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Research Article

Fusible modules

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Abstract

In this paper, we extend the concept of fusibility to the module-theoretic setting by introducing fusible modules. Let R be a ring with identity, M a right R-module and $0 \neq m \in M$. Then, m is called *fusible* if it can be expressed as the sum of a torsion element and a torsion-free element in M. The module M is said to be *fusible* if every non-zero element of M is fusible. We investigate some properties of fusible modules. It is proved that the class of fusible modules is between the classes of torsion-free and nonsingular modules.

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1. Introduction

Let R be a ring with identity. Ghashghaei and McGovern introduced fusible elements of rings in [1], that is, a non-zero element $a \in R$ said to be left (right) fusible if there exist a left (right) zero-divisor element z and a non-left (non-right) zero-divisor element r such that $a = z + r$. An element of R which is both left and right fusible is called fusible. A ring R is said to be left (right) fusible if every non-zero element of R is left (right) fusible. It is shown that every regular element and every idempotent element are fusible.

Motivated by the study of fusible rings, in this paper, we introduce the notion of a fusible module, as a module-theoretic analogue of a fusible ring. A module M is called *fusible* if every non-zero element of M decomposes into a sum of a torsion element and a torsion-free element. Some examples of fusible modules, such as vector spaces and flat modules over a domain, are given. It is shown that the class of fusible modules lies strictly between the class of torsion-free modules and the class of nonsingular modules. Examples that delineate the concepts and results are provided. We deal with the fusibility of direct sums and direct products of fusible modules. We also focus on the modules over polynomial rings and power series rings in terms of fusibility. In [1], it is proved that every commutative fusible ring is reduced. Inspired by this result, we investigate the relations between fusible modules and reduced modules which are defined in [3]. In this direction, it is proved that every fusible module over a duo ring (in particular, a commutative ring) is reduced.

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Throughout the paper, all rings R are associative with an identity and modules are right R-modules. Let M be a module and N a subset of M. The right ideal $r_R(N) = \{r \in R \mid Nr = 0\}$ is called the right annihilator of N in R. If the subset N is a singleton, say $N = \{m\}$, then we simply write $r_R(m)$. Let $T_R(M)$ stand for the torsion elements of M, i.e., $T_R(M) = \{m \in M \mid r_R(m) \neq 0\}$ and $T_R^*(M)$ denote the torsion-free elements of M, that is, $T_R^*(M) = \{m \in M \mid r_R(m) = 0\}$.

2. Fusible modules

Analogous to the fusible property of a ring, in this section, we introduce the notion of the fusible property for a module and obtain some of its basic properties.

Definition 2.1. Let M be a module and $0 \neq m \in M$. Then, m is called *fusible* if it can be written as $m = m_1 + m_2$ such that $m_1 \in T_R(M)$ and $m_2 \in T_R^*(M)$. The module M is said to be *fusible* if every non-zero element of M is fusible.

Remark 2.2. Note that $0 \in T_R(M)$ since $r_R(0) \neq 0$. For any torsion-free $m \in M$, m has a fusible decomposition $m = 0 + m$.

Theorem 2.3. *If M is a torsion-free module, then it is fusible. The converse holds if $T_R(M)$ is a subgroup of M.*

Proof. Let M be a torsion-free module. Then, M is fusible by Remark 2.2. For the converse, assume that M is fusible and $T_R(M)$ is a subgroup of M. Let $0 \neq m \in T_R(M)$. Hence there exist $m_1 \in T_R(M)$ and $m_2 \in T_R^*(M)$ such that $m = m_1 + m_2$. Thus, $m - m_1 = m_2$ and so $m - m_1 \in T_R^*(M)$ since $m_2 \in T_R^*(M)$. This is a contradiction, then we get $T_R(M) = 0$. □

We now give some sources of examples for fusible modules. They are obtained by Theorem 2.3.

Example 2.4. (1) Every vector space is fusible.
(2) Every flat module over a domain is fusible.
(3) Every module over a von Neumann regular domain is fusible.
(4) If R is a domain, then R is a fusible R-module.

We have some examples of fusible elements in modules.

Example 2.5. Let k be a field of characteristic $p > 0$ and G be an elementary abelian p-group of order p^2 generated by x, y. If V is the 3-dimensional right kG-module with basis $\{e_1, e_2, e_3\}$ multiplication defined by $e_1x = e_1$, $e_2x = e_2$, $e_3x = e_1 + e_3$, $e_1y = e_1$, $e_2y = e_2$, $e_3y = e_2 + e_3$, then $S = \mathrm{End}_{kG}(V)$ is isomorphic to the matrix ring $R = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{bmatrix} \mid a, b, c \in k \right\}$, by [2, 19.15, page 299]. Hence

(i) the elements of the form $A = \begin{bmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{bmatrix} \in R$ with $a \neq 0$ are fusible,

(ii) the torsion element $v = e_1 + e_2 + e_3 \in V$ is fusible.

Proof. (i) Let $A = \begin{bmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{bmatrix} \in R$ with $a \neq 0$. Then, $A = B + C$ where $B = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \in T_R(R)$ and $C = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in T_R^*(R)$.
(ii) Since $e_1(x - y) = 0$, $e_2(x - y) = 0$ and $e_3(x - y)^2 = 0$, $v = e_1 + e_2 + e_3 \in T_{kG}(V)$. If

the characteristic of k is 2, then $e_1(x+y) = 2e_1 = 0$ and $(e_2+e_3)(x+y) = e_1+e_2$. Hence $e_1 \in T_{kG}(V)$ and $e_2+e_3 \in T^*_{kG}(V)$. In this case, $v \in V$ is fusible. In a similar way, it can be proved for $p \geq 3$. □

Remark 2.6. Torsion modules are not fusible since the set of torsion-free elements is empty. Thus, $\mathbb{Z}_n$ is not a fusible $\mathbb{Z}$-module, for every positive integer n.

