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Research Article

Topological fundamental groupoids: Brown's topology

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Abstract

In this paper, we generalize the Brown's topology on the fundamental groupoids. For a locally path connected space X and a totally disconnected normal subgroupoid M of πX, we define a topology on the quotient groupoid $\frac{\pi X}{M}$ which is a generalization of what introduced by Brown for locally path connected and semilocally simply connected spaces. We prove that $\frac{\pi X}{M}$ equipped with this topology is a topological groupoid. Also, we will find a class of subgroupoids of topological groupoids whose their related quotient groupoids will be topological groupoids. By using this, we show that our topology on $\frac{\pi X}{M}$ is equivalent to the quotient of the Lasso topology on the topological fundamental groupoids, $\frac{\pi^L X}{M}$.

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1. Introduction

Brown and Danesh-Naruie [2] have defined the "lifted topology" on the quotient groupoid $\frac{\pi X}{M}$, where X is a loclly path connected and semilocally simply connected space and M is a totally disconnected normal subgroupoid of πX. By this topology that we call it Brown's topology, $\frac{\pi X}{M}$ becomes a locally trivial topological groupoid with discrete object groups.

Out of the category of locally nice spaces, the authors have determined the Lasso topology on the fundamental groupoid that makes it a topological groupoid [11]. Lasso topology on the fundamental groupoid is a generalization of the Lasso topology on the fundamental group and the universal path space [4, 9].

Although quotients of topological groups by normal subgroups are topological groups, but this has not yet been proven for topological groupoid and of course there is no counterexample [5, 7]. It is notable that in [5, 7] the normal subgroupoid is not necessarily

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totally disconnected. The problem that appears in the proof is that for a normal subgroup N of a topological group G, the quotient map $q : G \longrightarrow \frac{G}{N}$ is an open map, but this fails for topological groupoids. So we can not directly conclude that $\frac{\pi^L X}{M}$ is a topological groupoid.

Here, for a locally path connected space X and a totally disconnected normal subgroupoid M of πX we generalize the Brown's topology on the quotient groupoid $\frac{\pi X}{M}$. More precisely, by using spanier groups of open covers of X, we introduce a topological basis for a topology on the $\frac{\pi X}{M}$ such that makes it a topological groupoid. Then by reconstructing the basis of the Brown's topology on the $\frac{\pi X}{M}$, we will show that for semilocally simply connected spaces, these topologies are equivalent.

In the sequel, by proving that quotients of topological groupoids by totally disconnected normal subgroupoids are topological groupoids, we show that the quotient of Lasso topology on $\frac{\pi X}{M}$ makes it a topological groupoid. Finally, we will prove that these topologies on $\frac{\pi X}{M}$ are equivalent.

2. Preliminaries

A groupoid G over G_0, denoted by (G, G_0), consists of a set of arrows G and a set of objects G_0, together with two maps $S, T : G \longrightarrow G_0$, called respectively the source and target maps, a map $1 : G_0 \longrightarrow G; x \mapsto 1_x$, called the unit map, a map $i : G \longrightarrow G; a \longmapsto a^{-1}$, called the inverse map and a map $m : G_2 \longrightarrow G; (a; b) \mapsto m(a; b) = ab$, called the composition map, where G_2 denotes the set of composable arrows: $G_2 = \{(a; b) \in G \times G |\ S(b) = T(a)\}$.

These structure maps satisfy the following conditions:

(i) $S(ab) = S(a)$ and $T(ab) = T(b)$ for all $(a; b) \in G_2$,

(ii) $a(bc) = (ab)c$ for all $a, b, c \in G$ such that $S(b) = T(a)$ and $S(c) = T(b)$,

(iii) $S(1_x) = T(1_x) = x$ for all $x \in G_0$,

(iv) $a1_{T(a)} = a$ and $1_{S(a)}a = a$ for all $a \in G$,

(v) each $a \in G$ has a two-sided inverse a^{-1} such that $S(a^{-1}) = T(a)$, $T(a^{-1}) = S(a)$ and $aa^{-1} = 1_{S(a)}$; $a^{-1}a = 1_{T(a)}$.

The set of arrows from x to y is denoted by $G(x, y)$ and, in particular, $G(x) := G(x, x)$ is called the object group (or vertex group) at x. Also, we denote $S^{-1}(x)$ by G_x and $T^{-1}(x)$ by G^x.

Definition 2.1. ([7]) A *topological groupoid* is a groupoid G together with topologies on G and G_0 such that the structure maps are continuous.

Definition 2.2. ([1]) Let G and G' be two groupoids with object sets G_0 and G'_0, respectively. A groupoid homomorphism is a pair (F, f) of maps $F : G \longrightarrow G'$ and $f : G_0 \longrightarrow G'_0$ which send each object x of G to an object $f(x)$ of G' and each arrow $a \in G(x, y)$ to an arrow $F(a) \in G'(f(x), f(y))$, respectively, such that $F(ab) = F(a)F(b)$ for every $(a, b) \in G_2$, $S \circ F = f \circ S$ and $T \circ F = f \circ T$.

Also, a topological groupoid homomorphism is a groupoid homomorphism which is continuous on both objects and arrows. For each $a \in G(x, y)$ the right translation $R_a : G^x \longrightarrow G^y$, defined by $R_a(b) = ba$ and the left translation $L_a : G_y \longrightarrow G_x$, defined by $L_a(b) = ab$ are homeomorphism.

Let X be a topological space. The fundamental groupoid πX has homotopy classes of paths in X as the set of morphisms and has the set X as its set of objects, and for any $x, y \in X$ the set $\pi X(x, y)$ is the set of homotopy classes of paths in X from x to y. Composition of morphisms $[\alpha], [\beta]$ is $[\alpha * \beta]$ and the identity in $\pi X(x, x)$ is the $e_x = [c_x]$. We can consider the object group at x, $\pi X(x)$, as the well-known fundamental group $\pi_1(X, x)$.

