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SOME RESULTS ON JACK'S LEMMA FOR ANALYTIC FUNCTIONS

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Abstract

In this paper, an upper bound will be found for the second coefficient in the Taylor expansion of the analytical function $p(z)$ using the Jack lemma. Also, the modulus of the angular derivative of the $I_p(z)=\frac{zp'(z)}{p(z)}$ function on the unit disc will be estimated from below.

Key Words: Analytic function, Jack's lemma, Schwarz lemma.

Özet

Bu çalışmada Jack lemması kullanılarak $p(z)$ analitik fonksiyonunun Taylor açılımında ikinci katsayı için bir üst sınır bulunacaktır. Ayrıca, $I_p(z)=\frac{zp'(z)}{p(z)}$ fonksiyonunun birim disk üzerindeki açısal türevinin modülü aşağıdan tahmin edilecektir.

Anahtar Kelimeler: Analitik fonksiyon, Jack's lemma, Schwarz lemma

1. Introduction

The Schwarz lemma is one of the outcomes for analytic functions from the unit disc itself in complex analysis. However, it is one of the simplest results catching the rigidity of analytic functions. Supppose $f(z)$ is an analytic function on the unit disc $D=\{z:|z|<1\}$ such that $f(0)=0$ and $|f(z)|<1$ for $z\in D$. According to the standard Schwarz Lemma, for any point z in the unit disc D, we have $|f(z)|\leq|z|$ for all $z\in D$ and $|f'(0)|\leq 1$. Also, if the equality $|f(z)|=|z|$ holds for any $z\neq 0$, or $|f'(0)|=1$ then f is a rotation; that is $f(z)=ze^{i\alpha}$, α real ([5], p.329). The Schwarz lemma has numerous uses in engineering [12, 13].

To demonstrate our findings, we shall utilize the following lemma [6].

Lemma 1 (Jack's Lemma) *Let* $f(z)$ *be a non-constant anaytic function in* D *with* $f(0)=0$*. If*

$$|f(z_0)|=\max\{|f(z)|:|z|\leq|z_0|\}$$

then there exists a real number $m\geq 1$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)}=m.$$

The functional $I_p(z)=\dfrac{zp'(z)}{p(z)}$ has an essential role in the theory of starlike functions, Jack's lemma and shares some properties with logarithmic residuum.

Let $\mathcal{A}$ denote the class of functions $p(z)=z+c_2z^2+c_3z^3+\ldots$ that are analytic in D. Also, let $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of all functions $p(z)$ satisfying

$$\Re\left(1+\frac{zp'(z)}{p(z)}+\frac{zp''(z)}{p'(z)}\right)>\frac{1}{2}.$$

The qualities of the class $\mathcal{M}$ are examined in this paper. In particular, an upper bound the modulus of the coefficient $c_2=\dfrac{p''(0)}{2!}$ for this class will be established. The purpose of this work is to use Jack's Lemma to explore some characteristics of the function $p(z)$, which belongs to the class of $\mathcal{M}$.

Let $p\in\mathcal{M}$ and consider the function

$$\varphi(z)=\frac{I_p(z)-1}{I_p(z)}, \tag{1.1}$$

where $I_p(z)=\dfrac{zp'(z)}{p(z)}=1+c_2z+\left(2c_3-c_2^2\right)z^2+\ldots$.

$\varphi(z)$ is an analytic function in D and $\varphi(0)=0$. Now, we show that $|\varphi(z)|<1$ in D.

We assume that there exists a $z_0\in D$ such that

$$\max_{|z|\leq|z_0|}|\varphi(z)|=|\varphi(z_0)|=1$$

From Jack's Lemma, we have

$$\varphi(z_0)=e^{i\alpha} \text{ and } \frac{z_0\varphi'(z_0)}{\varphi(z_0)}=m.$$

