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A DIFFERENT VIEWPOINT ABOUT THE WEAK CONVERGENCE VIA IDEALS AND Δ^m SEQUENCES

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Abstract. In this study, we use generalized difference sequences $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ to obtain more general results about weak convergence and we investigate the concept of $\Delta^m\mathcal{I}-$weak convergence where $m \in \mathbb{N}$. We also define weak $\Delta^m\mathcal{I}-$limit points and weak $\Delta^m\mathcal{I}-$cluster points.

1. Introduction

In this part, we give a short literature data about our basic concepts difference sequences, $\mathcal{I}-$convergence and weak convergence. Difference sequences have defined in 1981 by Kızmaz [19] and he has defined $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ spaces where l_∞, c and c_0 are bounded, convergent and null sequence spaces, respectively. He obtained some relations between these spaces for example $c_0(\Delta) \subseteq c(\Delta) \subseteq l_\infty(\Delta)$.

Following these definitions, Et [9] , Et and Çolak [10] , Et and Başarır [11] , Et and Nuray [12] , Gümüş and Nuray [17] , Aydın and Başar [1] , Başarır [2] , Bektaş et. al. [3] , Et and Eşi [13] , Savaş [25], Dems [7], Dündar and Çakan [8], Nabiev et. al. [22] and many others searched various properties of this concept. Et and Çolak [10] generalized Kızmaz's results for Δ^m sequences such that,

$$\begin{aligned} c_0(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c_0\} \\ c(\Delta^m) &= \{x = (x_k) : \Delta^m x \in c\} \\ l_\infty(\Delta^m) &= \{x = (x_k) : \Delta^m x \in l_\infty\} \end{aligned}$$

where $m \in \mathbb{N}$ and $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1})$ that is $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$. They proved that these spaces are Banach spaces with the norm

$$\|.\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty .$$

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Now, lets talk about the concept of $\mathcal{I}-$convergence shortly and give some basic definitons.

The idea of $\mathcal{I}-$convergence for single sequences was introduced by Kostyrko, Salat and Wilezyński [21]. We can say that the concept is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. $\mathcal{I}-$convergence of real sequences coincides with the ordinary convergence if $\mathcal{I}$ is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if $\mathcal{I}$ is the ideal of subsets of $\mathbb{N}$ of natural density zero. Nowadays, it has become one of the most active areas of research in classical analysis. Savaş and Das defined generalized statistical convergence via ideals [26] .

We first need to recall the definitions of some other notions.

Definition 1.1. [21] A non-empty set $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal on $\mathbb{N}$ if;

(i) $B \in \mathcal{I}$ whenever $B \subseteq A$ for some $A \in \mathcal{I}$ (closed unders subsets).
(ii) $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ (closed under unions).

An ideal is called proper if $\mathbb{N} \notin \mathcal{I}$ and is called admissible if it is proper and contains all finite subsets.

Many concepts mentioned in this exposition are more frequently defined using limit along a filter. Filter is a dual notion of ideal.

Definition 1.2. [21] A non-empty set $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter on $\mathbb{N}$ if;

(i) $B \in \mathcal{F}$ whenever $B \supseteq A$ for some $A \in \mathcal{F}$ (closed unders supersets).
(ii) $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ (closed under intersections).

Proposition 1.1. $\{\mathbb{N} \backslash A : A \in \mathcal{I}\}$ *is a filter if and only if* $\mathcal{I}$ *is an ideal.*

Remark 1.1. Generally we will use ideals in our proofs but if the notion is more familiar for filters, we will use the notion of filter.

Definition 1.3. [21] Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a proper ideal on $\mathbb{N}$. The real sequence $x = (x_k)$ is said to be $\mathcal{I}-$convergent to $x \in \mathbb{R}$ provided that for each $\varepsilon > 0$,

$$A(\varepsilon) = \{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\} \in \mathcal{I}.$$

There are lots of examples about $\mathcal{I}-$convergence in Kostyrko, Salát and Wilezyński's paper. We just want to give some well known examples.

Example 1.1. If $\mathcal{I} = \mathcal{I}_f$=$\{A \subseteq \mathbb{N} : A \text{ is finite}\}$ then l_f-convergence gives the usual convergence.