If $\mathcal{U}$ is an open cover of X, the subgroup of $\pi_1(X, x)$ consisting of the homotopy classes of loops that can be represented by a product of the following type:

$$\prod_{j=1}^{n} u_j v_j u_j^{-1},$$

where the u_j's are arbitrary paths starting at the base point x and each v_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$. This group is called the *Spanier group with respect to* $\mathcal{U}$, denoted by $\pi(\mathcal{U}, x)$ [8, 10].

Definition 2.3. ([1]) Let (G, G_0) be a groupoid. A normal subgroupoid of (G, G_0) is a wide subgroupoid (M, G_0) such that for any $x, y \in G_0$ and any $a \in G(x, y)$ we have $aM(x)a^{-1} \in M(y)$. A totally disconnected subgroupoid (M, G_0) of (G, G_0) is a subgroupoid such that for any $x \neq y \in G_0$, we have $M(x, y) = \emptyset$.

Definition 2.4. ([1,5,6]) If M is a normal subgroupoid of the groupoid G, $\dfrac{G}{M}$ is a groupoid in which $Ob(\dfrac{G}{M}) = Ob(G)$ and the elements of $\dfrac{G}{M}$ are equivalence classes of elements of G under the relation $a \sim b$ if and only if $a = xby$ for some x, y in M. We denote this elements by aM or a_M.

Throughout this paper, all spaces are connected and locally path connected. Also, all homotopies between paths are relative to end points.

3. Main results

3.1. Generalization of the Brown's topology

Let X be a topological space and M be a totally disconnected, normal subgroupoid of πX. The quotient groupoid $\dfrac{\pi X}{M}$ has X as objects set and elements in $\dfrac{\pi X}{M}(x, y)$ are $[\alpha]_M = \{[\alpha * \beta]; [\beta] \in M(y)\}$ where $[\alpha] \in \pi X(x, y)$. Clearly $([\alpha]_M)^{-1} = ([\alpha]^{-1})_M = [\alpha^{-1}]_M$ and composition of $[\alpha]_M$, $[\beta]_M$ is $[\alpha * \beta]_M$, where $\alpha(1) = \beta(0)$.

Remark 3.1. It is notable that if $[\alpha]_M = [\beta]_M$ then there exists $\gamma \in M$ such that $\alpha \simeq \beta * \gamma$.

For an open cover $\mathcal{U}$ of X and for $x, y \in X$, let $[\alpha] \in \pi X(x, y)$ and $U, V \in \mathcal{U}$ be open neighborhoods of x, y, respectively. we define

$$B([\alpha]_M, \mathcal{U}, U, V) = \{[\lambda * \mu * \alpha * \eta * \mu' * \lambda']_M \mid [\eta] \in M(\alpha(1)), \lambda(I) \subseteq U, \lambda'(I) \subseteq V\},$$

where $[\mu] \in \pi(\mathcal{U}, \alpha(0))$, $[\mu]' \in \pi(\mathcal{U}, \alpha(1))$.

We recall that for two open covers $\mathcal{U}, \mathcal{U}'$ of X, $\mathcal{U} \subseteq \mathcal{U}'$ means that if $U \in \mathcal{U}$ then $U \in \mathcal{U}'$. The following proposition is useful when we are working with Spanier groups.

Proposition 3.2. *Let X be a topological space and $x \in X$. Then*

(i) For every open cover $\mathcal{U}$ of X, $\pi(\mathcal{U}, x)$ is a normal subgroup of $\pi_1(X, x)$.
(ii) If $\mathcal{U} \subseteq \mathcal{U}'$, then $\pi(\mathcal{U}, x) \subseteq \pi(\mathcal{U}', x)$.
(iii) $\mathcal{U} \cap \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$ is an open cover of X and $\mathcal{U} \cap \mathcal{V} \subseteq \mathcal{U}$.
(iv) If $\mathcal{V}$ is an open cover of Y and $f : X \longrightarrow Y$ is a continuous function with $f(x) = y$, then $\mathcal{U} := f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X and $f_*(\pi(\mathcal{U}, x)) \subseteq \pi(\mathcal{V}, y)$.

Proof. Easily come from definitions. □

Proposition 3.3. *Let $\mathcal{U}$ be an open cover of a given space X and M be a totally disconnected, normal subgroupoid of πX. Let $U,\ U' \in \mathcal{U}$ are open neighborhoods of $\alpha(0)$ and $V,\ V' \in \mathcal{U}$ are open neighborhoods of $\alpha(1)$ such that $U' \subseteq U$ and $V' \subseteq V$, then $B([\alpha]_M, \mathcal{U}, U', V') \subseteq B([\alpha]_M, \mathcal{U}, U, V)$.*

Proof. Since $U' \subseteq U$, pathes in U' are pathes in U. Now, according to the definition, the proof is clear. □

Lemma 3.4. *Let $\mathcal{U}$ be an open cover of X and M be the totally disconnected, normal subgroupoid of πX. If $[\alpha] \in \pi X(x, y)$ and $\eta \in M(\alpha(0))$ then there exists $\eta' \in M(\alpha(1))$ such that $\eta * [\alpha] \simeq [\alpha] * \eta'$.*

Proof. This comes from normality of M. □

Lemma 3.5. *Let $\mathcal{U}$ be an open cover of a given space X and M be a totally disconnected, normal subgroupoid of πX, then $[\gamma]_M \in B([\alpha]_M, \mathcal{U}, U, V)$ if and only if there exist $[\eta] \in M(\alpha(1))$, $[\mu] \in \pi(\mathcal{U}, \alpha(0))$, $[\mu'] \in \pi(\mathcal{U}, \alpha(1))$ and pathes λ, λ' in U, V such that $\gamma \simeq \lambda * \mu * \alpha * \eta * \mu' * \lambda'$.*

