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Research Article

Stability, Neimark-Sacker Bifurcation Analysis of a Prey-Predator Model with Strong Allee Effect and Chaos Control

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Abstract

In this study, the dynamical behaviors of a prey–predator model with strong Allee effect are investigated. Existence of the fixed points of the model are examined and topological classification of the coexistence fixed point is obtained. By selecting bifurcation parameter as a β, it is demonstrated that the model can experience a Neimark-Sacker bifurcation at the coexistence fixed point. Bifurcation theory is used to present the Neimark-Sacker bifurcation conditions of existence and the direction of the bifurcation. Additionally, some numerical simulations are provided to back up the analytical result. OGY feedback control method is implemented to control chaos in presented model due to emergence of Neimark-Sacker bifurcation. Following that, the model's bifurcation diagram, maximum Lyapunov exponents and the triangle-shaped stability zone are provided.

Keywords: Prey-predator model, strong Allee effect, Neimark-Sacker bifurcation analysis

Güçlü Allee Etkili Av-Avcı Modelinin Kararlılığı, Neimark-Sacker Çatallanma Analizi ve Kaos Kontrol

Öz

Bu çalışmada, güçlü Allee etkisi içeren bir av-avcı modelinin dinamik davranışları araştırılmıştır. Modelin sabit noktalarının varlığı incelenmiştir ve her iki türün bir arada olduğu denge noktasının topolojik sınıflandırması elde edilmiştir. β çatallanma parametresi olarak seçildiğinde, modelin her iki türün bir arada olduğu denge noktasında bir Neimark-Sacker çatallanması olacağı gösterilmiştir. Çatallanma teorisi, Neimark-Sacker çatallanma varoluş koşullarını ve çatallanmanın yönünü sunmak için kullanılır. Ek olarak, bazı sayısal simülasyonlar, analitik sonucu desteklemek için sunulmuştur. Sunulan modelde Neimark-Sacker çatallanmasının ortaya çıkması nedeniyle oluşan kaosu kontrol etmek için OGY geri besleme kontrol yöntemi uygulanmaktadır. Bunu takiben, modelin çatallanma diyagramı, maksimum Lyapunov üstelleri ve üçgen şeklindeki kararlılık bölgesi verilmiştir.

Anahtar Kelimeler: Av-avcı modeli, güçlü Allee etkisi, Neimark-Sacker çatallanma analizi.

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1. Introduction

In the literature on bio-mathematics, the study of prey-predator systems that demonstrate interactions between two prey-predator species has been a significant topic. Many researchers have recently investigated the intricate dynamics of prey-predator systems [1, 2, 3, 4, 5, 6, 7]. The Lotka-Volterra model, created by Lotka and Volterra, is the original and most basic prey-predator model. Many researchers have altered this model because it ignored a number of real situations.

Ecological aspects like emigration and immigration, functional responses, diffusion, time delays, etc. have been introduced. Allee effect is one of the key ecological variables that can significantly alter the prey-predator system [7, 8]. When simulating the interactions between predators and prey, it is crucial to take into account the increase of the prey population to its carrying capacity in the absence of predators. Consider the development of logistics. The per capita growth rate of prey achieves its maximum while their population density is low and starts to decrease as prey density increases, according to this logistic growth. However, such growth rates are not always advantageous for lower densities [9, 10]. There are various biological causes for this.

The Allee term derives from an experimental research conducted by renowned ecologist Warder Clyde Allee. Allee discovered in the 1930s that many natural populations, including those of plants, birds, marine invertebrates, insects and mammals, frequently experience individual fitness losses at lower critical densities. It describes an association between any metric of species fitness and population size that is favorable. Allee effect can be split into two groups: strong effects and weak effects. For the strong Allee effect, there is a population threshold below which the species becomes extinct. On the other hand, the weak Allee effect appears when the growth rate slows down while still being positive at low population density [11, 12, 13, 14]. Many researchers have been interested in the dynamics of predator-prey models that incorporate the Allee effect in prey development rate [15, 16, 17, 18, 19, 20, 21, 22, 23].

It is not well recognized that two or more Allee effects can occur simultaneously in the same population. For the management of endangered or exploited populations, it is crucial to consider the presence and interplay of various Allee effects. This work presents a mathematical investigation of the stability of a prey-predator system with strong Allee effect.

The following discrete-time predator-prey model has been considered by the author in [24]:

$$\begin{aligned} N_{t+1} &= \beta N_t(1-N_t) - N_t P_t \\ P_{t+1} &= \frac{1}{\alpha} N_t P_t \end{aligned} \tag{1}$$

where N_t and P_t represent the numbers of prey and predator, respectively. The α, β parameters are positive real numbers.

The current work looks at a discrete-time prey-predator model where the predator population outnumbers the prey population and the prey population is susceptible to a strong Allee effect. The stability and bifurcation analysis of the the prey-predator model with weak Allee effect will be examined in another study.

2. Preliminaries

The Model and Its Fixed Points

Modification of the model (1) is taken into consideration in this study,

$$\begin{aligned} N_{n+1} &= \beta N_n(1-N_n)\left(\frac{N_n-\zeta}{N_n+\eta}\right) - N_nP_n \\ P_{n+1} &= \frac{1}{\alpha}N_nP_n \end{aligned} \tag{2}$$

where N_n and P_n represent the densities of prey and predator, respectively, α and β are the intrinsic growth rates of the predator and prey, η is the non-fertile prey population, ζ is the Allee coefficient.

By biological setting, we have $0 < \eta < 1$ and $-\eta < \zeta < 1$, so $\zeta + \eta > 0$. If $0 < \zeta < 1$, the Allee effect is called strong, while if $-\eta < \zeta < 0$, it is called weak Allee effect.