In the sequel, we investigate under what conditions fusible modules are torsion-free. In order to do that, we need the following definition. Recall from [4] that a ring R is called *lineal* if its right annihilator lattice is linearly ordered.

Definition 2.7. A module M is said to *satisfy the comparability relation between annihilators of subsets* if for any subset N_1 and N_2 of M, we have $r_R(N_1) \subseteq r_R(N_2)$ or $r_R(N_2) \subseteq r_R(N_1)$.

Examples 2.8. (1) The $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ satisfies the comparability relation between annihilators of subsets.
(2) Every uniserial module M, i.e., for any submodules K, L of M, either $K \subseteq L$ or $L \subseteq K$, satisfies the comparability relation between annihilators of subsets.

It is well known that the subset $T_R(M)$ need not be a subgroup of a module M. We now determine under what conditions $T_R(M)$ is a subgroup of M. In the light of Theorem 2.3, we obtain the next result.

Corollary 2.9. *If a module M satisfies the comparability relation between annihilators of subsets, then $T_R(M)$ is a subgroup of M.*

Proof. Let m_1, $m_2 \in T_R(M)$ and n_1, n_2 be positive integers satisfying $m_1n_1 = 0$ and $m_2n_2 = 0$. So $n_1 \in r_R(m_1)$ and $n_2 \in r_R(m_2)$. By hypothesis, $r_R(m_1) \subseteq r_R(m_2)$ or $r_R(m_2) \subseteq r_R(m_1)$. If $r_R(m_1) \subseteq r_R(m_2)$, then $(m_1+m_2)n_1 = 0$. The case $r_R(m_2) \subseteq r_R(m_1)$ is treated similarly, and so we have $(m_1+m_2)n_2 = 0$. In either case, $m_1+m_2 \in T_R(M)$. This completes the proof. □

Proposition 2.10. *Let M be a fusible module. If M satisfies the comparability relation between annihilators of subsets, then M is torsion-free.*

Proof. Assume that M satisfies the comparability relation between annihilators of subsets. Let $0 \neq m \in M$. Suppose that $m \in T_R(M)$. Since M is fusible, m has a fusible decomposition $m = m_1 + m_2$ such that $m_1 \in T_R(M)$ and $m_2 \in T^*_R(M)$. By assumption, $r_R(m) \subseteq r_R(m_1)$ or $r_R(m_1) \subseteq r_R(m)$. Let $0 \neq t \in r_R(m) \subseteq r_R(m_1)$. Then, $0 = mt = m_1t + m_2t$ entails $m_2t = 0$ since $m_1t = 0$. Let $0 \neq s \in r_R(m_1) \subseteq r_R(m)$. Similarly, we get $m_2s = 0$. In either case we reach a contradiction. So $r_R(m) = 0$. It follows that M is torsion-free. □

For a module M, the submodule $Z(M) = \{m \in M \mid r_R(m) \text{ is essential in } R\}$ is called *singular submodule* and M is said to be *nonsingular* in case $Z(M) = 0$. The next result shows that the class of fusible modules is a subclass of the class of nonsingular modules.

Theorem 2.11. *Let M be a fusible module. Then, M is nonsingular.*

Proof. Assume that there exists $0 \neq m \in Z(M)$ and we get a contradiction. Being $m \in Z(M)$ yields that $r_R(m)$ is an essential right ideal of R. Also, M being fusible implies $m = m_1 + m_2$ for some $m_1, m_2 \in M$ with $r_R(m_1) \neq 0$ and $r_R(m_2) = 0$. Since $r_R(m_1) \neq 0$, $r_R(m) \cap r_R(m_1) \neq 0$. Let $0 \neq t \in r_R(m) \cap r_R(m_1)$. Since $m - m_1 = m_2$ implies $0 \neq t \in r_R(m) \cap r_R(m_1) \subseteq r_R(m - m_1) = r_R(m_2)$. So $r_R(m_2) \neq 0$. This is the required contradiction. Therefore $Z(M) = 0$. □

The following result is an immediate consequence of Theorem 2.11.

Corollary 2.12. *If every R-module is fusible, then R is semisimple.*

There are nonsingular modules which are not fusible.

Example 2.13. Let $R = U_2(\mathbb{Z}_2)$ denote the ring of all upper triangular 2×2 matrices over the ring $\mathbb{Z}_2$. We claim that $Z(R_R) = 0$ and the right R-module R is not fusible. Let e_{ij} denote 2×2 matrix units. Then, $r_R(e_{11}) = e_{22}R$, $r_R(e_{11} + e_{12}) = (e_{12} + e_{22})R$, $r_R(e_{12}) = (e_{11} + e_{12})R$, $r_R(e_{12} + e_{22}) = (e_{11} + e_{12})R$ and $r_R(e_{22}) = (e_{11} + e_{12})R$ are all direct summands. Hence R is nonsingular. Assume that $e_{12} = a + b$ has a decomposition with $a \in T_R(R)$ and $b \in T_R^*(R)$ where $a = \begin{bmatrix} r & s \\ 0 & t \end{bmatrix}$ and $b = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$. By $b \in T_R^*(R)$, we have $xz = \overline{1}$. It entails that $x = z = \overline{1}$, in turn we get $r = t = \overline{1}$. This is a contradiction.

Next, we investigate the basic properties of the fusibility of modules. First, we give some examples to show that submodules and quotient modules do not inherit the fusible property. Also this property does not pass from a ring to modules over it.