Example 1.2. If $\mathcal{I} = \mathcal{I}_\delta$ =$\{A \subseteq \mathbb{N} : \delta(A\ ) = 0\}$ then $l_\delta-$convergence gives the statistical convergence.

Et and Nuray [12] have introduced the Δ^m-statistical convergence in their study and the set of all Δ^m-statistical convergent sequences was denoted by $S(\Delta^m)$. Following this study, Gümüş and Nuray [17] have extended Δ^m-statistical convergence to Δ^m-ideal convergence.

Definition 1.4. [17] Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a proper ideal on $\mathbb{N}$. The real sequence $x = (x_k)$ is said to be Δ^m-ideal convergent to $x \in \mathbb{R}$ provided that for each $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |\Delta^m x_k - x| \geq \varepsilon\} \in \mathcal{I}.$$

The set of all Δ^m-ideal convergent sequences is denoted by $c_{\mathcal{I}}(\Delta^m)$.

Example 1.3. If $\mathcal{I} = \mathcal{I}_f$ then $c_{\mathcal{I}_f}(\Delta^m) = c(\Delta^m)$.

Example 1.4. $\mathcal{I} = \mathcal{I}_\delta$ then $c_{\mathcal{I}_\delta}(\Delta^m) = S(\Delta^m)$.

Now, we need to recall some definitions about weak convergence.

Definition 1.5. [4] Let B be a Banach space, (x_k) be a $B-$valued sequence and $x \in B$. The sequence (x_k) is weakly convergent to x provided that for any f in the continuous dual B^* of B,

$$\lim_k f(x_k - x) = 0.$$

In this case we write $W - \lim x_k = x$.

Let B be a Banach space, (x_k) be a $B-$valued sequence and $x \in B$. The sequence (x_k) is weakly C_1-convergent to x provided that for any f in the continuous dual B^* of B,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} f(x_k - x) = 0.$$

In 2000, Connor et al. [5] , have introduced a new concept of weak statistical convergence and have characterized Banach spaces with seperable duals via statistical convergence. Bhardwaj and Bala studied about weak statistical convergence [4]. Pehlivan and Karaev [24] have also used the idea of weak statistical convergence in strengthening a result of Gokhberg and Klein on compact operators.

Following Connor et al. we define weak statistical convergence as follows:

Definition 1.6. [5] Let B be a Banach space, (x_k) be a $B-$valued sequence and $x \in B$. The sequence (x_k) is weakly statistically convergent to x provided that for any f in the continuous dual B^* of B the sequence $(f(x_k - x))$ is statistically convergent to x i.e.

$$\lim_n \frac{1}{n} \left|\{k \leq n : |f(x_k - x)| \geq \varepsilon\}\right| = 0.$$

In this case we write $W - st - \lim x_k = x$.

In 2011, Nuray [23] has defined the weak $\mathcal{I}-$convergence as follows and has defined the set of all weak $\mathcal{I}-$convergent sequences by $W\mathcal{I}$.

Definition 1.7. [23] Let B be a Banach space, (x_k) be a $B-$valued sequence and $x \in B$. The sequence (x_k) is weak $\mathcal{I}-$convergent to x provided that for any f in the continuous dual B^* of B the sequence $(f(x_k - x))$ is weak $\mathcal{I}-$convergent to x that is,

$$\{k \in \mathbb{N} : |f(x_k - x)| \geq \varepsilon\} \in \mathcal{I}.$$

Taking the above examples, if $\mathcal{I} = \mathcal{I}_f$ then we have the usual weak convergence and if $I = I_f$ then weak I_f-convergence gives the usual weak convergence. After the definition of weak $\mathcal{I}-$convergence Gümüş has defined the weak $\mathcal{I}-$statistical convergence [18] .

2. Weak $\Delta^m\mathcal{I}-$Convergence

In this section, we define weak $\Delta^m\mathcal{I}-$convergence and we give some inclusion theorems. In our all subsequent definitions, let B be a Banach space, $(\Delta^m x_k)$ be a $B-$valued sequence, $x \in B$ and $\mathcal{I}$ be an admissible ideal.