The $\frac{N_n-\zeta}{N_n+\eta}$ term represents the multiple Allee effect which means two or more Allee effects act simultaneously on the single population.

We now explore the discrete-time prey-predator model's stability and the existence of fixed points, including the Allee effect on prey.

If we write

$$N_n \quad = N_{n+1} = N^*, \;\; P_n \quad = P_{n+1} = P^* \tag{3}$$

in system (2), the following system can be obtained as follows:

$$\begin{aligned} N^* &= \beta N^*(1-N^*)(\frac{N^*-\zeta}{N^*+\eta}) - N^*P^* \\ P^* &= \frac{1}{\alpha}N^*P^* \end{aligned} \tag{4}$$

A simple calculation reveals that the system (2) has the following three fixed points:

$$D_1 = (0,0)$$

$$D_2 = \left(\frac{\beta\zeta+\beta-1+\sqrt{\beta^2\zeta^2-2\beta^2\zeta+\beta^2-2\beta\zeta-4\beta\eta-2\beta+1}}{2\beta}, 0\right)$$

$$D_3 = \left(\alpha, \beta(1-\alpha)\left(\frac{\alpha-\zeta}{\alpha+\eta}\right)-1\right)$$

3. Main Theorem and Proof

Lemma 3.1 *For the system (2), the following statements hold true:*

i) *The system (2) always has an axial fixed point* $D_1 = (0,0)$.

ii) *The system (2) has an axial fixed point* $D_2 = \left(\frac{\beta\zeta+\beta-1+\sqrt{\beta^2\zeta^2-2\beta^2\zeta+\beta^2-2\beta\zeta-4\beta\eta-2\beta+1}}{2\beta}, 0\right)$ *if*

$$\eta \le \frac{\beta^2(\zeta-1)^2-2\beta(\zeta+1)+1}{4\beta}$$

iii) *The system (2) has an coexistence fixed point* $D_3 = \left(\alpha, \beta(1-\alpha)(\frac{\alpha-\zeta}{\alpha+\eta})-1\right)$ *if the following condition holds:*

$$0 < \alpha < 1 \qquad \beta > \left(\frac{\alpha+\eta}{(1-\alpha)(\alpha-\zeta)}\right) \tag{5}$$

3.1. Local Stability Analysis

As D_3 is biologically significant, stability and bifurcation analysis has been studied for this fixed point only. The Jacobian matrix of the system (2) evaluated at the fixed point D_3 is as following:

$$J(D_3) = \begin{pmatrix} \frac{-\beta\alpha^3+(-2\beta\eta+1)\alpha^2+((2+(\zeta+1)\beta)\eta+\beta\zeta)\alpha+\eta^2}{(\alpha+\eta)^2} & -\alpha \\ -\frac{\beta\alpha^2-\beta\alpha\zeta-\beta\alpha+\beta\zeta+\alpha+\eta}{(\alpha+\eta)\alpha} & 1 \end{pmatrix}$$

Moreover, the characteristic polynomial of $J(D_3)$ is given by:

$$\begin{aligned} F(\lambda) = \lambda^2 \quad & - \quad \left(\frac{-\beta\alpha^3 + (-2\beta\eta + 2)\alpha^2 + ((4 + (\zeta + 1)\beta)\eta + \beta\zeta)\alpha + 2\eta^2}{(\alpha + \eta)^2}\right)\lambda \\ & + \quad \frac{\beta(-2\alpha^3 + (\zeta - 3\eta + 1)\alpha^2 + 2\eta(\zeta + 1)\alpha - \zeta\eta)}{(\alpha + \eta)^2} \end{aligned} \tag{6}$$

Then, by simple computations it follows that

$$F(1) = -\frac{\beta\alpha^2 - \beta\alpha\zeta - \beta\alpha + \beta\zeta + \alpha + \eta}{\alpha + \eta}, \tag{7}$$

$$F(-1) = \frac{-3\beta\alpha^3 + (3 + (\zeta - 5\eta + 1)\beta)\alpha^2 + (((3\zeta + 3)\eta + \zeta)\beta + 6\eta)\alpha - \beta\zeta\eta + 3\eta^2}{(\alpha + \eta)^2}, \tag{8}$$

and

$$F(0) = \frac{\beta\left(-2\alpha^3 + (\zeta - 3\eta + 1)\alpha^2 + 2\eta(\zeta + 1)\alpha - \zeta\eta\right)}{(\alpha + \eta)^2} \tag{9}$$

The following lemma will be used to discuss the dynamics of coexistence fixed point of system.

Lemma 3.2 *$F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants and let $F(1) > 0$. Suppose λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then the following statements hold.*

(i) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $F(0) < 1$.

(ii) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $F(0) > 1$.

(iii) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$*, or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*, if and only if* $F(-1) < 0$.

(iv) λ_1 *and* λ_2 *are a pair of conjugate complex roots and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $B^2 - 4C < 0$ *and* $F(0) = 1$.

(v) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $B \neq 0, 2$.

Assume that λ_1 and λ_2 are the roots of the characteristic polynomial at the coexistence fixed point (N, P). Then, the point (N, P) is called sink if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ and it is locally asymptotically stable. (N, P) is called source if $|\lambda_1| > 1$ and $|\lambda_2| > 1$ and it is locally unstable. The point (N, P) is called saddle if $|\lambda_1| < 1$ and $|\lambda_2| > 1$, or $|\lambda_1| > 1$ and $|\lambda_2| < 1$. And, (N, P) is called non-hyperbolic if either $|\lambda_1| = 1$ or $|\lambda_2| = 1$.