Example 2.14. (1) $\mathbb{Z}_6$ is a fusible $\mathbb{Z}_6$-module but the submodule $\langle \bar{2} \rangle$ of $\mathbb{Z}_6$ is not a fusible $\mathbb{Z}_6$-module.
(2) $\mathbb{Z}$ is a fusible $\mathbb{Z}$-module but $\mathbb{Z}/6\mathbb{Z}$ is not a fusible $\mathbb{Z}$-module.
(3) Although $\mathbb{Z}_{(2)}$ is a fusible $\mathbb{Z}$-module, $\mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)}$ is not fusible. Indeed, consider $\frac{a}{b} + 2\mathbb{Z}_{(2)} \in \mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)}$ and $2 \in \mathbb{Z}$. Then, $\left(\frac{a}{b} + 2\mathbb{Z}_{(2)}\right)2 = 0 + 2\mathbb{Z}_{(2)}$. Hence there is not a torsion-free element, and so we can not find any fusible decomposition of non-zero elements of the $\mathbb{Z}$-module $\mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)}$.
(4) In spite of the fact that $\mathbb{Z}_{(2)}$ is a fusible $\mathbb{Z}_{(2)}$-module, $\mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)}$ is not fusible by a discussion similar to (3).
(5) By Theorem 2.11, the quotient module with respect to an essential module need not be fusible.

One may suspect that every direct summand of a fusible module is fusible. However, the following example erases the possibility.

Example 2.15. Consider $M = \mathbb{Z}_6$ as a $\mathbb{Z}_6$-module. Then, $M = M_1 \oplus M_2$ where $M_1 = \{\overline{0}, \overline{2}, \overline{4}\}$ and $M_2 = \{\overline{0}, \overline{3}\}$. Note that $T_R(M) = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$ and $T_R^*(M) = \{\overline{1}, \overline{5}\}$. Since $\overline{1} = \overline{0} + \overline{1}$, $\overline{2} = \overline{3} + \overline{5}$, $\overline{3} = \overline{2} + \overline{1}$, $\overline{4} = \overline{3} + \overline{1}$ and $\overline{5} = \overline{0} + \overline{5}$, M is a fusible $\mathbb{Z}_6$-module. Although, $\overline{2}$ has a unique fusible decomposition $\overline{2} = \overline{3} + \overline{5}$ in M, this decomposition is not in M_1 since $\overline{3} \notin M_1$. Similarly, even though, $\overline{3} = \overline{2} + \overline{1}$ has a unique fusible decomposition $\overline{3} = \overline{2} + \overline{1}$ in M, this decomposition is not in M_2 since $\overline{2} \notin M_2$. It entails that neither M_1 nor M_2 is fusible.

Proposition 2.16. *Let R be an integral domain. Then, every submodule of fusible R-modules is fusible.*

Proof. Let M be a fusible R-module and N be a submodule of M. Then, $T_R(M)$ is a subgroup of M since R is an integral domain. Hence M is torsion-free by Theorem 2.3, and so N is torsion-free. Thus, N is fusible. □

Lemma 2.17. *Let $\{M_i\}_{i \in I}$ be a family of R-modules and $(m_i) \in \bigoplus_{i \in I} M_i$. If any $m_k \in T_R^*(M_k)$, that is $m_k a = 0$ implies $a = 0$, then $(m_i) \in T_R^*(\oplus_{i \in I} M_i)$.*

Proof. Suppose that there exists $r \in R$ such that $(m_i)r = 0$ for some $r \in R$. Then, $m_i r = 0$ for each $i \in I$. This contradicts $m_k r \neq 0$. □

Corollary 2.18. *If $(m_i) \in T_R(\oplus_{i \in I} M_i)$, then $m_i \in T_R(M)$ for each $i \in I$.*

Theorem 2.19. *Let R_i be a ring and M_i be an R_i-module for $i = 1, 2$. Then, $M = M_1 \oplus M_2$ is a fusible $R = R_1 \oplus R_2$-module if and only if M_i is a fusible R_i-module for $i = 1, 2$.*

Proof. Let $0 \neq (m_1, 0) \in M_1 \oplus M_2$. Since $0 \neq m_1 \in M_1$, there exists a decomposition $m_1 = z_1 + y_1$ with $z_1 \in T_{R_1}(M_1)$, $y_1 \in T^*_{R_1}(M_1)$. Let $0 \neq z'_1 \in R_1$ with $z_1 z'_1 = 0$. Consider $0 \neq m_2 \in M_2$ due to $M_2 \neq 0$. There exists a decomposition $m_2 = a_2 + b_2$ with $a_2 \in T_{R_2}(M_2)$, $b_2 \in T^*_{R_2}(M_2)$. Then, we have $(m_1, 0) = (z_1, -b_2) + (y_1, b_2)$. Note that $(y_1, b_2) \in T^*_R(M)$ by Lemma 2.17. Also, being $(z_1, -b_2)(z'_1, 0) = (0, 0)$ implies that $(z_1, -b_2) \in T_R(M)$. Thus, $(m_1, 0) = (z_1, -b_2) + (y_1, b_2)$ is the fusible decomposition of $(m_1, 0)$. Similarly, $0 \neq (0, m_2) \in M_1 \oplus M_2$ is fusible.
Let $0 \neq m_1 \in M_1$ and $0 \neq m_2 \in M_2$ with $m_1 = a_1 + b_1$ and $m_2 = a_2 + b_2$ where $a_i \in T_{R_i}(M_i)$ and $b_i \in T^*_{R_i}(M_i)$ for $i = 1, 2$. Hence there is $0 \neq a'_i \in R_i$ such that $a_i a'_i = 0$ for each $i = 1, 2$. Note that $(m_1, m_2) = (a_1, a_2) + (b_1, b_2)$. On the one hand, we have $(b_1, b_2) \in T^*_R(M)$ by Lemma 2.17. On the other hand, $(a_1, a_2) \in T_R(M)$ by the fact that $(a_1, a_2)(a'_1, a'_2) = (0, 0)$. Therefore M is fusible.
Conversely, let $m \in M_1$. Since M is fusible, there are $(m_i) \in T_R(M)$, $(n_i) \in T^*_R(M)$ such that $(m, 0) = (m_1, m_2) + (n_1, n_2)$. Then, $m = m_1 + n_1$ is the fusible decomposition of m in M_1. Hence M_1 is fusible. A similar proof reveals that M_2 is fusible. □

However, despite all our efforts we have not succeeded in answering positively the following question for modules over an arbitrary ring.