Proof. It follows from the definition, Remark 3.1 and Lemma 3.4 . □

Lemma 3.6. *Let $\mathcal{U}$ be an open cover of X, M be a totally disconnected, normal subgroupoid of πX and $U, V \in \mathcal{U}$ are open neighborhoods of $x = \alpha(0)$, $y = \alpha(1)$, respectively. If $[\gamma]_M \in B([\alpha]_M, \mathcal{U}, U, V)$ then*

$$B([\alpha]_M, \mathcal{U}, U, V) = B([\gamma]_M, \mathcal{U}, U, V).$$

Proof. If $[\gamma]_M \in B([\alpha]_M, \mathcal{U}, U, V)$ then by Lemma 3.5, $\gamma \simeq \lambda * \mu * \alpha * \eta * \mu' * \lambda'$, where $\eta \in M(\alpha(1))$, $\lambda(I) \subseteq U$, $\lambda'(I) \subseteq V$, $\mu \in \pi(\mathcal{U}, \alpha(0))$ and $\mu' \in \pi(\mathcal{U}, \alpha(1))$. Assume that $[z]_M \in B([\gamma]_M, \mathcal{U}, U, V)$. Then there exist $\eta' \in M(\gamma(1))$, $\delta(I) \subseteq U$, $\delta'(I) \subseteq V$, $\nu \in \pi(\mathcal{U}, \gamma(0))$ and $\nu' \in \pi(\mathcal{U}, \gamma(1))$ such that

$$\begin{aligned} z &\simeq \delta * \nu * \gamma * \eta' * \nu' * \delta' \\ &\simeq \delta * \nu * \lambda * \mu * \alpha * \eta * \mu' * \lambda' * \eta' * \nu' * \delta'. \end{aligned}$$

By Lemma (3.4) there exist $\theta_1 \in \pi(\mathcal{U}, \lambda(1))$, $\theta_2 \in M(\lambda'(0))$ and $\theta_3 \in \pi(\mathcal{U}, \lambda'(0))$ such that $\nu * \lambda \simeq \lambda * \theta_1$, $\lambda' * \eta' \simeq \theta_2 * \lambda'$ and $\lambda' * \nu' \simeq \theta_3 * \lambda'$. Therefore,

$$z \simeq (\delta * \lambda) * (\theta_1 * \mu) * (\alpha * \eta) * (\mu' * \theta_2 * \theta_3) * (\lambda' * \delta').$$

Since $\theta := \theta_1 * \mu \in \pi(\mathcal{U}, \alpha(0))$, $\theta' := \mu' * \theta_1 * \theta_2 \in \pi(\mathcal{U}, \alpha(1))$, $\sigma := (\delta * \lambda)(I) \subseteq U$ and $\sigma' := (\lambda' * \delta')(I) \subseteq V$, $z \simeq \sigma * \theta * \alpha * \eta * \theta' * \sigma'$ which implies that $[z]_M \in B([\alpha]_M, \mathcal{U}, U, V)$ and hence

$$B([\gamma]_M, \mathcal{U}, U, V) \subseteq B([\alpha]_M, \mathcal{U}, U, V).$$

Conversely, we have $\alpha \simeq \mu^{-1} * \lambda^{-1} * \gamma * \lambda'^{-1} * \mu'^{-1} * \eta^{-1}$ because $\gamma \simeq \lambda * \mu * \alpha * \eta * \mu' * \lambda'$. By Lemma (3.4) there exists $[\eta'] \in M(\gamma(1))$ such that $\alpha \simeq \mu^{-1} * \lambda^{-1} * \gamma * \eta' * \lambda'^{-1} * \mu'^{-1}$. Also, there exist $[\theta] \in \pi(\mathcal{U}, \gamma(0))$ and $[\theta'] \in \pi(\mathcal{U}, \gamma(1))$ such that $\alpha \simeq \lambda^{-1} * \theta * \gamma * \eta' * \theta' * \lambda'^{-1}$ and therefore $[\alpha] \in B([\gamma]_M, \mathcal{U}, U, V)$. Now, if apply the first part then

$$B([\alpha]_M, \mathcal{U}, U, V) \subseteq B([\gamma]_M, \mathcal{U}, U, V).$$

□

Now, we are ready to define a topology on the $\dfrac{\pi X}{M}$. But, we need to prove the following proposition.

Proposition 3.7. *For a given space X and any totally disconnected, normal subgroupoid M of πX, the family*

$$\{B([\alpha]_M, \mathcal{U}, U, V) |\ \mathcal{U} \text{ is an open cover of } X;\ U, V \in \mathcal{U},\ [\alpha] \in \pi X(x, y),\ x \in U,\ y \in V\}$$

form a basis for a topology on the quotient groupoid $(\dfrac{\pi X}{M})$.

Proof. For every $[\alpha] \in \pi X(x,y)$, let $U = V = X$ and $\mathcal{U} = \{X\}$. Then it is easy to see that $[\alpha] \in B([\alpha]_M, \mathcal{U}, U, V)$. If $[\alpha]_M \in B([\gamma]_M, \mathcal{U}, U, V) \cap B([\delta]_M, \mathcal{U}', U', V')$ then $B([\alpha]_M, \mathcal{U}, U, V) = B([\gamma]_M, \mathcal{U}, U, V)$ and $B([\alpha]_M, \mathcal{U}', U', V') = B([\delta]_M, \mathcal{U}', U', V')$, by Lemma (3.6). Since $\alpha(0) \in U \cap U'$ and $\alpha(1) \in V \cap V'$, $B([\alpha]_M, \mathcal{U} \cap \mathcal{U}', U \cap U', V \cap V')$ is basis element containing $[\alpha]_M$. Also,

$$B([\alpha]_M, \mathcal{U} \cap \mathcal{U}', U \cap U', V \cap V') \subseteq B([\alpha]_M, \mathcal{U}, U, V) \cap B([\alpha]_M, \mathcal{U}', U', V'),$$

as desired. □

We denote the quotient fundamental groupoid with this topology by $(\frac{\pi X}{M})^B$. After defining topology, we are interested to know does the quotient groupoid become a topological groupoid with this topology?

The next theorem answer this question.