Definition 2.1. The sequence (x_k) is weak Δ^m-convergent to x provided that for any f in the continuous dual B^* of B,

$$\lim_k f(\Delta^m x_k - x) = 0.$$

The set of all weak Δ^m-convergent sequences is denoted by $Wc(\Delta^m)$.

Definition 2.2. The sequence (x_k) is weak Δ^m-statistically convergent to x provided that for any f in the continuous dual B^* of B and every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |f(\Delta^m x_k - x)| \geq \varepsilon\}| = 0.$$

The set of all weak Δ^m-statistically convergent sequences is denoted by $WS(\Delta^m)$.

Definition 2.3. The sequence (x_k) is weak $\Delta^m\mathcal{I}-$convergent to x provided that for any f in the continuous dual B^* of B and every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |f(\Delta^m x_k - x)| \geq \varepsilon\} \in \mathcal{I}.$$

In this case we write $x_k \to x(Wc_{\mathcal{I}}(\Delta^m))$. The set of all weak $\Delta^m\mathcal{I}-$convergent sequences is denoted by $Wc_{\mathcal{I}}(\Delta^m)$.

Example 2.1. $Wc_{\mathcal{I}_f}(\Delta^m) = Wc(\Delta^m)$.

Example 2.2. $Wc_{\mathcal{I}_\delta}(\Delta^m) = WS(\Delta^m)$.

After the above definitions, lets give a main theorem which explains the relation between weak Δ^m-convergence and weak $\Delta^m\mathcal{I}-$convergence.

Theorem 2.1. *Let (x_k) is weak Δ^m-convergent to x. Then, (x_k) is weak $\Delta^m\mathcal{I}-$convergent to x.*

Proof. Let (x_k) is weak Δ^m-convergent to x. It means $f(\Delta^m x_k)$ is convergent to $f(x)$ for all $f \in B^*$. Then, $f(\Delta^m x_k)$ is $\mathcal{I}-$convergent to $f(x)$ that is, $(x_k) \in Wc_{\mathcal{I}}(\Delta^m)$. □

We give the following example to show that the inverse of this theorem is not generally true.

Example 2.3. Let $(f(\Delta^m x_k)) = \begin{cases} 1, & n \text{ is square} \\ 0, & \text{otherwise} \end{cases}$.

Then, $(x_k) \in Wc_{\mathcal{I}_\delta}(\Delta^m)$ but $(x_k) \notin Wc(\Delta^m)$.

Before the following theorem, reader should be warned at this point that, from the $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$ formula, we can easily prove that $\Delta^m(x_k + y_k) = \Delta^m(x_k) + \Delta^m(x_k)$ and $\Delta^m(\lambda x_k) = \lambda \Delta^m(x_k)$.

Theorem 2.2. *Let $\mathcal{I}$ be an admissible ideal, $(\Delta^m x_k)$ and $(\Delta^m y_k)$ be $B-$valued sequences and $x, y \in B$.*

(*i*) $x_k \to x(Wc_{\mathcal{I}}(\Delta^m))$ and $y_k \to y(Wc_{\mathcal{I}}(\Delta^m))$ then $x_k + y_k \to x + y(Wc_{\mathcal{I}}(\Delta^m))$.
(*ii*) $x_k \to x(Wc_{\mathcal{I}}(\Delta^m))$ and $\lambda \in \mathbb{R}$ then $\lambda x_k \to \lambda x(Wc_{\mathcal{I}}(\Delta^m))$.

Proof. (*i*) Assume that $x_k \to x(Wc_{\mathcal{I}}(\Delta^m))$ and $y_k \to y(Wc_{\mathcal{I}}(\Delta^m))$. Lets define the sets A_1 and A_2 such that,

$$A_1 = \left\{k \in \mathbb{N} : |f(\Delta^m x_k - x)| < \frac{\varepsilon}{2}\right\}$$

and

$$A_2 = \left\{ k \in \mathbb{N} : |f(\Delta^m y_k - y)| < \frac{\varepsilon}{2} \right\}.$$

It is obvious that A_1 and A_2 are in $\mathcal{F}(\mathcal{I})$. If we remember the properties of the filter, $A_1 \cap A_2 \in \mathcal{F}(\mathcal{I})$ and $A_1 \cap A_2 \neq \varnothing$. Since $f \in B^*$, for all $k \in A_1 \cap A_2$,

$$\begin{array}{rcl} |f(\Delta^m(x_k + y_k) - (x + y))| & = & |f(\Delta^m x_k - x) + f(\Delta^m y_k - y)| \\ & \leq & |f(\Delta^m x_k - x)| + |f(\Delta^m y_k - y)| \\ & < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ & = & \varepsilon. \end{array}$$

It proves (i).