Question. Is a (finite) direct sum of fusible R-modules a fusible R-module?

We now determine the condition for which the answer is positive.

Theorem 2.20. *Let M_i be a fusible R-module for $i = 1, 2, \dots, n$. If $T_R(\bigoplus_{i=1}^{n} M_i)$ is a subgroup of $\bigoplus_{i=1}^{n} M_i$, then $\bigoplus_{i=1}^{n} M_i$ is fusible.*

Proof. Let $n = 2$ and $0 \neq (m_1, m_2) \in M_1 \oplus M_2$. Consider the following cases:
Case I. Assume that $m_1 \neq 0$ and $m_2 = 0$. Since M_1 is fusible, there exist $m'_1 \in T_R(M_1)$ and $m''_1 \in T^*_R(M_1)$ such that $m_1 = m'_1 + m''_1$. Then, $(m_1, 0) = (m'_1, 0) + (m''_1, 0)$ with $(m'_1, 0) \in T_R(M_1 \oplus M_2)$, $(m''_1, 0) \in T^*_R(M_1 \oplus M_2)$.
Case II. Let $m_1 = 0$ and $m_2 \neq 0$. The proof is similar to Case I.
Case III. Assume that $m_1 \neq 0$ and $m_2 \neq 0$. The fusibility of M_1, M_2 yields that there exist $m'_i \in T_R(M_i)$ and $m''_i \in T^*_R(M_i)$ such that $m_i = m'_i + m''_i$ for $i = 1, 2$. It follows that $(m_1, m_2) = (m'_1, m'_2) + (m''_1, m''_2)$. On the one hand, having $(m'_1, 0), (0, m'_2) \in T_R(M_1 \oplus M_2)$ implies $(m'_1, m'_2) \in T_R(M_1 \oplus M_2)$ by hypothesis. On the other hand, $(m''_1, m''_2) \in T^*_R(M_1 \oplus M_2)$.
Therefore (m_1, m_2) is fusible, and so $M_1 \oplus M_2$ is fusible. The proof is completed by induction on n. □

If R is an integral domain, then the set of all torsion elements of an R-module is a subgroup. Hence we have the following result.

Corollary 2.21. *Let R be an integral domain. If M_i is a fusible R-module for $i = 1, 2, \dots, n$, then $\bigoplus_{i=1}^{n} M_i$ is fusible.*

We now provide some examples illustrating the above results.

Example 2.22. Let M denote the abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then,

(1) M is not a fusible $\mathbb{Z}$-module,
(2) M is a fusible $\mathbb{Z}_2$-module,
(3) M is a fusible $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-module.

Proof. (1) Clear by the fact that M is a torsion $\mathbb{Z}$-module.
(2) It is obvious since M is a torsion-free $\mathbb{Z}_2$-module.

(3) Note that $(\bar{1},\bar{1}) \in T^*_{\mathbb{Z}_2\oplus\mathbb{Z}_2}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ and $(\bar{0},\bar{1})$, $(\bar{1},\bar{0}) \in T_{\mathbb{Z}_2\oplus\mathbb{Z}_2}(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. The fusible decompositions of non-zero elements of M are $(\bar{1},\bar{0}) = (\bar{0},\bar{1}) + (\bar{1},\bar{1})$, $(\bar{0},\bar{1}) = (\bar{1},\bar{0}) + (\bar{1},\bar{1})$ and $(\bar{1},\bar{1}) = (\bar{0},\bar{0}) + (\bar{1},\bar{1})$. □

Next, we investigate the fusibility of elements in direct sums and also the fusibility of direct sums of fusible modules.

Example 2.23. (1) Let $R = M_2(\mathbb{Z}_2)$ be the ring, $N = M_2(\mathbb{Z}_2)$ and $M = N \oplus N$ denote the R-modules, and $A = \begin{bmatrix} \bar{1} & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ \bar{1} & 0 \end{bmatrix} \in N$. Then, A and B have fusible decompositions $A = A_{ii} + A_{ij}$ and $B = B_{ii} + B_{ij}$ in N with A_{ii}, $B_{ii} \in T_R(N)$ and A_{ij}, $B_{ij} \in T^*_R(N)$ but the corresponding couples (A_{ii}, B_{ii}) need not belong to $T_R(M)$ in general.
(2) Let $N = M_2(\mathbb{Z}_2)$ as an $R = U_2(\mathbb{Z}_2)$-module. Then,
(i) N is fusible.
(ii) Let $M = N \oplus N$. Then, M is fusible.

Proof. (1) Let $A = \begin{bmatrix} \bar{1} & 0 \\ 0 & 0 \end{bmatrix}$ have a fusible decomposition as $A = C + D$ where $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in T_R(N)$ and $D = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in T^*_R(N)$. Since $D \in T^*_R(N)$, $xt = \bar{1}$ or $yz = \bar{1}$. If $xt = \bar{1}$, then $x = t = \bar{1}$, so $a = 0$ and $d = \bar{1}$. Similarly, if $yz = \bar{1}$, then $y = z = \bar{1}$, so $b = \bar{1}$ and $c = \bar{1}$. By considering possibilities in this way we may reach the following decompositions:
$A = \begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{1} & 0 \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} 0 & \bar{1} \\ 0 & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \bar{1} & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{1} & 0 \\ \bar{1} & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix} + \begin{bmatrix} 0 & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}$,
$B = \begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{1} \\ \bar{1} & 0 \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{1} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & 0 \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ 0 & \bar{1} \end{bmatrix} = \begin{bmatrix} 0 & \bar{1} \\ 0 & \bar{1} \end{bmatrix} + \begin{bmatrix} 0 & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}$. Consider the couple $X = \left(\begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix}, \begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix} \right)$. Since $\left(\begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix}, \begin{bmatrix} 0 & \bar{1} \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \bar{1} & \bar{1} \\ 0 & 0 \end{bmatrix} = 0$, we have $X \in T_R(M)$. On the other hand, take the couple $Y = \left(\begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ 0 & 0 \end{bmatrix} \right)$. Then, $\left(\begin{bmatrix} 0 & 0 \\ 0 & \bar{1} \end{bmatrix}, \begin{bmatrix} \bar{1} & \bar{1} \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0$ implies $a = b = c = d = 0$. So $Y \notin T_R(M)$.
Another one is similar.
(2) (i) Consider $N = M_2(\mathbb{Z}_2)$ as an $R = U_2(\mathbb{Z}_2)$-module. We list fusible decompositions of non-zero torsion elements in M as follows:

$$\begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix}, \quad \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix},$$
$$\begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}, \quad \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix},$$
$$\begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{1} \end{bmatrix}, \quad \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{bmatrix},$$
$$\begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{1} & \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{bmatrix}, \quad \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix},$$
$$\begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{0} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}.$$

The other 6 elements except 0 are regular. Hence N is a fusible R-module.
(ii) It can easily be seen that $M = N \oplus N$ is fusible. □

We now focus on the direct product of modules in terms of the fusibility.

Lemma 2.24. *Let $\{R_i\}_{i\in I}$ be a family of rings and $\{M_i\}_{i\in I}$ be a family of modules with each M_i is an R_i-module. Let $M = \prod_{i\in I} M_i$ and $R = \prod_{i\in I} R_i$. Then, $T^*_R(M) = \prod_{i\in I} T^*_{R_i}(M_i)$.*

Proof. Note that $(m_i)(r_i) = (m_ir_i)$ for any $(m_i) \in M$ and $(r_i) \in R$. Let $(m_i) \in T^*_R(M)$ and $(r_i) \in R$ with $m_ir_i = 0$ for each $i \in I$. Then, $(m_i)(r_i) = 0$, and so $(r_i) = 0$. This implies that $r_i = 0$, so $m_i \in T^*_{R_i}(M_i)$ for each $i \in I$. Hence $T^*_R(M) \subseteq \prod_{i\in I} T^*_{R_i}(M_i)$. For the reverse inclusion, let $m = (m_i) \in \prod_{i\in I} T^*_{R_i}(M_i)$ and $r = (r_i) \in R$ with $mr = 0$. It follows that $m_ir_i = 0$ for each $i \in I$. Hence $r_i = 0$ for each $i \in I$. Thus, $r = 0$, and therefore $m \in T^*_R(M)$. □

Theorem 2.25. *Let $\{R_i\}_{i\in I}$ be a family of rings, $\{M_i\}_{i\in I}$ be a family of modules with each M_i is an R_i-module, $M = \prod_{i\in I} M_i$ and $R = \prod_{i\in I} R_i$. Then, M is a fusible R-module if and only if M_i is a fusible R_i-module for each $i \in I$.*

Proof. Assume that M is a fusible R-module. Let $x_i \in M_i$. Define $0 \neq (m_i) \in M$ such that $m_i = x_i$ and $m_k = 0$ if $k \neq i$. By assumption, $(m_i) = (z_i) + (y_i)$ where $(z_i) \in T_R(M)$, $(y_i) \in T^*_R(M)$. Then, there exists $0 \neq (t_i) \in R$ with $(z_i)(t_i) = 0$. Then, $z_it_i = 0$ for $i \in I$. On the one hand, for each $k \in I$ with $k \neq i$, $z_k + y_k = 0$ and $x_i = z_i + y_i$. By Lemma 2.24, $(y_i) \in T^*_R(M)$ yields $y_i \in T^*_{R_i}(M_i)$ for each $i \in I$. On the other hand, $z_k + y_k = 0$ implies $y_kt_k = 0$ for each $k \in I$ with $k \neq i$. So $t_k = 0$ for each $k \in I$ with $k \neq i$. Hence $t_i \neq 0$ and $x_i = z_i + y_i$ is a fusible decomposition of $x_i \in M_i$. Thus, each M_i is a fusible R_i-module.

Conversely, suppose that for each $i \in I$, M_i is a fusible R_i-module. Let $(m_i) \in M$. For every $0 \neq m_i \in M_i$, there exists a decomposition $m_i = z_i + y_i$ with $z_i \in T_{R_i}(M_i)$, $y_i \in T^*_{R_i}(M_i)$. Let $0 \neq z'_i \in R_i$ with $z_iz'_i = 0$. Define $(u_i) \in T_R(M)$ and $(v_i) \in T^*_R(M)$ such that $(m_i) = (u_i) + (v_i)$ as follows: If $m_i \neq 0$, then $u_i = z_i$ and $v_i = y_i$. If $m_i = 0$, then consider $0 \neq n_i \in M_i$ due to $M_i \neq 0$. There exists a decomposition $n_i = a_i + b_i$ with $a_i \in T_{R_i}(M_i)$, $b_i \in T^*_{R_i}(M_i)$. For all $i \in I$ with $m_i = 0$, we define $u_i = -b_i$ and $v_i = b_i$. Then, on the one hand, we claim $(u_i) \in T_R(M)$. In fact, we define $(u'_i) \in R$ such that $(u_i)(u'_i) = 0$ as follows: If $m_i \neq 0$, we let $u'_i = z'_i$, otherwise, that is $m_i = 0$, we let $u'_i = 0$. Then, $(u_i)(u'_i) = 0$. Hence $(u_i) \in T_R(M)$. On the other hand, $(v_i) \in T^*_R(M)$ by Lemma 2.24. Thus, (m_i) has a fusible decomposition as $(m_i) = (u_i) + (v_i)$. Therefore M is fusible. □

Let R be a ring. Consider the polynomial ring $R[x]$ and the power series ring $R[[x]]$ over R. Let α be an endomorphism of R, that is, $\alpha\colon R \to R$ is a ring homomorphism with $\alpha(1) = 1$. We denote by $R[x;\alpha]$ the ring of skew polynomials in an indeterminate x. Every element of $R[x;\alpha]$ can be uniquely written in the form $f(x) = \sum_{i=0}^{n} a_ix^i$ and the multiplication is completely determined by $xa = \alpha(a)x$. Similarly, $R[[x;\alpha]]$ is the ring of skew formal power series and every element can be uniquely written in the form $f(x) = \sum_{i=0}^{\infty} a_ix^i$ and the multiplication is the same as $xa = \alpha(a)x$.

