1}{k}B_{i-k}{}^{n-(r+1)}(x)\right]\right]\end{aligned}$$

Proof: From the induction method, the theorem can be obviously proven.

Theorem 2.13. *By taking the Eq.(2.2), written the Bernstein polynomial in a special form can be written as:*

$$\begin{aligned} {B_i}^n(z) &= (1-x).{B_i}^{n-1}(x) + x.{B_{i-1}}^{n-1}(x) \\ &+\varepsilon.y.\left[\begin{array}{l} (-1).{B_i}^{n-1}(x) + (1-x).\left[{B_{i-1}}^{n-2}(x) - {B_i}^{n-2}(x)\right] \\ +{B_{i-1}}^{n-1}(x) + z.\left[{B_{i-2}}^{n-2}(x) - {B_{i-1}}^{n-2}(x)\right] \end{array}\right]. \end{aligned}$$

Proof: If we take ${B_i}^n(x) = (1-x).{B_i}^{n-1}(x) + x.{B_{i-1}}^{n-1}(x)$ and

$$\begin{aligned} \frac{d}{dx}{B_i}^n(x) &= (-1).{B_i}^{n-1}(x) + (1-x).\left[{B_{i-1}}^{n-2}(x) - {B_i}^{n-2}(x)\right] \\ &+{B_{i-1}}^{n-1}(x) + z.\left[{B_{i-2}}^{n-2}(x) - {B_{i-1}}^{n-2}(x)\right], \end{aligned}$$

then by substitution above equations on the Eq.(2.2), we obtain

$$\begin{aligned} {B_i}^n(z) &= {B_i}^n(x) + y\frac{d{B_i}^n(x)}{dx}\varepsilon \\ &= (1-x).{B_i}^{n-1}(x) + x.{B_{i-1}}^{n-1}(x) \\ &+y.\varepsilon.\left[\begin{array}{l} (-1).{B_i}^{n-1}(x) + (1-x).\left[{B_{i-1}}^{n-2}(x) - {B_i}^{n-2}(x)\right] \\ +{B_{i-1}}^{n-1}(x) + z.\left[{B_{i-2}}^{n-2}(x) - {B_{i-1}}^{n-2}(x)\right] \end{array}\right]. \end{aligned}$$

Theorem 2.14. $\sum\limits_{i=0}^{n} {B_i}^n(z) = \sum\limits_{i=0}^{n-1} {B_i}^{n-1}(x) + \varepsilon.y.\sum\limits_{i=0}^{n-1} \frac{d{B_i}^{n-1}(x)}{dx}$

Proof: For $i = 1, 2, ..., n$ the sum of ${B_i}^n(z)$ is

$$\begin{aligned} \sum_{i=0}^{n} {B_i}^n(z) &= \sum_{i=0}^{n} {B_i}^n(x) + \sum_{i=0}^{n} \varepsilon y \frac{d{B_i}^n(x)}{dx} \\ &= \sum_{i=0}^{n} {B_i}^n(x) + \varepsilon y \sum_{i=0}^{n} \frac{d{B_i}^n(x)}{dx} \\ &= \sum_{i=0}^{n} {B_i}^n(x) + \varepsilon y \frac{d\sum\limits_{i=0}^{n} {B_i}^n(x)}{dx} \\ &= \sum_{i=0}^{n-1} {B_i}^{n-1}(x) + \varepsilon y \frac{d\left(\sum\limits_{i=0}^{n-1} {B_i}^{n-1}(x)\right)}{dx} \\ &= \sum_{i=0}^{n-1} {B_i}^{n-1}(x) + \varepsilon.y.\sum_{i=0}^{n-1} \frac{d{B_i}^{n-1}(x)}{dx}, \end{aligned}$$

here $\sum\limits_{i=0}^{n} {B_i}^n(x) = \sum\limits_{i=0}^{n-1} {B_i}^{n-1}(x)$.

Theorem 2.15. *The multiplication of two dual-variable Bernstein polynomials is*

$$\begin{aligned} {B_i}^n(z).{B_k}^m(z) &= \frac{\begin{pmatrix} n \\ i \end{pmatrix}.\begin{pmatrix} m \\ k \end{pmatrix}}{\begin{pmatrix} n+m \\ i+k \end{pmatrix}} {B_{i+k}}^{n+m}(x) \\ + \quad & \varepsilon y\left[n.\left({B_{i-1}}^{n-1}(x) - {B_i}^{n-1}(x)\right){B_k}^m(x) + {B_i}^n(x).\left({B_{k-1}}^{m-1}(x) - {B_k}^{m-1}(x)\right)\right] \end{aligned}$$

Proof: By using the multiplication properties of the dual-variable functions, we get

$$B_i{}^n(z).B_k{}^m(z) = B_i{}^n(x).B_k{}^m(x) + \varepsilon y \frac{dB_i{}^n(x)\,B_k{}^m(x)}{dx}$$

$$= \frac{\binom{n}{i}.\binom{m}{k}}{\binom{n+m}{i+k}} B_{i+k}{}^{n+m}(x) + \varepsilon y\left[\frac{d}{dx}B_i{}^n(x).B_k{}^m(x) + B_i{}^n(x).\frac{d}{dx}B_k{}^m(x)\right]$$

$$= \frac{\binom{n}{i}.\binom{m}{k}}{\binom{n+m}{i+k}} B_{i+k}{}^{n+m}(x)$$

$$+ \quad \varepsilon y\left[n.\left(B_{i-1}{}^{n-1}(x) - B_i{}^{n-1}(x)\right)B_k{}^m(x) + B_i{}^n(x).\left(B_{k-1}{}^{m-1}(x) - B_k{}^{m-1}(x)\right)\right].$$

3. Conclusion

In our study, we defined a generalized dual-variable Bernstein polynomials in the dual space, and calculated many usefull properties of the dual-variable Bernstein polynomials. In the future, the concept of the dual-variable Bernstein polynomials will be lead a new kind of the Bezier curves which can be called "the Bezier curve with dual-variable Bernstein basis".

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