(ii) Let $x_k \to x(Wc_{\mathcal{I}}(\Delta^m))$ and $\lambda \in \mathbb{R}$. Using the same technique, for all $k \in A_1$ and every $\varepsilon > 0$

$$\begin{array}{rcl} |f(\Delta^m(\lambda x_k) - \lambda x)| & = & |f(\lambda \Delta^m(x_k - x)| \\ & = & |\lambda| \, |f(\Delta^m x_k - x)| \\ & < & |\lambda| \, \frac{\varepsilon}{2}. \end{array}$$

As $\varepsilon > 0$ is arbitrary, it follows that $\{k \in \mathbb{N} : |f(\Delta^m(\lambda x_k) - \lambda x)| < \eta\} \in \mathcal{F}(\mathcal{I})$ for any $\eta > 0$. Then, we have the proof. □

Remark 2.1. Since $\Delta^m(x_k.y_k) \neq \Delta^m(x_k).\Delta^m(x_k)$, we can not say that $x_k.y_k \to x.y(Wc_{\mathcal{I}}(\Delta^m))$ when $x_k \to x(Wc_{\mathcal{I}}(\Delta^m))$ and $y_k \to y(Wc_{\mathcal{I}}(\Delta^m))$.

Definition 2.4. Let $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$. If,

$$\{k + 1 : k \in A\} \in \mathcal{I}$$

for any $A \in \mathcal{I}$, then $\mathcal{I}$ is said to be translation invariant ideal.

Example 2.4. $\mathcal{I}_\delta$ is a translation invariant ideal.

Corollary 2.1. *If $\mathcal{I}$ is translation invariant and $(x_k) \in Wc_{\mathcal{I}}(\Delta^m)$ then $(x_{k+1}) \in Wc_{\mathcal{I}}(\Delta^m)$.*

Proposition 2.1. *Suppose that $\mathcal{I}$ is an admissible translation invariant ideal and $m \in \mathbb{N}$. Then,*

$$Wc_{\mathcal{I}}(\Delta^{m-1}) \subseteq Wc_{\mathcal{I}}(\Delta^m).$$

Proof. Suppose that $x \in Wc_{\mathcal{I}}(\Delta^{m-1})$ and it means $\left(\Delta^{m-1} x_k\right) \in Wc_{\mathcal{I}}$. Since $\mathcal{I}$ is translation invariant we have $\left(\Delta^{m-1} x_{k+1}\right) \in Wc_{\mathcal{I}}$. From the definition of difference sequences we can write

$$(\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right).$$

Then we obtain $(\Delta^m x_k) \in Wc_{\mathcal{I}}$ i.e. $x \in Wc_{\mathcal{I}}(\Delta^m)$. □

Theorem 2.3. *Let $\mathcal{I}$ be a proper ideal in $\mathbb{N}$. If there is a weak $\Delta^m \mathcal{I}-$convergent sequence y such that,*

$$\{k \in \mathbb{N} : f(\Delta^m x_k) \neq f(\Delta^m y_k)\} \in \mathcal{I}$$

then x is also weak $\Delta^m \mathcal{I}-$convergent.

Proof. Assume that $\{k \in \mathbb{N} : f(\Delta^m x_k) \neq f(\Delta^m y_k)\} \in \mathcal{I}$ and y is weak $\Delta^m\mathcal{I}-$convergent to x. For each $\varepsilon > 0$,

$$\begin{array}{rcl} \{k \in \mathbb{N} : |f(\Delta^m x_k - x)| \geq \varepsilon\} & \subseteq & \{k \in \mathbb{N} : f(\Delta^m x_k) \neq f(\Delta^m y_k)\} \\ & \cup & \{k \in \mathbb{N} : |f(\Delta^m y_k - x)| \geq \varepsilon\} \end{array}$$

As the right hand side of inclusion is in ideal, we have that

$$\{k \in \mathbb{N} : |f(\Delta^m x_k - x)| \geq \varepsilon\} \in \mathcal{I}.$$

□

Definition 2.5. Let $\mathcal{I}$ be a proper ideal in $\mathbb{N}$. For each $\varepsilon > 0$ there is a number $n_0(\varepsilon)$ such that $\{k \in \mathbb{N} : |f(\Delta^m x_k - \Delta^m x_{n_0})| \geq \varepsilon\} \in \mathcal{I}$ then, x is called by weak $\Delta^m\mathcal{I}-$Cauchy sequence.