Thus, we obtain

$$\Re\left(1+\frac{z_0p'(z_0)}{p(z_0)}+\frac{z_0p''(z_0)}{p'(z_0)}-\frac{1}{2}\right)=\Re\left(1+\frac{2}{1-\varphi(z_0)}+\frac{z_0\varphi'(z_0)}{1-\varphi(z_0)}-\frac{1}{2}\right)=\Re\left(\frac{1}{2}+\frac{2}{1-e^{i\alpha}}+\frac{me^{i\alpha}}{1-e^{i\alpha}}\right)$$

$$=\Re\left(\frac{1}{2}+\frac{2+me^{i\alpha}}{1-e^{i\alpha}}\right)=\Re\left(\frac{1}{2}+\frac{2+m(\cos\theta+i\sin\theta)}{1-(\cos\theta+i\sin\theta)}\right)=\frac{1}{2}(1-m)\leq 0.$$

This contradicts the $p\in M$. This implies that there is no point in doing so $z_0\in D$ such that $\max_{|z|\leq|z_0|}|\varphi(z)|=|\varphi(z_0)|=1$. Hence, we take $|\varphi(z)|<1$ for $z\in D$. The Schwarz Lemma gives us

$$\varphi(z)=\frac{c_2z+\left(2c_3-c_2^2\right)z^2+\ldots}{1+c_2z+\left(2c_3-c_2^2\right)z^2+\ldots},$$

$$\frac{\varphi(z)}{z}=\frac{c_2+\left(2c_3-c_2^2\right)z+\ldots}{1+c_2z+\left(2c_3-c_2^2\right)z^2+\ldots}$$

and

$$|p''(0)|\leq 2.$$

This inequality is sharp with the following function

$$p(z)=\frac{z}{1-z}.$$

Then

$$z+c_2z^2+c_3z^3+\ldots=\frac{z}{1-z},$$

$$1+c_2z+c_3z^2+\ldots=\frac{1}{1-z},$$

$$c_2z+c_3z^2+\ldots=\frac{z}{1-z}$$

and

$$c_2+c_3z+\ldots=\frac{1}{1-z}.$$

Passing to limit $(z\to 0)$ in the last equality yields $c_2=1$.

As a result, we have the following lemma.

Lemma 2 *If* $p\in M$ *, then we have*

$$|p''(0)|\leq 2. \tag{1.2}$$

This result is sharp with the function

$$p(z)=\frac{z}{1-z}.$$

An important consequence of the Schwarz lemma is the evaluation of the modulus of the derivative of the function at the boundary of the unit disk from below. The boundary version of Schwarz Lemma is given as follows [10, 14]:

Lemma 4 *If* $f(z)$ *extends continuously to some boundary point* τ with $|\tau|=1$, $|f(z)|<1$ for $z\in D$, $f(0)=0$ *and if* $|f(\tau)|=1$ *and* $f'(\tau)$ *exists, then*

$$|f'(\tau)|\geq\frac{2}{1+|f'(0)|} \qquad (1.3) \qquad (1.4)$$

and

$$|f'(\tau)|\geq 1 \qquad (1.4)$$

In addition, the equality in (1.4) holds if and only if $f(z)=ze^{i\alpha}$, *where* α *is a real number. Also, the equality in (1.3) holds if and only if f is of the form* $f(z)=-z\frac{a-z}{1-\bar{a}z}, \forall z\in D$, *for some constant* $a\in(-1,0]$.

These inequalities are important in the literature and still continue to be studied among current issues [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13].

Let us give the following lemma for proofs of our work [14].

Lemma 5 (Julia-Wolff lemma) *Let* f *be an analytic function in* D, $f(0)=0$ *and* $|f(z)|<1$ for $z\in D$. *If, in addition, the function* f *has an angular limit* $f(\tau)$ *at* $\tau\in\partial D$, $|f(\tau)|=1$, *then the angular derivative* $f'(\tau)$ *exists and* $1\leq|f'(\tau)|\leq\infty$.

2. Main Results

In this section, the derivative of the function at point 1 is evaluated from below. Some of the coeffcients in the Taylor expansion of the function are used in this evaluation.

Theorem 1 *Let* $p\in M$. *Suppose that, for* $1\in\partial D$, p *has an angular limit* $p(\tau)$ *at the point* τ, $p'(\tau)=\frac{p(\tau)}{2\tau}$. *Then we have*

$$\left|I_p'(\tau)\right|\geq\frac{1}{4}. \qquad (2.1)$$

The inequality (2.1) *is sharp with extremal function*

$$p(z)=\frac{z}{1-z}.$$

Proof. Let

$$\varphi(z)=\frac{I_p(z)-1}{I_p(z)},$$

where $I_p(z)=\dfrac{zp'(z)}{p(z)}$.