Theorem 3.8. *Let X be a topological space and M be any totally disconnected normal subgroupoid of πX. Then $(\frac{\pi X}{M})^B$ is a topological groupoid.*

Proof. **Continuity of the initial map (S).** Let $\alpha \in \pi X(x,y)$ and U be an open neighborhood of $S(\alpha) = x$. For every open cover $\mathcal{U}$ of X, $\mathcal{W} = \mathcal{U} \cup \{U\}$ is an open cover of X. If $V \in \mathcal{W}$ contains y, $[\alpha]_M \in O := B([\alpha]_M, \mathcal{W}, U, V)$. We have $\gamma \simeq \lambda * \mu * \alpha * \theta * \mu' * \lambda'$ where $\lambda, \mu, \theta, \mu', \lambda'$ are descriptive components and $\lambda(I) \subseteq U$. Then $S([\gamma]_M) = \lambda(0) \in U$, and hence $S(O) \subseteq U$ which implise continuity of S. Similarly, the final map T is continuous.

Continuity of unit map 1. Let $\mathcal{U}$ be an open cover of X and $x \in X$. Let $B([\alpha]_M, \mathcal{U}, U, V)$ be a basic open neighborhood of $[1_x]_M$. Thus there exists $\eta \in M(\alpha(1))$ such that $1_x \simeq \lambda_1 * \mu_1 * \alpha * \eta * \mu_2 * \lambda_2$ where $\mu_1 \in \pi(\mathcal{U}, \alpha(0))$, $\mu_2 \in \pi(\mathcal{U}, \alpha(1))$, $\lambda_1(I) \subseteq U$ and $\lambda_2(I) \subseteq V$. Note that $U \cap V \neq \emptyset$ because $\lambda_1(0) = \lambda_2(1) = x$. Let N be the path component of $U \cap V$ containing x. For every $x' \in N$, there exists a path φ in N from x' to x and there exists $\eta' \in M(x')$ such that $1_{x'} \simeq \gamma_1 * \theta_1 * 1_{x'} * \eta' * \theta_2 * \gamma_2$ where $\gamma_1(I) \subseteq N \subseteq U$, $\gamma_2(I) \subseteq N \subseteq V$, $\theta_1 \in \pi(\mathcal{U}, x')$, $\theta_2 \in \pi(\mathcal{U}, x')$ and $\gamma_1(0) = \gamma_2(1) = x'$. Since $1_{x'} \simeq \varphi * 1_x * \varphi^{-1}$, we have

$$1_{x'} \simeq \gamma_1 * \theta_1 * \varphi * 1_x * \varphi^{-1} * \eta' * \theta_2 * \gamma_2.$$

By Lemma (3.4) there exist $\theta \in M(x)$ such that $\varphi^{-1} * \eta' \simeq \theta * \varphi^{-1}$ and there are $\nu \in \pi(\mathcal{U}, \varphi(1))$ and $\nu' \in \pi(\mathcal{U}, \varphi^{-1}(0))$ such that $\theta_1 * \varphi \simeq \varphi * \nu$ and $\varphi^{-1} * \theta_2 \simeq \nu' * \varphi^{-1}$. So

$$1_{x'} \simeq (\gamma_1 * \varphi) * \nu * 1_x * \theta * \nu' * (\varphi^{-1} * \gamma_2).$$

Where $(\gamma_1 * \varphi)(I) \subseteq U$ and $(\varphi^{-1} * \gamma_2)(I) \subseteq V$. This implies that

$$[1_{x'}]_M \in B([1_x]_M, \mathcal{U}, U, V).$$

Continuity of the inverse map i. Let $O := B([\alpha^{-1}]_M, \mathcal{U}, U, V)$ be a basic open neighborhood of $[\alpha^{-1}]_M$ in $\frac{\pi X}{M}$. Clearly $[\alpha]_M \in O' := B([\alpha]_M, \mathcal{U}, V, U)$. To show that O' is desired neighborhood of $[\alpha]_M$, it is sufficient to prove that $i(O') \subseteq O$. If $[\gamma]_M \in O'$ is an arbitrary element, then there exists $\eta \in M(\alpha(1))$ such that $\gamma \simeq \lambda_1 * \mu_1 * \alpha * \eta * \mu_2 * \lambda_2$ where λ_1, λ_2 are paths in V, U and $\mu_1 \in \pi(\mathcal{U}, \alpha(0))$, $\mu_2 \in \pi(\mathcal{U}, \alpha(1))$. Since $\gamma^{-1} \simeq \lambda_2^{-1} * \mu_2^{-1} * \eta^{-1} * \alpha^{-1} * \mu_1^{-1} * \lambda_1^{-1}$ and by Lemma 3.4 there exist $\eta' \in M(\alpha(0))$ such that $\eta^{-1} * \alpha^{-1} \simeq \alpha^{-1} * \eta'$, hence

$$\gamma^{-1} \simeq \lambda_2^{-1} * \mu_2^{-1} * \alpha^{-1} * \eta' * \mu_1^{-1} * \lambda_1^{-1}.$$

Since $\lambda_2^{-1}(I) \subseteq U$ and $\lambda_1^{-1}(I) \subseteq V$, $[\gamma^{-1}]_M \in B([\alpha^{-1}]_M, \mathcal{U}, U, V)$ that means $i([\gamma]_M) \in O$, as desired.

Continuity of the multiplication map m. Let $[\alpha]_M \in \frac{\pi X}{M}(x,y)$, $[\beta]_M \in \frac{\pi X}{M}(y,z)$ and let $B([\alpha * \beta]_M, \mathcal{U}, U, V)$ be a basic open neighborhood of $[\alpha * \beta]_M$. For any $W \in \mathcal{U}$ containing y, $B([\alpha]_M, \mathcal{U}, U, W)$ and $B([\beta]_M, \mathcal{U}, W, V)$ are basic open neighborhoods of