Theorem 2.4. *If x is weak $\Delta^m\mathcal{I}-$convergent sequence then, x is weak $\Delta^m\mathcal{I}-$Cauchy sequence.*

Proof. Suppose that x is weak $\Delta^m\mathcal{I}-$convergent and $\varepsilon > 0$. Then,

$$A = \left\{k \in \mathbb{N} : |f(\Delta^m x_k - x)| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}).$$

Lets choose $n_0 \in A$. In this case, $|f(\Delta^m x_k - x)| < \frac{\varepsilon}{2}$. We can write,

$$\begin{array}{rcl} |f(\Delta^m x_k - \Delta^m x_{n_0})| & < & |f(\Delta^m x_k - x)| + |f(\Delta^m x_{n_0} - x)| \\ & < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{array}$$

Then we have the proof. □

3. Weak $\Delta^m\mathcal{I}^*-$Convergence

In this part, we define weak $\Delta^m\mathcal{I}^*-$convergence and we will investigate the inclusion with weak $\Delta^m\mathcal{I}-$convergence.

Definition 3.1. Let B be a Banach space, $(\Delta^m x_k)$ be $B-$valued sequence and $x \in B$. The sequence (x_k) is weak $\Delta^m\mathcal{I}^*-$convergent to x if and only if for any f in the continuous dual B^* of B, there exists a set $M = \{n_1 < n_2 < ... < n_k < ...\} \subseteq \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that $\lim_k f(\Delta^m x_{n_k} - x) = 0$. $Wc_{\mathcal{I}^*}(\Delta^m)$ denotes the set of all weak $\Delta^m\mathcal{I}^*-$convergent sequences.

Theorem 3.1. *Let $\mathcal{I}$ be an admissible ideal. If (x_k) is weak $\Delta^m\mathcal{I}^*-$convergent to x then (x_k) is weak $\Delta^m\mathcal{I}-$convergent to x.*

Proof. By assumption there is a set $D \in \mathcal{I}$ such that

$$M = \mathbb{N} \backslash D = \{n_1 < n_2 < ... < n_k < ...\}$$

and we have

$$\lim_k f(\Delta^m x_{n_k} - x) = 0$$

Let $\varepsilon > 0$. From the definition of limit, there exists $k_0 \in \mathbb{N}$ such that $|f(\Delta^m x_{n_k} - x)| < \varepsilon$ for each $k > k_0$. Since $\mathcal{I}$ is admissible,

$$\{k \in \mathbb{N} : |f(\Delta^m x_{n_k} - x)| \geq \varepsilon\} \subset D \cup \{n_1 < n_2 < ... < n_{k_0}\} \in \mathcal{I}.$$

□

To say that the inverse of the theorem satisfies, we need to remind the concept of (AP) property.

Definition 3.2. An admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Theorem 3.2. *Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}$ has property (AP), then we say that if (x_k) is weak $\Delta^m\mathcal{I}-$convergent to x then (x_k) is weak $\Delta^m\mathcal{I}^*-$convergent to x.*