Theorem 3.3 *For coexistence fixed point D_3 of system (2) the following holds true:*
(i) D_3 is a sink if and only if

$$3\beta\alpha^3 + 5\eta\beta\alpha^2 + \beta\zeta\eta < (3 + \zeta\beta + \beta)\alpha^2 + (3\zeta\eta\beta + 3\eta\beta + \zeta\beta + 6\eta)\alpha + 3\eta^2$$

and

$$\beta(\zeta+1)\alpha(\alpha+2\eta) < (\alpha+\eta)^2 + 2\beta\alpha^3 + 3\eta\beta\alpha^2 + \beta\zeta\eta$$

(ii) D_3 is a source if and only if

$$3\beta\alpha^3 + 5\eta\beta\alpha^2 + \beta\zeta\eta < (3+\zeta\beta+\beta)\alpha^2 + (3\zeta\eta\beta + 3\eta\beta + \zeta\beta + 6\eta)\alpha + 3\eta^2$$

and

$$\beta(\zeta+1)\alpha(\alpha+2\eta) > (\alpha+\eta)^2 + 2\beta\alpha^3 + 3\eta\beta\alpha^2 + \beta\zeta\eta$$

(iii) D_3 is a saddle if and only if

$$3\beta\alpha^3 + 5\eta\beta\alpha^2 + \beta\zeta\eta > (3+\zeta\beta+\beta)\alpha^2 + (3\zeta\eta\beta + 3\eta\beta + \zeta\beta + 6\eta)\beta + 3\eta^2$$

(iv) The roots of Eq.(6) are complex with modulus one if and only if

$$\beta\alpha^3 + (2\beta\eta-2)\alpha^2 + ((-4+(-\zeta-1)\beta)\eta - \beta\zeta)\alpha - 2\eta^2 < 4\beta(2\alpha^3 - \alpha^2\zeta + 3\alpha^2\eta - 2\alpha\zeta\eta - \alpha^2 - 2\alpha\eta + \zeta\eta)(\alpha+\eta)^2$$

and

$$\beta(\zeta+1)\alpha(\alpha+2\eta) = (\alpha+\eta)^2 + 2\beta\alpha^3 + 3\eta\beta\alpha^2 + \beta\zeta\eta$$

(v) Assume that λ_1 and λ_2 be roots of Eq.(6), then $\lambda_1 = -1$ and $|\lambda_2| \neq 1$ if and only if

$$3\beta\alpha^3 + 5\eta\beta\alpha^2 + \beta\zeta\eta = (3+\zeta\beta+\beta)\alpha^2 + (3\zeta\eta\beta + 3\eta\beta + \zeta\beta + 6\eta)\alpha + 3\eta^2$$

$$\beta\alpha^3 + (2\beta\eta-2)\alpha^2 + ((-4+(-\zeta-1)\beta)\eta - \beta\zeta)\alpha - 2\eta^2 \neq 0$$

and

$$\beta\alpha^3 + (2\beta\eta-4)\alpha^2 + ((-8+(-\zeta-1)\beta)\eta - \beta\zeta)\alpha - 4\eta^2 \neq 0$$

3.2. Neimark–Sacker Bifurcation Analysis

We examine the Neimark-Sacker bifurcation conditions for the system (2) at coexistence fixed point $\left(\alpha, \beta(1-\alpha)(\frac{\alpha-\zeta}{\alpha+\eta}) - 1\right)$ in this section. In addition, the direction of the Neimark-Sacker bifurcation is analyzed. From Eq.(6) it follows that $F(\lambda) = 0$ has two complex conjugate roots with modulus one, if the following conditions are satisfied:

$$\begin{aligned} \Delta &= \left(\beta\alpha^3 + (2\beta\eta-2)\alpha^2 + ((-4+(-\zeta-1)\beta)\eta - \beta\zeta)\alpha - 2\eta^2\right)^2 \\ &\quad -4\beta(\alpha+\eta)^2\left(2\alpha^3 + (-\zeta+3\eta-1)\alpha^2 + (-2\zeta\eta-2\eta)\alpha + \zeta\eta\right) < 0 \end{aligned}$$

$$\beta = \frac{(\alpha+\eta)^2}{-2\alpha^3 + (\zeta-3\eta+1)\alpha^2 + 2\eta(\zeta+1)\alpha - \eta\zeta}, \zeta \neq \frac{\alpha(2\alpha^2+3\alpha\eta-\alpha-2\eta)}{\alpha^2+2\alpha\eta-\eta} \tag{10}$$

Assume that

$$\Omega_{NS} = \left\{(\beta,\alpha,\zeta,\eta) \in \mathbb{R}_+^4 : \Delta < 0, \quad \beta = \frac{(\alpha+\eta)^2}{-2\alpha^3 + (\zeta-3\eta+1)\alpha^2 + 2\eta(\zeta+1)\alpha - \eta\zeta}, \quad \zeta \neq \frac{\alpha(2\alpha^2+3\alpha\eta-\alpha-2\eta)}{\alpha^2+2\alpha\eta-\eta}\right\}$$

The coexistence fixed point of system (2) undergoes Neimark-Sacker bifurcation as a result of parameter change in the small neighborhood of the set Ω_{NS}. Set $\beta_2 = \dfrac{(\alpha+n)^2}{-2\alpha^3 + (\zeta-3\eta+1)\alpha^2 + 2\eta(\zeta+1)\alpha - \eta\zeta}$

such that $\zeta \neq \dfrac{\alpha(2\alpha^2+3\alpha\eta-\alpha-2\eta)}{\alpha^2+2\alpha\eta-\eta}$ and assume that $(\beta_2,\alpha,\zeta,\eta)\in\Omega_{NS}$, then system (2) can be expressed by following two-dimensional map:

$$\begin{pmatrix}N\\P\end{pmatrix}\rightarrow\begin{pmatrix}\beta_2 N(1-N)(\dfrac{N-\zeta}{N+\eta})-NP\\ \dfrac{1}{\alpha}NP\end{pmatrix} \tag{11}$$