$[\alpha]_M$ and $[\beta]_M$, respectively. If $[\alpha_1]_M \in B([\alpha]_M, \mathfrak{U}, U, W)$ and $[\beta_1]_M \in B([\beta]_M, \mathfrak{U}, W, V)$ are arbitrary elements where $\alpha_1(1) = \beta_1(0)$, then there exists $\eta_1 \in M(\alpha(1))$ such that $\alpha_1 \simeq \lambda_1 * \mu_1 * \alpha * \eta_1 * \mu_2 * \lambda_2$ where $\lambda_1, \lambda_2, \mu_1$ and μ_2 are descriptive components and also there exists $\eta_2 \in M(\beta(1))$ such that $\beta_1 \simeq \gamma_1 * \nu_1 * \beta * \eta_2 * \nu_2 * \gamma_2$ where $\gamma_1, \gamma_2, \nu_1, \nu_2$ are descriptive components. But, $m([\alpha_1]_M, [\beta_1]_M) = [\alpha_1 * \beta_1]_M$. Thus,

$$\begin{aligned} \alpha_1 * \beta_1 &\simeq \lambda_1 * \mu_1 * \alpha * \eta_1 * \mu_2 * \lambda_2 * \gamma_1 * \nu_1 * \beta * \eta_2 * \nu_2 * \gamma_2 \\ &\simeq \lambda_1 * \mu_1 * (\alpha * \beta) * \beta^{-1} * \eta_1 * \mu_2 * \lambda_2 * \gamma_1 * \nu_1 * \beta * \eta_2 * \nu_2 * \gamma_2. \end{aligned}$$

By Lemma 3.4, there exists $\theta \in M(z)$ such that

$$\beta^{-1} * \eta_1 * \mu_2 * \lambda_2 * \gamma_1 * \nu_1 * \beta * \eta_2 \simeq \theta * \beta^{-1} * \mu_2 * \lambda_2 * \gamma_1 * \nu_1 * \beta.$$

Also, $\lambda_2(I), \gamma_1(I) \subseteq W \in \mathfrak{U}$ implies that $[\lambda_2 * \gamma_1] \in \pi(\mathfrak{U}, y)$. Hence $[\mu_2 * \lambda_2 * \gamma_1 * \nu_1] \in \pi(\mathfrak{U}, y)$ and so $[\beta^{-1} * \mu_2 * \lambda_2 * \gamma_1 * \nu_1 * \beta] \in \pi(\mathfrak{U}, z)$. Therefore we can write

$$\alpha_1 * \beta_1 \simeq \lambda_1 * \mu_1 * (\alpha * \beta) * \theta * (\beta^{-1} * \mu_2 * \lambda_2 * \gamma_1 * \nu_1 * \beta) * \nu_2 * \gamma_2.$$

The result is $[\alpha_1 * \beta_1]_M \in B([\alpha * \beta]_M, \mathfrak{U}, U, V)$. Hence

$$m(B([\alpha]_M, \mathfrak{U}, U, W), B([\beta]_M, \mathfrak{U}, W, V)) \subseteq B([\alpha * \beta]_M, \mathfrak{U}, U, V).$$

□

In the following we show that $(\frac{\pi X}{M})^B$ is a generalization of the Brown topology on $\frac{\pi X}{M}$ [2], when X is semilocally simply connected. At first, we reconstruct the Brown topology by some changes to the symbols.

For a semilocally simply connected space X and normal totally disconnected subgroupoid M of πX, let $\mathfrak{U}$ be the open cover of X consisting of all open, path-connected subsets U of X such that $i_*\pi_1(U, x) = \{e_x\}$, when $i : U \to X$ is the inclusion and $x \in U$. For every $U \in \mathfrak{U}$ and every $x \in U$, let $\widetilde{U}_x = \{[\lambda]_M \mid \lambda(0) = x, \lambda(1) \in U\} \subseteq \frac{\pi X}{M}$. If $[\alpha]_M \in \frac{\pi X}{M}(x, y)$ and $U, V \in \mathfrak{U}$, define

$$B([\alpha]_M, U, V) = \{[\beta]_M \mid [\beta]_M = [\lambda^{-1} * \alpha * \lambda']_M, [\lambda]_M \in \widetilde{U}_x, [\lambda']_M \in \widetilde{V}_y\}.$$

Theorem 3.9. ([2]) For a semillocally simply connected space X and any totally disconnected normal subgroupoid M of πX, the family

$$\{B([\alpha]_M, U, V) | \ [\alpha] \in \pi X(x, y), \ x \in U, \ y \in V\}$$

form a basis for a topology on quotientl groupoid $\frac{\pi X}{M}$ that makes $\frac{\pi X}{M}$ a topological groupoid.

Since X is semillocally simply connected space, there exists open cover $\mathfrak{U}$ consisting of the open path connected subsets U of X such that $i_*\pi_1(U, x) = \{e_x\}$, when $i : U \to X$ is the inclusion and $x \in U$. For every $x \in X$, $\pi(\mathfrak{U}, x)$ is the trivial group. For every $[\alpha]_M \in \frac{\pi X}{M}$ and every $U, V \in \mathfrak{U}$ containing $\alpha(0) = x$ and $\alpha(1) = y$, respectively,

$$B([\alpha]_M, \mathfrak{U}, U, V) = \{[\lambda * \alpha * \eta * \lambda']_M \mid \lambda(I) \subseteq U, \lambda'(I) \subseteq V, \eta \in M(\alpha(1))\}.$$

By Lemma(3.4), there exists $\eta' \in M(\lambda'(1))$ such that $\eta * \lambda' \simeq \lambda' * \eta'$. Therefore $[\lambda * \alpha * \eta * \lambda']_M = [\lambda * \alpha * \lambda' * \eta']_M$. Since $\eta' \in M(\lambda'(1))$, $[\lambda * \alpha * \lambda' * \eta']_M = [\lambda * \alpha * \lambda']_M$.

Proposition 3.10. *For a semillocally simply connected space X and normal, totally disconnected subgroupoid M of πX, the topology of $(\frac{\pi X}{M})^B$ is the Brown's topology on $\frac{\pi X}{M}$.*

3.2. Some quotients of topological groupoids

An example of the differences between topological groups and topological groupoids is quotienting. For topological groups, if M is a normal subgroup of the topological group G, the identification mapping $q : G \longrightarrow \frac{G}{M}$ is an open mapping and this makes it easy to prove the continuity of the composition map in $\frac{G}{M}$. But for topological groupoid G and normal subgroupoid M, this conclusion breaks down because the identification map is not necessarily an open map. Although in the case that G is locally compact hausdorff topological groupoid and M is a compact normal subgroupoid, Brown and Hardy [3] have proved that $q : G \longrightarrow \frac{G}{M}$ is an identification map. Brown and Hardy [3] have introduced a topology on the $\frac{G}{M}$ by a different construction making it a topological groupoid in which under some conditions agree with the quotient topology.