Proof. Suppose that $\mathcal{I}$ satisfies condition (AP) and $(x_k) \in Wc_{\mathcal{I}}(\Delta^m)$. Then for every $\varepsilon > 0$, $\{k \in \mathbb{N} : |f(\Delta^m x_k - x)| \geq \varepsilon\} \in \mathcal{I}$. Put

$$A_1 = \{k \in \mathbb{N} : |f(\Delta^m x_k - x)| \geq 1\}$$

and

$$A_k = \left\{k \in \mathbb{N} : \frac{1}{k} \leq |f(\Delta^m x_k - x)| \leq \frac{1}{k-1}\right\}$$

for $k \geq 2$, $k \in \mathbb{N}$. Obviously $A_j \cap B_j = \varnothing$ for $i \neq j$. By condition (AP), there exists a sequence of sets $(B_k)_{k\in\mathbb{N}}$ such that $A_j \Delta B_j$ are finite sets for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. It is sufficient to prove that for $M = \mathbb{N}\backslash B$ we have $\lim_{\substack{k\to\infty \\ k\in M}} f(\Delta^m x_k - x) = 0$. Let $\eta > 0$ and choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \eta$. Then,

$$\{k \in \mathbb{N} : |f(\Delta^m x_k - x)| \geq \eta\} \subset \bigcup_{j=1}^{n+1} A_j.$$

Since $A_j \Delta B_j$ $(j = 1, 2, ..., n+1)$ are finite sets there exists $k_0 \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{n+1} B_j \cap \{k \in \mathbb{N} : k > k_0\} = \bigcup_{j=1}^{n+1} A_j \cap \{k \in \mathbb{N} : k > k_0\}. \tag{3.1}$$

If $k > k_0$ and $k \notin B$, then $k \notin \bigcup_{j=1}^{n+1} B_j$ and by (3.1), $k \notin \bigcup_{j=1}^{n+1} A_j$. But then, $|f(\Delta^m x_k - x)| < \frac{1}{k+1} < \eta$; so we have the proof. □

4. Weak $\Delta^m\mathcal{I}-$Limit Points And Weak $\Delta^m\mathcal{I}-$Cluster Points

The notion of limit is one of the central notions in mathematical analysis. It was generalized by mathematicians in various ways. After identification statistical convergence by Fast [14], the question was how to define the statistical limit points and statistical cluster points. Fridy [15] answered this question and he defined these concepts. Later, these concepts were also identified for ideals. Demirci [6] and Koystro et. al. [20] studied about $I-$convergence and extremal $I-$limit points. Talo and Dndar [27] investigated these concepts for fuzzy numbers. Nuray [23] combined these concepts with weak convergence and he defined weak $\mathcal{I}-$limit points and weak $\mathcal{I}-$cluster points.

Definition 4.1. Let B be a Banach space, $(\Delta^m x_k)$ be a $B-$valued sequence and $\lambda \in B$. Let f in the continuous dual B^* of B. λ is said to be a weak $\Delta^m\mathcal{I}-$limit point of (x_k) provided that there exists a set $M = \{n_1 < n_2 < ... < n_k < ...\} \subseteq \mathbb{N}$

such that $M \notin \mathcal{I}$ and $\lim_k f(\Delta^m x_{n_k} - x) = 0$. The set of all weak $\Delta^m\mathcal{I}-$limit points denoted by $W\Delta^m\mathcal{I}(\Lambda_x)$.

Example 4.1. Let $\mathcal{I} = \mathcal{I}_\delta$ and $(f(\Delta^m x_k)) = \begin{cases} 1, & \text{if } k \text{ is square} \\ 0, & \text{otherwise} \end{cases}$.

Then $W\Delta^m\mathcal{I}(\Lambda_x) = \{0\}$.

Definition 4.2. Let B be a Banach space, $(\Delta^m x_k)$ be a $B-$valued sequence and $\gamma \in B$. Let f in the continuous dual B^* of B. γ is said to be a weak $\Delta^m\mathcal{I}-$cluster point of (x_k) if and only if for each $\varepsilon > 0$ we have

$$\{k \in \mathbb{N} : |f(\Delta^m x_k - \gamma)| < \varepsilon\} \notin \mathcal{I}.$$

The set of all weak $\Delta^m\mathcal{I}-$cluster points denoted by $W\Delta^m\mathcal{I}(\Gamma_x)$.