Let $\overline{\beta}$ denotes the bifurcation parameter, then corresponding perturbed mapping of (11) is given as follows:

$$\begin{pmatrix}N\\P\end{pmatrix}\rightarrow\begin{pmatrix}(\beta_2+\overline{\beta})N(1-N)(\dfrac{N-\zeta}{N+\eta})-NP\\ \dfrac{1}{\alpha}NP\end{pmatrix} \tag{12}$$

where $||\overline{\beta}||\ll 1$ denotes the small bifurcation parameter. Next, the transformations $t=N-\alpha$ and $s=P-\beta(1-\alpha)(\frac{\alpha-\zeta}{\alpha+\eta})-1$ are considered, then from the map (12) we have

$$\begin{pmatrix}t\\s\end{pmatrix}\rightarrow\begin{pmatrix}m_{11}&m_{12}\\m_{21}&m_{22}\end{pmatrix}\begin{pmatrix}t\\s\end{pmatrix}+\begin{pmatrix}f_1(t,s)\\f_2(t,s)\end{pmatrix} \tag{13}$$

where

$$f_1(t,s)=m_{13}t^3+m_{14}t^2+m_{15}ts+O(|t|+|s|)^4$$

$$f_2(t,s)=m_{25}ts+O(|t|+|s|)^4$$

$$m_{11}=-\frac{\eta^2+\alpha(-2\alpha(\beta_2+\overline{\beta})+(\zeta+1)(\beta_2+\overline{\beta})+2)\eta+\alpha(-(\beta_2+\overline{\beta})\alpha^2+\alpha+(\beta_2+\overline{\beta})\zeta)}{(\alpha+\eta)^2},\quad m_{12}=-\alpha$$

$$m_{21}=\frac{(\beta_2+\overline{\beta})(1-\alpha)(\alpha-\zeta)+(\alpha+\eta)}{\alpha(\alpha+\eta)}\quad m_{22}=1$$

$$m_{13}=-\frac{\eta(\beta_2+\overline{\beta})(\eta+1)(m\zeta+\eta)}{(\alpha+\eta)^4},\quad m_{14}=-\frac{(\alpha^3+3\alpha^2\eta+3\alpha\eta^2-\zeta\eta^2-\zeta\eta-\eta^2)(\beta_2+\overline{\beta})}{(\alpha+\eta)^3}$$

$$m_{15}=-1\quad m_{25}=\frac{1}{\alpha}$$

The characteristic equation of Jacobian matrix of linearized system of (13) evaluated at the fixed (0, 0) can be written as follows:

$$\lambda^2-A(\overline{\beta})\lambda+B(\overline{\beta})=0, \tag{14}$$

where

$$A(\overline{\beta})=2-\frac{(\beta_2+\overline{\beta})(\alpha^3+2\alpha^2\eta-\alpha\zeta\eta-\alpha\zeta-\alpha\eta)}{(\alpha+\eta)^2}$$

and

$$B(\overline{\beta})=\frac{(\beta_2+\overline{\beta})\left(-2\alpha^3+(\zeta-3\eta+1)\alpha^2+2\eta(\zeta+1)\alpha-\zeta\eta\right)}{(\alpha+\eta)^2}$$

Since $(\beta_2,\alpha,\zeta,\eta)\in\Omega_{NS}$, therefore the complex conjugate roots of Eq. (14) are given by:

$$\lambda_{1,2}=\frac{A(\overline{\beta})\pm i\sqrt{4B(\overline{\beta})-\left(A(\overline{\beta})\right)^2}}{2}$$

Then, we obtain that

$$|\lambda_1|=|\lambda_2|=\sqrt{\frac{(\beta_2+\overline{\beta})\left(-2\alpha^3+(\zeta-3\eta+1)\alpha^2+2\eta(\zeta+1)\alpha-\zeta\eta\right)}{(\alpha+\eta)^2}}$$

Moreover, in order to obtain the non-degeneracy conditions, we assume that $\zeta \neq \dfrac{\alpha(2\alpha^2+3\alpha\eta-\alpha-2\eta)}{\alpha^2+2\alpha\eta-\eta}$ then it follows that

$$\left(\frac{d|\lambda_1|}{d\overline{\beta}}\right)_{\overline{\beta}=0} = \left(\frac{d|\lambda_2|}{d\overline{\beta}}\right)_{\overline{\beta}=0} = -\frac{2\alpha^3-(\zeta-3\eta+1)\alpha^2-2\eta(\zeta+1)\alpha+\zeta\eta}{2(\alpha+\eta)^2} \neq 0$$

Moreover, we have $-2 < A(0) < 2$ because $(\beta_2, \alpha, \zeta, \eta) \in \Omega_{NS}$. On the other hand, we have $A(0) = 2 + \dfrac{\alpha(\alpha^2+2\alpha\eta-\zeta\eta-\zeta-\eta)}{2\alpha^3+(-\zeta+3\eta-1)\alpha^2+(-2\zeta\eta-2\eta)\alpha+\zeta\eta}$. Assume that $\zeta \neq \dfrac{\alpha(2\alpha^2+3\alpha\eta-\alpha-2\eta)}{\alpha^2+2\alpha\eta-\eta}$, $A(0) \neq 0$ and $A(0) \neq -1$, that is,

$$\frac{\alpha(\alpha^2+2\alpha\eta-\zeta\eta-\zeta-\eta)}{2\alpha^3+(-\zeta+3\eta-1)\alpha^2+(-2\zeta\eta-2\eta)\alpha+\zeta\eta} \neq -2, -3 \tag{15}$$