Now, we want to show that when M is a totally disconnected normal subgroupoid of G, the quotient groupoid $\frac{G}{M}$ is a topological groupoid.

Proposition 3.11. *If G is a topological groupoid and M is a totally disconnected normal subgroupoid of G. Then the quotient map $q : G \longrightarrow \frac{G}{M}$ is open.*

Proof. We must show that $q(U)$ is open, for every open set $U \subseteq G$. We claim that $q^{-1}(q(U)) = UM$. If $c \in q^{-1}(q(U))$ then $q(c) \in q(U)$ and there exists $a \in U$ such that $q(c) = q(a)$ or equivalently $cM = aM$. Since $c1_y \in cM(y)$, there exists $k \in M(y)$ such that $c1_y = ak$ or $c = ak$ and hence $c \in UM$.

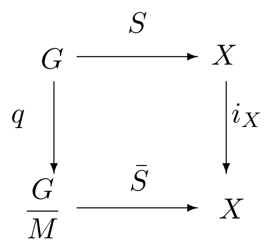
Conversely, let $c \in UM$. There exist $a \in U$ and $k \in M(y)$ such that $c = ak$. Then $cM = akM = aM$ and so $q(c) = q(a)$ which implies that $c \in q^{-1}(q(a))$ and hence $c \in q^{-1}(q(U))$.

Now, $UM = \bigcup_{g \in M} Ug$ and also for every $g \in M(x)$, $Ug = U^x g = R_g(U^x)$.

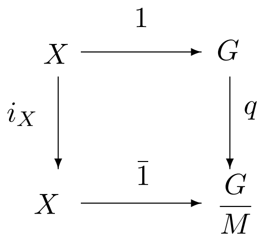
Since M is a totally disconnected full subgroupoid of G, $g \in M$ means that there exists $x \in G_0$ such that $g \in M(x)$ and also $R_g : G^x \longrightarrow G^x$ is a homeomorphism. U^x is open in G^x because U is open in G. Hence Ug and so UM is open in G. According to the quotient topology, $q(U)$ is open in $\frac{G}{M}$ and therefore q is open map. □

Theorem 3.12. *If G is a topological groupoid and M is a totally disconnected normal subgroupoid of G, then $\frac{G}{M}$ is a topological groupoid.*

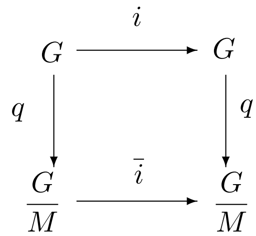
Proof. **Continuity of initial map $\bar{S}$.** For the initial map $\bar{S} : \frac{G}{M} \longrightarrow X$ that $\bar{S}(aM) = a(0)$, continuity comes from the continuity of the initial map $S : G \longrightarrow X$, continuity of the identity map i_X and the fact that q is open map.



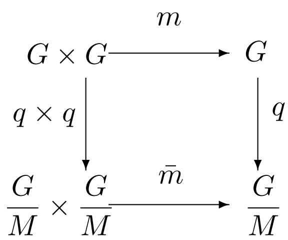
Continuity of unit map $\bar{1}$. Let $\bar{1} : X \longrightarrow \frac{G}{M}$ be $\bar{1}(x) = 1_x M = M$. According to the following commutative diagram and continuity of i_X and 1, the map $\bar{1}$ is also continuous, because q is quotient map.



Continuity of inverse map $\bar{i}$. Similarly and by using the following diagram, $\bar{i}$ is continuous.



Continuity of multiplication map $\bar{m}$. By Proposition 3.11, the quotient map q is open which implies that $q \times q$ is a quotient map. Now, consider the following diagram. Since multiplication map m of topological groupoid G is continuous and q, $q \times q$ are quotient maps, $\bar{m}$ is continuous.



□

3.3. Quotient of lasso topology

The authors in [11] have introduced lasso topology on the fundamental groupoid and have shown that this topology makes the fundamental groupoids as a topological groupoid and denoted it by $\pi^L X$.

In the previous section, we have proved that quotients of topological groupoids by totally disconnected normal subgroupoids are also topological groupoids. Hence for every totally disconnected normal subgroupoid M of $\pi^L X$, $\dfrac{\pi^L X}{M}$ is a topological groupoid. Here we will show that the generalized Brown topology on $\dfrac{\pi X}{M}$ is equivalent to the quotient of lasso topology on $\dfrac{\pi X}{M}$.

Definition 3.13. ([11]) Let $\mathcal{U}$ be an open cover of a given space X and for $x, y \in X$, let $[\alpha] \in \pi X(x, y)$. If $U, V \in \mathcal{U}$ are open neighborhoods of x, y respectively, then $N([\alpha], \mathcal{U}, U, V) =$

$$\{[\beta] \in \pi X \mid \beta \simeq \lambda * \mu * \alpha * \mu' * \lambda', \mu \in \pi(\mathcal{U}, x), \mu' \in \pi(\mathcal{U}, y), \lambda : I \longrightarrow U, \lambda' : I \longrightarrow V\}.$$

The lasso topology on the fundamental groupoids has a basis as follow:

$$B = \{N([\alpha], \mathcal{U}, U, V); \mathcal{U} \textit{ is an open cover of } X, U, V \in \mathcal{U}, [\alpha] \in \pi X(x, y)\}.$$

The following lemma shows that $q(B)$ will be a basis for the quotient of the lasso topology on $\dfrac{\pi X}{M}$.