For a module M, consider polynomial extensions:

- $M[x;\alpha] := \left\{\sum_{i=0}^{s} m_ix^i \mid s \geq 0, m_i \in M\right\}$,
- $M[[x;\alpha]] := \left\{\sum_{i=0}^{\infty} m_ix^i \mid m_i \in M\right\}$.

Let $m(x) = \sum_{i=0}^{\infty} m_ix^i \in M[[x;\alpha]]$ and $f(x) = \sum_{i=0}^{\infty} a_ix^i \in R[[x;\alpha]]$. Define

$$m(x)f(x) = \sum_t \Big(\sum_{i+j=t} m_i\alpha^i(a_j)\Big)x^t. \tag{2.1}$$

With the usual addition and multiplication defined by (2.1), $M[[x;\alpha]]$ becomes a module over $R[[x;\alpha]]$. A similar definition as (2.1), $M[x;\alpha]$ is a module over the ring $R[x;\alpha]$. In this direction we have the following result.

Theorem 2.26. *Let R be a domain and σ be a ring endomorphism of R which is not injective. If $M[x;\sigma]$ is a fusible $R[x;\sigma]$-module, then M is a fusible R-module.*

Proof. Assume that $M[x;\sigma]$ is a fusible $R[x;\sigma]$-module. Let $0 \neq m \in M$ and consider $m(x) = m$ in $M[x;\sigma]$. Since $M[x;\sigma]$ is fusible, there exist $d(x) = d_0 + d_1x + \cdots + d_nx^n \in T_{R[x;\sigma]}(M[x;\sigma])$, $e(x) = e_0 + e_1x + \cdots + e_tx^t \in T^*_{R[x;\sigma]}(M[x;\sigma])$ with $m(x) = d(x) + e(x)$. Then, $m = d_0 + e_0$ and

$$d_1 + d_2x + \cdots + d_nx^{n-1} + e_1 + e_2x + \cdots + e_tx^{t-1} = 0.$$

Consider the following cases:
(1) Let $d_0 \neq 0$ and $e_0 \neq 0$. Since $d(x) \in T_{R[x;\sigma]}(M[x;\sigma])$, there exists $0 \neq f(x) = f_0 + f_1x + f_2x^2 + \cdots + f_kx^k \in R[x;\sigma]$ with $d(x)f(x) = 0$. It follows that $d_0f_0 = 0$. If $f_0 \neq 0$, there is nothing to do. If $f_0 = 0$, then $d_0f_1 = 0$. Similarly, if $f_1 \neq 0$, there is nothing to do. If $f_1 = 0$, then $d_0f_2 = 0$. By continuing in this way, we reach that there exists $0 \neq f_i \in R$ such that $d_0f_i = 0$. Thus, $d_0 \in T_R(M)$. Next, we claim that $e_0 \in T^*_R(M)$. Assume that $e_0s = 0$ for $s \in R$. Then, $e(x)s = e_1xs + \cdots + e_tx^ts = g(x)x$ where $g(x) \in M[x;\sigma]$. Multiplying the latter by r from the right, we get $e(x)sr = 0$. Hence $sr = 0$ since $e(x) \in T^*_{R[x;\sigma]}(M[x;\sigma])$. The ring R being a domain and $r \neq 0$ imply $s = 0$. Thus, $e_0 \in T^*_R(M)$. Hence $m = d_0 + e_0$ is a fusible decomposition of m in M.
(2) Let $d_0 \neq 0$ and $e_0 = 0$. Then, $m = d_0$. However this is not a fusible decomposition of m in $M[x;\sigma]$. So this is not the case.
(3) Let $d_0 = 0$ and $e_0 \neq 0$. Then, $m = e_0 \in T^*_R(M)$, so $m \in M$ is fusible.
Therefore M is fusible. □

Corollary 2.27. *Let R be a domain and σ be a ring endomorphism of R which is not injective. If $M[[x;\sigma]]$ is a fusible right $R[[x;\sigma]]$-module, then M is a fusible right R-module.*

Theorem 2.28. *Let M be a fusible R-module. Then, the following hold.*

(i) *Every element of $M[x;\alpha]$ with have constant term is fusible.*
(ii) *If α is an isomorphism, then $M[x;\alpha]$ is a fusible $R[x;\alpha]$-module.*

Proof. (i) Let $m(x) = m_0 + m_1x + m_2x^2 + \cdots + m_nx^n \in M[x;\alpha]$ with $m_0 \neq 0$. Since M is fusible, there exist $z_0 \in T_R(M)$ and $d_0 \in T^*_R(M)$ such that $m_0 = z_0 + d_0$. Let $n_0(x) = z_0$ and $d_0(x) = d_0 + m_1x + m_2x^2 + \cdots + m_nx^n$. Then, $n_0(x) \in T_{R[x;\alpha]}(M[x;\alpha])$ and $d_0(x) \in T^*_{R[x;\alpha]}(M[x;\alpha])$. In fact, let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_tx^t \in R[x;\alpha]$ and assume that $d_0(x)f(x) = 0$ and $t \leq n$. Then, we have the following:

(1) $d_0a_0 = 0$. It implies $a_0 = 0$ since $d_0 \in T^*_R(M)$.
(2) $d_0a_1 + m_1\alpha(a_0) = 0$. Hence $a_1 = 0$.
(3) $d_0a_2 + m_1\alpha(a_1) + m_2\alpha^2(a_0) = 0$, so $a_2 = 0$.
(4) $d_0a_3 + m_1\alpha(a_2) + m_2\alpha^2(a_1) + m_3\alpha^3(a_0) = 0$, and $a_3 = 0$.