Conditions in Eq.(15) and with $(\beta_2, \alpha, \zeta, \eta) \in \Omega_{NS}$ make sure that $A(0) \neq \pm 2, 0, -1$, and in a result we have $\lambda_1^m, \lambda_2^m \neq 1$ for all $m = 1, 2, 3, 4$ at $\overline{\beta} = 0$. Hence roots of Eq.(14) do not lie in the intersection of the unit circle with the coordinate axes when $\overline{\beta} = 0$. In order to obtain the normal form of Eq.(13) at $\overline{\beta} = 0$, assuming that $\xi = \dfrac{A(0)}{2}$ and $\tau = \dfrac{\sqrt{4B(0)-(A(0))^2}}{2}$. Furthermore, let us consider the following transformation:

$$\begin{pmatrix} t \\ s \end{pmatrix} \rightarrow \begin{pmatrix} m_{12} & 0 \\ \xi - m_{11} & -\tau \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{16}$$

Under transformation Eq.(16), the normal form of Eq.(13) can be written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \xi & -\tau \\ \tau & \xi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \overline{f}(x,y) \\ \overline{g}(x,y) \end{pmatrix} \tag{17}$$

where

$$\overline{f}(x,y) = \frac{m_{13}}{m_{12}}t^3 + \frac{m_{14}}{m_{12}}t^2 + \frac{m_{15}}{m_{12}}ts + O((|x|+|y|)^4),$$

$$\overline{g}(x,y) = \left(\frac{(\xi-m_{11})m_{13}}{m_{12}\tau} - \frac{m_{23}}{\tau}\right)t^3 + \left(\frac{(\xi-m_{11})m_{14}}{m_{12}\tau} - \frac{m_{24}}{\tau}\right)t^2 + \left(\frac{(\xi-m_{11})m_{15}}{m_{12}\tau} - \frac{m_{25}}{\tau}\right)ts + O((|x|+|y|)^4),$$

$t = m_{12}x$ and $s = (\xi - m_{11})x - \tau y$. Next, we consider the following real number:

$$L = \left(\left[-Re\left(\frac{(1-2\lambda_1)\lambda_2^2}{1-\lambda_1}\kappa_{20}\kappa_{11}\right) - \frac{1}{2}|\kappa_{11}|^2 - |\kappa_{02}|^2 + Re(\lambda_2\kappa_{21})\right]\right)_{\overline{\beta}=0}$$

where

$$\kappa_{20} = \frac{1}{8}\left[\overline{f}_{xx} - \overline{f}_{yy} + 2\overline{g}_{xy} + i(\overline{g}_{xx} - \overline{g}_{yy} - 2\overline{f}_{xy})\right]$$

$$\kappa_{11} = \frac{1}{4}\left[\overline{f}_{xx} + \overline{f}_{yy} + i(\overline{g}_{xx} + \overline{g}_{yy})\right]$$

$$\kappa_{02} = \frac{1}{8}\left[\overline{f}_{xx} - \overline{f}_{yy} - 2\overline{g}_{xy} + i(\overline{g}_{xx} - \overline{g}_{yy} + 2\overline{f}_{xy})\right]$$

$$\kappa_{21} = \frac{1}{16}\left[\overline{f}_{xxx} + \overline{f}_{xyy} + \overline{g}_{xxy} + \overline{g}_{yyy} + i(\overline{g}_{xxx} + \overline{g}_{xyy} - \overline{f}_{xxy} - \overline{f}_{yyy})\right]$$

Moreover, the partial derivatives of $\overline{f}$ and $\overline{g}$ evaluated at $\overline{\beta} = 0$ are given by:

$$\overline{f}_{xx} = 2m_{14}m_{12} + 2(\xi - m_{11})m_{15}, \quad \overline{f}_{yy} = \overline{g}_{yy} = 0 \quad \overline{g}_{xy} = -(\xi - m_{11})m_{15} + m_{25}m_{12}$$

$$\overline{g}_{xx} = 2\left(\frac{(\xi-m_{11})m_{12}(m_{14}-m_{25}) - m_{24}m_{12}^2 + (\xi-m_{11})^2 m_{15}}{\tau}\right), \quad \overline{f}_{xy} = -\tau m_{15}, \quad \overline{f}_{xxx} = 6m_{12}^2 m_{13},$$

$$\overline{g}_{xxx} = 6\left(\frac{(\xi-m_{11})m_{13}m_{12}^2 - m_{23}m_{12}^3}{\tau}\right) \quad \overline{g}_{yyy} = \overline{g}_{xyy} = \overline{f}_{yyy} = \overline{f}_{xxy} = 0$$

Arguing as in [5, 6, 25, 26, 27] we have the following result which gives parametric conditions for existence and direction of Neimark-Sacker bifurcation for coexistence fixed point of system Eq.(2).