Lemma 3.14. *For a given space Y and equivalence relation $\sim$ on Y, let $q: Y \longrightarrow \frac{Y}{\sim}$ be an open quotient map and B be a basis for the topology on Y. Then $q(B)$ is a basis for the quotient topology on $\frac{Y}{\sim}$.*

Proof. It is straight. □

Corollary 3.15. *For a given topological space X and any totally disconnected normal subgroupoid M of πX, the family*
$\{q(N([\alpha],\mathfrak{U},U,V)) = N([\alpha]_M,\mathfrak{U},U,V); \mathfrak{U}$ *is an open cover of* $X, [\alpha] \in \pi X(x,y)\}$, *forms a basis for the quotient topology on* $\frac{\pi^L X}{M}$.

Remark 3.16. Not that $N([\alpha],\mathfrak{U},U,V) = \{[\beta]_M \mid \beta \simeq \lambda * \mu * \alpha * \mu' * \lambda'\}$ and since $[\beta]_M = \{\beta * \eta \mid \eta \in M(\beta(1))\}$, $[\gamma]_M \in N([\alpha]_M,\mathfrak{U},U,V)$ means that $\gamma \simeq \lambda * \mu * \alpha * \mu' * \lambda' * \eta$ where λ is a path in U and λ' is a path in V such that $\lambda(1) = \alpha(0)$, $\lambda'(0) = \alpha(1)$ and $\mu \in \pi(\mathfrak{U},\alpha(0))$, $\mu' \in \pi(\mathfrak{U},\alpha(1))$, $\eta \in M(\gamma(1))$.

Theorem 3.17. *For a given space X and totally disconnected normal subgroupoid M of πX, $(\frac{\pi X}{M})^B = \frac{\pi^L X}{M}$.*

Proof. As we know $obj(\frac{\pi X}{M}) = X$ and $\frac{\pi^L X}{M}(x,y) = \{[\alpha]_M \mid [\alpha] \in \pi^L X(x,y)\}$. Let's prove it first that $\frac{\pi^L X}{M}$ is finer than $(\frac{\pi X}{M})^B$.

Let $[\gamma]_M \in B([\alpha]_M,\mathfrak{U},U,V)$. We need to find an open neighborhood $N([\alpha]_M,\mathfrak{U},U,V)$ of $[\gamma]_M$ in $\frac{\pi^L X}{M}$ such that $[\gamma]_M \in N([\alpha]_M,\mathfrak{U},U,V) \subseteq B([\alpha]_M,\mathfrak{U},U,V)$. Since $[\gamma]_M \in B([\alpha]_M,\mathfrak{U},U,V)$, there exists $\eta \in M(\alpha(1))$ such that

$$\gamma \simeq \lambda * \mu * \alpha * \eta * \mu' * \lambda',$$

where $\lambda(I) \subseteq U$, $\lambda'(I) \subseteq V$, $\mu \in \pi(\mathfrak{U},\alpha(0))$ and $\mu' \in \pi(\mathfrak{U},\alpha(1))$.

By multiple uses of Lemma 3.4, there exists $\xi \in M(\gamma(1))$ such that

$$\gamma \simeq \lambda * \mu * \alpha * \mu' * \lambda' * \xi.$$

So $[\gamma]_M \in N([\alpha]_M,\mathfrak{U},U,V)$ by Remark 3.16.

Now, for every $[\beta]_M \in N([\alpha]_M,\mathfrak{U},U,V)$, there exists $\theta \in M(\beta(1))$ such that $\beta \simeq \delta * \nu * \alpha * \nu' * \delta' * \theta$, where $\nu \in \pi(\mathfrak{U},\alpha(0))$, $\nu' \in \pi(\mathfrak{U},\alpha(1))$ and δ,δ' are paths in U,V, respectively. By Lemma 3.4, there exists $\eta' \in M(\alpha(1))$ such that $\delta' * \theta \simeq \eta' * \delta'$. So we can write $\beta \simeq \delta * \nu * \alpha * \eta' * \eta'^{-1} * \nu' * \eta' * \delta'$. Since $\bar{\nu} := (\eta'^{-1} * \nu' * \eta') \in \pi(\mathfrak{U},\alpha(1))$, $\beta \simeq \delta * \nu * \alpha * \eta' * \bar{\nu} * \delta'$ and hence $[\beta]_M \in B([\alpha]_M,\mathfrak{U},U,V)$, as desired.

Conversely, we will prove that $(\frac{\pi X}{M})^B$ is finer than $\frac{\pi^L X}{M}$.

Let $[\gamma]_M \in N([\alpha]_M,\mathfrak{U},U,V)$, then there exists $\eta \in M(\gamma(1))$ such that

$$\gamma \simeq \lambda * \mu * \alpha * \mu' * \lambda' * \eta,$$

where λ,λ' are paths in U,V, respectively and $\mu \in \pi(\mathfrak{U},\alpha(0))$, $\mu' \in \pi(\mathfrak{U},\alpha(1))$.

By Lemma 3.4, there exist $\eta' \in M(\alpha(1))$ such that $(\mu' * \lambda') * \eta \simeq \eta' * (\mu' * \lambda')$ and so

$$\gamma \simeq \lambda * \mu * \alpha * \eta' * \mu' * \lambda'.$$

Hence, $[\gamma]_M \in B([\alpha]_M,\mathfrak{U},U,V)$.

We show that $B([\alpha]_M,\mathfrak{U},U,V) \subseteq N([\alpha]_M,\mathfrak{U},U,V)$.

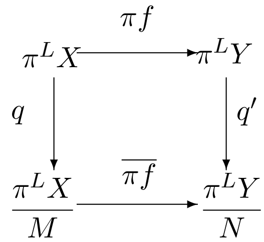
Let $[\beta]_M \in B([\alpha]_M,\mathfrak{U},U,V)$. Then there exist $\xi \in M(\alpha(1))$ such that $\beta \simeq \delta * \nu * \alpha * \xi * \nu' * \delta'$, where δ,δ' are paths in U,V, respectively and $\nu \in \pi(\mathfrak{U},\alpha(0))$, $\nu' \in \pi(\mathfrak{U},\alpha(1))$.

By using Lemma 3.4, there exists $\xi' \in M(\beta(1))$ such that $\beta \simeq \delta * \nu * \alpha * \nu' * \delta' * \xi'$. Hence $[\beta]_M \in N([\alpha]_M, \mathcal{U}, U, V)$. $\square$

The following propositions specify some properties of the induced maps on the topological fundamental groupoids.