Continuing in this way, we may reach
(n) $d_0a_{n-1} + m_1\alpha(a_{n-2}) + m_2\alpha^2(a_{n-3}) + \cdots + m_n\alpha^n(a_0) = 0$ gives $a_{n-1} = 0$. It follows that $a_n = 0$ and so $f(x) = 0$. Now assume that $t > n$. After the n-th step, we have $d_0a_{n+1} = d_0a_{n+2} = \cdots = d_0a_t = 0$. This yields $a_{n+1} = a_{n+2} = \cdots = a_t = 0$. Thus, $f(x) = 0$. Therefore $m(x) = n_0(x) + d_0(x)$ is a fusible decomposition of $m(x)$.
(ii) Assume that $0 \neq m(x) = (m_0 + m_1x + m_2x^2 + \cdots + m_nx^n)x^l \in M[x;\alpha]$ with $l \neq 0$. Then, $m_0 \neq 0$, and so there exist $z_0 \in T_R(M)$ and $d_0 \in T^*_R(M)$ such that $m_0 = z_0 + d_0$. Hence $n_0(x) = z_0x^l \in T_{R[x;\alpha]}(M[x;\alpha])$ is clear, and $d_0(x) = (d_0 + m_1x + m_2x^2 + \cdots + m_nx^n)x^l \in T^*_{R[x;\alpha]}(M[x;\alpha])$. In fact, let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_tx^t \in R[x;\alpha]$ and assume that $d_0(x)f(x) = 0$ and $t \leq n$. Note that $x^la_i = \alpha^{l-i}(a_i)x^l$. By $d_0(x)f(x) = 0$, we have the following:

(1) $d_0\alpha^l(a_0) = 0$. It implies $a_0 = 0$ since α is an isomorphism. It implies $a_0 = 0$ since $d_0 \in T^*_R(M)$.
(2) $d_0\alpha^l(a_1) + m_1\alpha^{l+1}(a_0) = 0$. Hence $a_1 = 0$.
(3) $d_0\alpha^l(a_2) + m_1\alpha^{l+1}(a_1) + m_2\alpha^{l+2}(a_0) = 0$, so $a_2 = 0$.

$$\vdots$$

(n-1) $d_0\alpha^l(a_{n-2}) + m_1\alpha^{l+1}(a_{n-3}) + \cdots + m_{n-1}\alpha^{l+(n-2)}(a_0) = 0$, and $a_{n-2} = 0$.

(n) $d_0\alpha^l(a_{n-1}) + m_1\alpha^{l+1}(a_{n-2}) + \cdots + m_n\alpha^{l+(n-1)}(a_0) = 0$, thus $a_{n-1} = 0$. It follows that $a_n = 0$ and so $f(x) = 0$.

Now assume that $t > n$. After the n-th step, we have $d_0a_{n+1} = d_0a_{n+2} = \cdots = d_0a_t = 0$. This yields $a_{n+1} = a_{n+2} = \cdots = a_t = 0$. Thus, $f(x) = 0$ and so $d_0(x) \in T^*_{R[x;\alpha]}(M[x;\alpha])$. Therefore $m(x) = n_0(x) + d_0(x)$ is a fusible decomposition of $m(x)$. The proof is completed. □

Theorem 2.29. *If M is a fusible R-module, then $M[x]$ is a fusible $R[x]$-module. The converse holds if R is an integral domain.*

Proof. Let M be a fusible R-module. Then, $M[x]$ is a fusible $R[x]$-module by taking $\alpha = 1_R$ in Theorem 2.28 (ii). For the converse, let R be an integral domain and $0 \neq m \in M$. Since $M[x]$ is a fusible $R[x]$-module, m has a fusible decomposition such as $m = f(x)+g(x)$ where $f(x) \in T_{R[x]}(M[x])$ and $g(x) \in T^*_{R[x]}(M[x])$ where $f(x) = a_0 + a_1x + \cdots + a_tx^t$, $g(x) = b_0+b_1x+\cdots+b_nx^n \in M[x]$. Then, by the equality of the polynomials, $m = a_0+b_0$. On the one hand, $a_0 \in T_R(M)$ as a similar discussion in the proof of Theorem 2.26. On the other hand, in case $b_0 \in T^*_R(M)$, there is nothing to do. Assume that $b_0 \in T_R(M)$. Since $M[x]$ is fusible, b_0 has a fusible decomposition such as $b_0 = f_1(x) + g_1(x)$ with $f_1(x) = c_0 + f_1'(x)x \in T_{R[x]}(M[x])$, $g_1(x) = d_0 + g_1'(x)x \in T^*_{R[x]}(M[x])$. Then, $b_0 = c_0 + d_0$. As it is done previously, $c_0 \in T_R(M)$. Assume that $d_0 \in T^*_R(M)$. Then, it is done since R is an integral domain, $a_0+c_0 \in T_R(M)$, and so $m = (a_0+c_0)+d_0$ is a fusible decomposition of m. Otherwise, $d_0 \in T_R(M)$ and it has a fusible decomposition in $M[x]$ as $d_0 = f_2(x)+g_2(x)$ with $f_2(x) = c_1 + f_2'(x)x \in T_{R[x]}(M[x])$, $g_2(x) = d_1 + g_2'(x)x \in T^*_{R[x]}(M[x])$. Then, $d_0 = c_1 + d_1$, as it done before $c_1 \in T_R(M)$. If $d_1 \in T^*_R(M)$, there is nothing to do. In fact, since $a_0 + c_0 + c_1 \in T_R(M)$ and $d_1 \in T^*_R(M)$ give a fusible decomposition of $m = (a_0 + c_0 + c_1) + d_1$ since the sum of the torsion elements are torsion in R. Otherwise, assume that $d_1 \notin T^*_R(M)$. Since $M[x]$ is fusible, d_1 has a fusible decomposition as $d_1 = f_3(x) + g_3(x)$ with $f_3(x) = c_2 + f_3'(x)x \in T_{R[x]}(M[x])$, $g_3(x) = d_2 + g_3'(x)x \in T^*_{R[x]}(M[x])$. Then, $d_1 = c_2 + d_2$. As previously done that $c_2 \in T_R(M)$. If $d_2 \in T^*_R(M)$, we are done. Otherwise, that is $d_2 \notin T^*_R(M)$, this procedure continuous. But this continuation leads us a contradiction. So this process stops at a finite step. Thus, m has a fusible decomposition in M. □

Reduced modules are defined and investigated in [3]. Recall that a module M is *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $(mR) \cap (Ma) = 0$. Note that a ring R is reduced if and only if the R-module R is reduced.