Theorem 3.4 *Suppose that Eq.* (15) *holds true and* $L \neq 0$*, then system* (2) *endures Neimark-Sacker bifurcation at its unique positive steady-state* $\left(\alpha, \beta(1-\alpha)(\frac{\alpha-\zeta}{\alpha+\eta})-1\right)$ *when the bifurcation parameter* β *varies in a small neighborhood of* $\beta_2 = \dfrac{(\alpha+\eta)^2}{-2\alpha^3+(\zeta-3\eta+1)\alpha^2+2\eta(\zeta+1)\alpha-\eta\zeta}$ *such that* $\zeta \neq \dfrac{\alpha(2\alpha^2+3\alpha\eta-\alpha-2\eta)}{\alpha^2+2\alpha\eta-\eta}$*. Furthermore, if* $L<0$*, then an attracting invariant closed curve bifurcates from the fixed point for* $\beta > \beta_2$*, and if* $L>0$*, then a repelling invariant closed curve bifurcates from the fixed point for* $\beta < \beta_2$*.*

3.3. Chaos Control

Population models, particularly those that deal with the biological reproduction of species, are thought to be critically dependent on the ability to control chaos and bifurcation. Discrete-time models often exhibit more complex behavior than continuous ones. To safeguard the public from unforeseen events, it is essential to put chaos control mechanisms into place. In this section, we will look at a feedback control strategy for reorienting the unstable trajectory toward the stable one. To accomplish this, we first use the OGY approach to run system (2) Ott et al.[28], proposed this strategy. See also [29] for further details on the OGY strategy. To apply the OGY method, we rewrite system (2) as follows:

$$\begin{aligned} N_{n+1} &= \beta N_n(1-N_n)(\frac{N_n-\zeta}{N_n+\eta}) - N_nP_n = f(N_n,P_n,\beta) \\ P_{n+1} &= \frac{1}{\alpha}N_nP_n = g(N_n,P_n,\beta) \end{aligned} \tag{18}$$

where the necessary chaos control is achieved by using only very small disturbances and the regulating parameter β. To do this, the parameter β is restricted to lie inside the range $\beta \in (\beta_0-\Psi, \beta_0+\Psi)$, where $\Psi > 0$ and β_0 denote the nominal value associated with the chaotic region, respectively. We employ the stabilizing feedback control strategy to direct the trajectory toward the desired orbit. Assuming that the unstable fixed point of system (2) is $(N^*,P^*) = \left(\alpha, \beta(1-\alpha)(\frac{\alpha-\zeta}{\alpha+\eta})-1\right)$. System (18) can be approximated concerning the unstable residual point in the chaotic zone brought on by the appearance of Neimark-Sacker bifurcation:

$$\begin{bmatrix} N_{n+1}-N^* \\ P_{n+1}-P^* \end{bmatrix} \approx J(x^*,y^*,\beta_0)\begin{bmatrix} N_n-N^* \\ P_n-P^* \end{bmatrix} + H\left[\beta-\beta_0\right], \tag{19}$$

where

$$J(N^*,P^*,\beta_0) = \begin{bmatrix} \dfrac{\partial f(N^*,P^*,\beta_0)}{\partial N} & \dfrac{\partial f(N^*,P^*,\beta_0)}{\partial P} \\ \dfrac{\partial g(N^*,P^*,\beta_0)}{\partial N} & \dfrac{\partial g(N^*,P^*,\beta_0)}{\partial P} \end{bmatrix} = \begin{bmatrix} \frac{-\beta_0\alpha^3+(-2\beta_0\eta+1)\alpha^2+((2+(\zeta+1)\beta_0)\eta+\beta_0\zeta)\alpha+\eta^2}{(\alpha+\eta)^2} & -\alpha \\ \frac{-\beta_0\alpha^2+\beta_0\alpha\zeta+\beta_0\alpha-\beta_0\zeta-\alpha-\eta}{(\alpha+\eta)\alpha} & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} \dfrac{\partial f(N^*,P^*,\beta_0)}{\partial \beta} \\ \dfrac{\partial g(N^*,P^*,\beta_0)}{\partial \beta} \end{bmatrix} = \begin{bmatrix} \frac{\alpha^2-\alpha\zeta-\alpha^3+\alpha^2\zeta}{\alpha+\eta} \\ 0 \end{bmatrix}$$

Moreover, system (18) is controllable provided that the following matrix

$$C = \left[B:JB\right] = \begin{bmatrix} \frac{\alpha^2-\alpha\zeta-\alpha^3+\alpha^2\zeta}{\alpha+\eta} & \frac{(-\alpha+\zeta)(-\beta_0\alpha^3+(-2\beta_0\eta+1)\alpha^2+((2+(\zeta+1)\beta_0)\eta+\beta_0\zeta)\alpha+\eta^2)(-1+\alpha)\alpha}{(\alpha+\eta)^3} \\ 0 & \frac{(-1+\alpha)(-\alpha+\zeta)(-\beta_0\alpha^2+(-1+(\zeta+1)\beta_0)\alpha-\beta_0\zeta-\eta)}{(\alpha+\eta)^2} \end{bmatrix} \tag{20}$$

is of rank 2. Moreover, taking $[\beta-\beta_0] = -K\begin{bmatrix} N_n - N^* \\ P_n - P^* \end{bmatrix}$ where $K = \begin{bmatrix}\rho_1 & \rho_2\end{bmatrix}$, then system (19) can be written as

$$\begin{bmatrix} N_{n+1} - N^* \\ P_{n+1} - P^* \end{bmatrix} \approx [J - BK]\begin{bmatrix} N_n - N^* \\ P_n - P^* \end{bmatrix} \tag{21}$$

Furthermore, the corresponding controlled system of Eq. (2) is given by

$$\begin{aligned} N_{n+1} &= (\beta_0 - \rho_1(N_n - N^*) - \rho_2(P_n - P^*))N_n(1-N_n)\left(\frac{N_n - \zeta}{N_n + \eta}\right) - N_nP_n \\ P_{n+1} &= \frac{1}{\alpha}N_nP_n \end{aligned} \tag{22}$$