Theorem 3.18. *Let X, Y are topological spaces and M, N are totally disconnected normal subgroupoid of πX, πY, respectively. If $f : X \longrightarrow Y$ is a continuous map such that $\pi f(M) \subseteq N$ then the map $\overline{\pi f} : (\frac{\pi X}{M})^B \longrightarrow (\frac{\pi Y}{N})^B$ defined by $\overline{\pi f}([\lambda]_M) = [f \circ \lambda]_N$ is continuous.*

Proof. By Theorem 3.17 it suffices to prove that $\overline{\pi f} : \frac{\pi^L X}{M} \longrightarrow \frac{\pi^L Y}{N}$ is continuous. Continuity of $\pi f : \pi^L X \longrightarrow \pi^L Y$ comes from the continuity of f [11], and q, q' are quotient maps . By commutativity of the following diagrom, continuity of $\overline{\pi f}$ is obvious.



$\square$

Recall that for paths λ, γ in X, Y, respectively, by $(\lambda, \gamma) : I \longrightarrow X \times Y$ we mean the path $(\lambda, \gamma)(t) = (\lambda(t), \gamma(t))$. It is straight that $(\lambda, \gamma) * (\alpha, \beta) = (\lambda * \alpha, \gamma * \beta)$, where the concatenations are defined.

Proposition 3.19. *Let X, Y are topological spaces and M, N are totally disconnected normal subgroupoids of $\pi X, \pi Y$, respectively. Then $(\frac{\pi(X \times Y)}{M \times N})^B$ is isomorphic as topological groupoid to $(\frac{\pi X}{M})^B \times (\frac{\pi Y}{N})^B$, equiped by the product topology.*

Proof. Since the projection maps $p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$ are continuous, by Proposition 3.18, the maps $\overline{\pi p_1}$ $(\frac{\pi(X \times Y)}{M \times N})^B \longrightarrow (\frac{\pi X}{M})^B$ and $\overline{\pi p_2} : (\frac{\pi(X \times Y)}{M \times N})^B \longrightarrow (\frac{\pi Y}{N})^B$ are continuous which induce the continuous map $\psi : (\frac{\pi(X \times Y)}{M \times N})^B \longrightarrow (\frac{\pi X}{Y})^B \times (\frac{\pi Y}{N})^B$. If $\varphi : (\frac{\pi X}{M})^B \times (\frac{\pi Y}{N})^B \longrightarrow (\frac{\pi(X \times Y)}{M \times N})^B$ is defined by $\varphi([\alpha]_M, [\beta]_N) = [(\alpha, \beta)]_{M \times N}$, it is easy to see that $\varphi \circ \psi = id$ and $\psi \circ \varphi = id$. Hence ψ is a bijection.

It remains to prove the continuity of φ. Let $W := B([(\lambda, \gamma)]_{M \times N}, \mathcal{U}, U, V)$ be a basic open neighborhood of $[(\lambda, \gamma)]_{M \times N}$, where $\mathcal{U}$ is an open cover of $X \times Y$ and $U, V \in \mathcal{U}$. If $\mathcal{U}_1 := p_1(\mathcal{U})$ and $\mathcal{U}_2 := p_2(\mathcal{U})$, then $\mathcal{U}_1$ is an open cover of X and $\mathcal{U}_2$ is an open cover of Y. Let $U_1 := p_1(U)$ and let $U_2 := p_2(U)$ where they are open neighborhoods of $\lambda(0)$ and $\gamma(0)$, respectively. Also, let $V_1 := p_1(V)$ and let $V_2 := p_2(V)$ where they are open neighborhoods of $\lambda(1)$ and $\gamma(1)$, respectively.

Now, let $O_1 := B([\lambda]_M, \mathcal{U}_1, U_1, V_1)$, $O_2 := B([\gamma]_N, \mathcal{U}_2, U_2, V_2)$. It is obvious that $[\lambda]_M \in O_1$ and $[\gamma]_N \in O_2$. We will show that $\varphi(O_1, O_2) \subseteq W$. Suppose that $[\lambda_1]_M \in O_1$. There exists $\eta_1 \in M(\lambda(1))$ such that

$$\lambda_1 \simeq \mu * \theta * \lambda * \eta_1 * \theta' * \mu',$$

where $\mu(I) \subseteq U_1$, $\mu'(I) \subseteq V_1$, $\theta \in \pi(\mathcal{U}_1, \lambda(0))$ and $\theta' \in \pi(\mathcal{U}_1, \lambda(1))$.

Let $[\gamma_1] \in O_2$. There exists $\eta_2 \in N(\gamma(1))$ such that

$$\gamma_1 \simeq \nu * \delta * \gamma * \eta_2 * \delta' * \nu',$$

where $\nu(I) \subseteq U_2$, $\nu'(I) \subseteq V_2$, $\delta \in \pi(\mathcal{U}_2, \gamma(0))$ and $\delta' \in \pi(\mathcal{U}_2, \gamma(1))$. Thus

$$\begin{aligned}\varphi([\lambda_1]_M, [\gamma_1]_N) &= [(\lambda_1, \gamma_1)]_{M\times N} \\ &= [(\mu, \nu) * (\theta, \delta) * (\lambda, \gamma) * (\eta_1, \eta_2) * (\theta', \delta') * (\mu', \nu')],\end{aligned}$$

where (μ, ν) is path in $U = U_1 \times U_2$ and (μ', ν') is path in $V = V_1 \times V_2$, $(\theta, \delta) \in \pi(\mathcal{U}, (\lambda(0), \gamma(0)))$, $(\theta', \delta') \in \pi(\mathcal{U}, (\lambda(1), \gamma(1)))$ and $(\eta_1, \eta_2) \in M \times N(\lambda(1), \gamma(1))$ Hence

$$[(\lambda_1, \gamma_1)]_{M\times N} \in B([(\lambda, \gamma)]_{M\times N}, \mathcal{U}, U, V).$$

Therefore φ is continuous. □

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