Proposition 4.1. *If x is a weak $\Delta^m\mathcal{I}-$cluster point of (x_k), then there is an ideal $\mathcal{I}$ such that (x_k) is weak $\Delta^m\mathcal{I}-$convergent to x.*

Theorem 4.1. *Let $\mathcal{I}$ be an admissible ideal. Then for each sequence $(\Delta^m x_k) \in B$ we have $W\Delta^m\mathcal{I}(\Lambda_x) \subseteq W\Delta^m\mathcal{I}(\Gamma_x)$.*

Proof. Assume that $\lambda \in W\Delta^m\mathcal{I}(\Lambda_x)$. Then, there exists a set

$$M = \{n_1 < n_2 < ... < n_k < ...\} \notin \mathcal{I}$$

such that

$$\lim_k f(\Delta^m x_{n_k} - x) = 0.$$

From the definition of usual convergence, for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have $|f(\Delta^m x_{n_k} - \lambda)| < \varepsilon$. Hence,

$$\{k \in \mathbb{N} : |f(\Delta^m x_k - \lambda)| < \varepsilon\} \supset M \backslash \{n_1 < n_2 < ... < n_{k_0}\}$$

and

$$\{k \in \mathbb{N} : |f(\Delta^m x_k - \lambda)| < \varepsilon\} \notin \mathcal{I}.$$

It means that $\lambda \in W\Delta^m\mathcal{I}(\Gamma_x)$. □

Proposition 4.2. *If the sequence (x_k) is $\Delta^m\mathcal{I}-$convergent to λ. Then,*

$$W\Delta^m\mathcal{I}(\Lambda_x) = W\Delta^m\mathcal{I}(\Gamma_x) = \{\lambda\}.$$

The inverse of this proposition is not generally true.

Example 4.2. Let $(f(\Delta^m x_k)) = (1+(-1)^k)$. Then, $W\Delta^m\mathcal{I}(\Lambda_x) = W\Delta^m\mathcal{I}(\Gamma_x) = \{0\}$ but (x_k) is not $\Delta^m\mathcal{I}-$convergent to 0.

Proposition 4.3. *Let $(\Delta^m x_k)$ and $(\Delta^m y_k)$ sequences satisfies*

$$\{k \in \mathbb{N} : (\Delta^m x_k) = (\Delta^m y_k)\} \notin \mathcal{I}.$$

Then,

$$W\Delta^m\mathcal{I}(\Lambda_x) = W\Delta^m\mathcal{I}(\Lambda_y) \text{ and } W\Delta^m\mathcal{I}(\Gamma_x) = W\Delta^m\mathcal{I}(\Gamma_y).$$

Proof. Suppose that $\{k \in \mathbb{N} : (\Delta^m x_k) = (\Delta^m y_k)\} \notin \mathcal{I}$ and $\lambda \in W\Delta^m\mathcal{I}(\Lambda_x)$. There is a set M such that,

$$\lim_{k\to\infty} \Delta^m x_{n_k} = \lambda \text{ and } M = \{n_1 < n_2 < ... < n_k < ...\} \notin \mathcal{I}.$$

From our assumption, this set defines a sequence such that

$$\lim_{l\to\infty} \Delta^m y_{m_l} = \lambda.$$

So $\lambda \in W\Delta^m\mathcal{I}(\Lambda_y)$. Using the same techniques we obtain $W\Delta^m\mathcal{I}(\Lambda_y) \subseteq W\Delta^m\mathcal{I}(\Lambda_x)$.

Now, we will prove the same property for cluster points. Let $\gamma \in W\Delta^m\mathcal{I}(\Gamma_x)$ then $\{k \in \mathbb{N} : |f(\Delta^m x_k - \gamma)| < \varepsilon\} \notin \mathcal{I}$. Hence,

$$\begin{aligned} \{k \in \mathbb{N} : |f(\Delta^m y_k - \gamma)| < \varepsilon\} \quad &\supseteq \quad \{k \in \mathbb{N} : (\Delta^m x_k) = (\Delta^m y_k)\} \\ &\cap \quad \{k \in \mathbb{N} : |f(\Delta^m x_k - \gamma)| < \varepsilon\} \end{aligned}$$

Since the right hand does not belong to $\mathcal{I}$, we have $\{k \in \mathbb{N} : |f(\Delta^m y_k - \gamma)| < \varepsilon\} \notin \mathcal{I}$ and it means $\gamma \in W\Delta^m\mathcal{I}(\Gamma_y)$. Using the same techniques we obtain $W\Delta^m\mathcal{I}(\Gamma_y) \subseteq W\Delta^m\mathcal{I}(\Gamma_x)$. □

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