Lemma 2.30. [3, Lemma 1.2] *Let M be an R-module. Then, the following are equivalent:*

(1) *M is reduced;*

(2) *For any $m \in M$ and $a \in R$;*

  (i) *$ma^2 = 0$ implies $ma = 0$;*

  (ii) *$ma = 0$ implies $mRa = 0$.*

We show that there are many reduced modules M that are not fusible.

Example 2.31. Let p be a prime number, $n \geq 2$ and $R = \mathbb{Z}_{p^n}$. Consider $M = p^{n-1}\mathbb{Z}_{p^n}$ as an R-module. Then, M is reduced which is not fusible.

Proof. Note that $M = p^{n-1}\mathbb{Z}_{p^n} = \{\overline{0}, \overline{p^{n-1}}, \overline{p^{n-1}2}, \ldots, \overline{p^{n-1}(p-1)}\}$. Let $0 \neq \overline{m} = \overline{tp^{n-1}} \in M$ and $\overline{s} \in R$ with $\overline{ms} = 0$. Since $0 \neq \overline{m} = \overline{tp^{n-1}}$, p does not divide t. This and the assumption $\overline{ms} = 0$ entail that p divides s. We claim that $(\overline{m}R) \cap (M\overline{s}) = 0$. Let $\overline{mr} = \overline{m_1s} \in (\overline{m}R) \cap (M\overline{s})$. Since p divides s, p also divides mr, and therefore p divides

r. Hence $\overline{mr} = 0$, and so $(\overline{m}R) \cap (M\overline{s}) = 0$. Thus, M is reduced. On the other hand, M is not fusible because for all $0 \neq \overline{m} \in M$, $\overline{mp} = 0$ for $\overline{p} \in R$. □

There are fusible R-modules that are not reduced.

Example 2.32. Consider the ring $R = M_2(\mathbb{Z}_2)$ as an R-module over itself, i.e., $M_R = M_2(\mathbb{Z}_2)$. Then, M is fusible but not reduced.

Proof. Since $\mathbb{Z}_2$ is fusible, [1, Theorem 2.18] entails that M is a fusible R-module. We claim that M is not reduced. For if, $m = \begin{bmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{bmatrix}$, $a = \begin{bmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{1}\end{bmatrix}$, then $ma = 0$. If M_R would be reduced, then $(mR) \cap (Ma)$ would be zero. Let $r = \begin{bmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{bmatrix} \in R$ and $m_1 = \begin{bmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{bmatrix} \in M$. Then, $mr = m_1 a \in (mR) \cap (Ma)$. In fact, $mr = \begin{bmatrix}\bar{1} & \bar{1}\\ \bar{0} & \bar{0}\end{bmatrix}\begin{bmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{bmatrix} = \begin{bmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{0}\end{bmatrix} = \begin{bmatrix}\bar{1} & \bar{0}\\ \bar{0} & \bar{0}\end{bmatrix}\begin{bmatrix}\bar{0} & \bar{1}\\ \bar{0} & \bar{1}\end{bmatrix} = m_1 a \neq 0$. It follows that M is not a reduced R-module. □

In commutative context, Ghashghaei and McGovern have shown that fusible rings are reduced in [1]. Now we investigate similar property for fusible modules. Recall that a ring R is called *duo* if every one sided ideal is two sided, equivalently, $aR \subseteq Ra$ and $Ra \subseteq aR$ for each $a \in R$.

Theorem 2.33. *Let M be a fusible R-module and R be a duo ring. Then, M is reduced.*

Proof. (i) Let $m \in M$ and $a \in R$. Suppose that $ma^2 = 0$. If $ma = 0$, there is nothing to show. Assume that $ma \neq 0$. Since M is fusible, there exist $z \in T_R(M)$, $y \in T^*_R(M)$ and $0 \neq z' \in R$ such that $ma = z + y$ and $zz' = 0$. Then, $za + ya = 0$ and $za \in zR \subseteq Rz$. Hence there exists $t \in R$ with $za = tz$. So $0 = za + ya = tz + ya$. Multiplying the latter from the right by z', we get $tzz' + yaz' = 0$. Hence $yaz' = 0$. Since $y \in T^*_R(M)$, $az' = 0$. Having $ma = z + y$ implies $maz' = zz' + yz'$, and so $yz' = 0$. Thus, $z' = 0$. This is a contradiction. It follows that $ma^2 = 0$ implies $ma = 0$.
(ii) Let $m \in M$ and $a \in R$ with $ma = 0$. Suppose that $(mR) \cap (Ma) \neq 0$. Set $0 \neq mr' = m'a \in (mR) \cap (Ma)$. Then, $r'a \in Ra \subseteq aR$. Hence $r'a = at$ for some $t \in R$. Multiplying $mr' = m'a$ from the right by a and using $r'a = at$ entails $m'a^2 = mr'a = mat = 0$. By (i), $m'a = 0$. Thus, $(mR) \cap (Ma) = 0$. Having $mRa \subseteq (mR) \cap (Ma)$ yields $mRa = 0$. Therefore by Lemma 2.30, M is reduced. □

The next result is an immediate consequence of Theorem 2.33.

Corollary 2.34. *Every fusible module over a commutative ring is reduced.*

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