Furthermore, if and only if both of the eigenvalues of the matrix $J-BK$ are contained within an open unit disk, fixed point $(N^*, P^*) = \left(\alpha, \beta(1-\alpha)\left(\frac{\alpha-\zeta}{\alpha+\eta}\right) - 1\right)$ is locally asymptotically stable. The controlled system (22)'s Jacobian matrix $J - BK$ may be expressed as follows:

$$J - BK = \begin{bmatrix} \frac{-\beta_0\alpha^3 + (-2\beta_0\eta+1)\alpha^2 + ((2+(\zeta+1)\beta_0)n + \beta_0\zeta)\alpha + \eta^2}{(\alpha+\eta)^2} + \frac{\alpha(\alpha-1)(\alpha-\zeta)\rho_1}{\alpha+\eta} & -\alpha + \frac{\alpha(\alpha-1)(\alpha-\zeta)\rho_2}{\alpha+n} \\ \frac{-\beta_0\alpha^2 + \beta_0\alpha\zeta + \beta_0\alpha - \beta_0\zeta - \alpha - \eta}{(\alpha+\eta)\alpha} & 1 \end{bmatrix}.$$

Let λ_1 and λ_2 are the eigenvalues of the characteristic equation of Jacobian matrix $J - BK$, then we have

$$\lambda_1 + \lambda_2 = \frac{-\beta_0\alpha^3 + (-2\beta_0\eta + 1)\alpha^2 + ((2 + (\zeta+1)\beta_0)\eta + \beta_0\zeta)\alpha + \eta^2}{(\alpha+\eta)^2} + \frac{\alpha(\alpha-1)(\alpha-\zeta)\rho_1}{\alpha+\eta}, \tag{23}$$

$$\begin{aligned} \lambda_1\lambda_2 = {} & \frac{\alpha(\alpha-1)(\alpha-\zeta)}{\alpha+\eta}\rho_1 + \frac{(-\alpha+\zeta)(\alpha-1)(-\beta_0\alpha^2 + (\beta_0\zeta + \beta_0 - 1)\alpha - \beta_0\zeta - \eta)}{(\alpha+\eta)^2}\rho_2 \\ & + \frac{\beta_0(-2\alpha^3 + (\zeta - 3\eta + 1)\alpha^2 + 2\eta(\zeta+1)\alpha - \zeta\eta)}{(\alpha+\eta)^2}. \end{aligned} \tag{24}$$

Next, in order to determine the lines of marginal stability for the corresponding controlled system, we choose $\lambda_1 = \pm 1$ and $\lambda_1\lambda_2 = 1$. Additionally, these limitations guarantee that the open unit disk contains λ_1 and λ_2. Inferring from Eq. (24) that $\lambda_1\lambda_2 = 1$, it is implied that:

$$\begin{aligned} L_1 := {} & \frac{\alpha^4 + (-\zeta + \eta - 1)\alpha^3 + ((-\zeta - 1)\eta + \zeta)\alpha^2 + \alpha\zeta\eta}{(\alpha+\eta)^2}\rho_1 \\ & + \frac{\beta_0\alpha^4 + (-2\beta_0\zeta - 2\beta_0 + 1)\alpha^3 + (\beta_0\zeta^2 + 4\beta_0\zeta + \beta_0 - \zeta + \eta - 1)\alpha^2 + ((-\zeta - 1)\eta - 2\beta_0\zeta + \zeta - 2\beta_0\zeta^2)\alpha + \zeta\eta + \beta_0\zeta^2}{(\alpha+\eta)^2}\rho_2 \\ & + \frac{-2\beta_0\alpha^3 + (\beta_0\zeta - 3\beta_0\eta + \beta_0 - 1)\alpha^2 + (2\beta_0\zeta + 2\beta_0 - 2)\eta\alpha - \beta_0\zeta\eta - \eta^2}{(\alpha+\eta)^2} = 0 \end{aligned}$$

Moreover, we assume that $\lambda_1 = 1$, then Eq. (23) and Eq. (24) yield that:

$$\begin{aligned} L_2 := {} & \frac{\beta_0\alpha^4 + (-2\beta_0\zeta - 2\beta_0 + 1)\alpha^3 + (\beta_0\zeta^2 + (4\beta_0 - 1)\zeta + \beta_0 + \eta - 1)\alpha^2 + (-2\beta_0\zeta^2 + (1 - 2\beta_0 - \eta)\zeta - \eta)\alpha + \zeta\eta + \beta_0\zeta^2}{(\alpha+\eta)^2}\rho_2 \\ & + \frac{-\beta_0\alpha^3 + (\beta_0\zeta - \beta_0\eta + \beta_0 - 1)\alpha^2 + ((\beta_0\eta - \beta_0)\zeta + \beta_0\eta - 2\eta)\alpha - \beta_0\zeta\eta - \eta^2}{(\alpha+\eta)^2} = 0 \end{aligned}$$

Finally taking $\lambda_1 = -1$, then from Eq. (23) and Eq. (24) we get

$$\begin{aligned} L_3 := {} & -\frac{2\alpha(\alpha-1)\zeta - 2\alpha^3}{\alpha+\eta}\rho_1 + -\frac{(-\alpha+\zeta)(\alpha-1)(-\beta_0\alpha^2 + (\beta_0\zeta + \beta_0 - 1)\alpha - \beta_0\zeta - \eta)}{(\alpha+\eta)^2}\rho_2 \\ & + \frac{-3\beta_0\alpha^3 + (3 + (\zeta - 5\eta + 1)\beta_0)\alpha^2 + (((3\zeta+3)\eta + \zeta)\beta_0 + 6\eta)\alpha - \beta_0\zeta\eta + 3\eta^2}{(\alpha+\eta)^2} = 0 \end{aligned}$$

The triangular area in the $\rho_1\rho_2$-plane bounded by the straight lines L_1, L_2, L_3 then contains stable eigenvalues for a given parametric value.

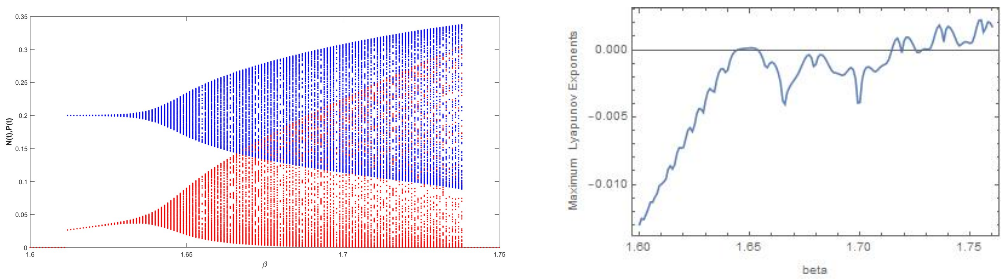
3.4. Numerical Simulations

Using the Maple 2021 and Matlab R2022b programs, numerical simulations are utilized to demonstrate the correctness of the theoretical research included in this section. Different parameter values were utilized during these simulations, and unique graphs were created for each.

Example 3.4.1: For the parameter values $\alpha = 0.2, \zeta = 0.001, \eta = 0.05$ and with initial condition $(N_0, P_0) = (0.24, 0.08)$, the coexistence fixed point of the system (2) is obtained as $D_3 = (0.2, 0.04709287)$. The critical value of Neimark-Sacker bifurcation point is obtained as $\beta = 1.644304130$. The characteristic equation of the Jacobian matrix evaluated at $(0.2, 0.04709287)$ is given by:

$$\lambda^2 - 1.952907130\lambda + 1 = 0.$$

The roots of this characteristic equation are $\lambda_1 = 0.9764535650 + 0.2157276881i$ and $\lambda_2 = 0.9764535650 - 0.2157276881i$. And also, $|\lambda_1| = |\lambda_2| = 1$. Therefore $(\beta, \alpha, \zeta, \eta) = (1.644304130, 0.2, 0.001, 0.05) \in \Omega_{NS}$. The bifurcation diagrams and corresponding maximum Lyapunov exponents (MLE) are plotted in Figure 1.



(a) Bifurcation diagram for N_n,P_n (b) Maximum Lyapunov exponents.

Figure 1: Bifurcation diagrams and MLE for system (2) with $\alpha = 0.2, \zeta = 0.001, \eta = 0.05$ and $(N_0, P_0) = (0.24, 0.08)$

Next, we implement OGY control strategy in order to control chaos due to emergence of Neimark-Sacker bifurcation. For this, we take $\beta_0 = 1.725, \alpha = 0.2, \zeta = 0.001, \eta = 0.05$ and $D_3 = (0.2, 0.09848)$, then corresponding controlled system is given by:

$$\begin{aligned} N_{n+1} &= (1.725 - \rho_1(N_n - 0.2) - \rho_2(P_n - 0.09848))N_n(1 - N_n)\left(\frac{N_n - 0.001}{N_n + 0.05}\right) - N_n P_n \qquad (25) \\ P_{n+1} &= \frac{1}{0.2} N_n P_n \end{aligned}$$

Then, characteristic polynomial of the Jacobian matrix of the controlled system (25) evaluated at $(0.2, 0.09848)$ is given by:

$$F(\lambda) = \lambda^2 - (1.950596 - 0.12736\rho_1)\lambda + 1.049076 - 0.12736\rho_1 + 0.062712064\rho_2 \qquad (26)$$

Additionally, the marginal stability lines are provided by for the controlled system (25)

$$L_1 := 0.049076 - 0.12736\rho_1 + 0.062712064\rho_2 = 0$$

$$L_2 := 0.09848 + 0.062712064\rho_2 = 0$$

$$L_3 := 3.999672 - 0.25472\rho_1 + 0.062712064\rho_2 = 0$$

Figure 2 depicts the triangle-shaped stability zone enclosed by these marginal stability lines L_1, L_2, and L_3.

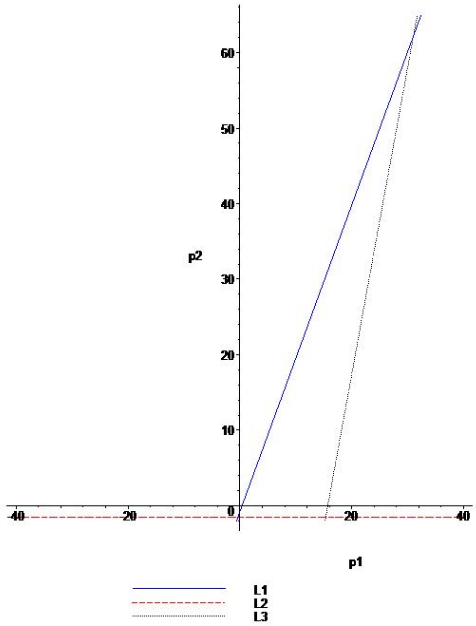


Figure 2: Triangular stability region bounded by L_1, L_2 and L_3 for the controlled system (25)

4. Conclusion

The stability and bifurcation analysis of a discrete-time predator-prey model with strong Allee effect are examined in this article. The system (2) has three fixed points, identified as D_1, D_2 and D_3, as we proved. Topological classifications of these fixed locations were given. We showed using bifurcation theory that the system (2) will experience Neimark-Sacker bifurcation at a specific coexistence fixed point if a varies around the set Ω_{NS}. The parametric conditions for coexistence fixed point D_3's direction Neimark-Sacker bifurcation were given. Finally, Neimark-Sacker bifurcation, chaos control and maximum Lyapunov exponent of the coexistence fixed point are verified with the help of numerical simulations. To support the theoretical conclusions, we also provided further numerical simulations using Maple 2021 and Matlab R2022b.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Conflicts of Interest

No conflict of interest or common interest has been declared by the authors.

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