Since $p'(\tau)=\dfrac{p(\tau)}{2\tau}$, we have $|\varphi(\tau)|=1$. So, from (1.4), we obtain

$$1\le|\varphi'(1)|=\frac{|I'_p(\tau)|}{|I_p(\tau)|^2}=\frac{|I'_p(\tau)|}{\frac{1}{4}}$$

and

$$|I'_p(\tau)|\ge\frac{1}{4}.$$

To show that the inequality (2.1) is sharp, take the an analytic function

$$p(z)=\frac{z}{1-z}.$$

Then, we take

$$p'(z)=\frac{1}{1-z},$$

$$I_p(z)=\frac{zp'(z)}{p(z)}=\frac{1}{1-z}$$

and

$$|I'_f(-1)|=\frac{1}{4}.$$

Theorem 2 *Using the same presumptions as in Theorem 1, we obtain*

$$|I'_p(\tau)|\ge\frac{1}{2+|p''(0)|}. \tag{2.2}$$

The equality in (2.2) *occurs for the function*

$$p(z)=\frac{z}{1-z}.$$

Proof. Let $\varphi(z)$ be as defined above. So, by (1.3), we take

$$\frac{2}{1+|\varphi'(0)|}\le|\varphi'(\tau)|=\frac{|I'_p(\tau)|}{\frac{1}{4}}.$$

Since

$$|\varphi'(0)|=|c_2|=\frac{|p''(0)|}{2},$$

we obtain

$$\frac{2}{1+\frac{|p''(0)|}{2}} \leq |\varphi'(\tau)| = \frac{|I_p'(\tau)|}{\frac{1}{4}}$$

and

$$|I_p'(\tau)| \geq \frac{1}{2+|p''(0)|}.$$

Let us show that this inequality is sharp. Let

$$p(z) = \frac{z}{1-z}.$$

Then, we take

$$|I_p'(-1)| = \frac{1}{4}.$$

On the other hand, we have

$$z + c_2 z^2 + c_3 z^3 + \ldots = \frac{z}{1-z}$$

and

$$c_2 + c_3 z + \ldots = \frac{1}{1-z}.$$

Passing to limit as z tends to 0 in the last equality, we get $c_2 = 1$.
Therefore, we obtain

$$\frac{1}{2+|p''(0)|} = \frac{1}{4}.$$

Theorem 3 *Using the same presumptions as in Theorem 1, we obtain*

$$|I_p'(\tau)| \geq \frac{1}{4}\left(1+\frac{2\left(1-|c_2|\right)^2}{1-|c_2|^2+2\left|c_3-c_2^2\right|}\right). \tag{2.3}$$

Proof. Let $\varphi(z)$ function be the same as (1.1) and $s(z)=z$. By the maximum principle, for each $z \in D$, we have the inequality $|\varphi(z)| \leq |s(z)|$. Therefore, we take

$$k(z) = \frac{\varphi(z)}{s(z)} = \frac{I_p(z)-1}{z\left(I_p(z)\right)} = \frac{c_2 + \left(2c_3 - c_2^2\right)z + \ldots}{1 + c_2 z + \left(2c_3 - c_2^2\right)z^2 + \ldots}.$$

In particular, we have

$$|k(0)| = |c_2|$$

and

$$|k'(0)| = 2\left|c_3 - c_2^2\right|.$$

The function

$$r(z) = \frac{k(z)-k(0)}{1-\overline{k(0)}k(z)}$$

is analytic in D, $r(0)=0$, $|r(z)|<1$ for $|z|<1$ and $|r(\tau)|=1$ for $\tau\in\partial D$. From (1.3), we obtain

$$\frac{2}{1+|r'(0)|}\leq |r'(\tau)|=\frac{1-|k(0)|^2}{\left|1-\overline{k(0)}k(\tau)\right|^2}|k'(\tau)|\leq\frac{1+|k(0)|}{1-|k(0)|}|k'(\tau)|=\frac{1+|k(0)|}{1-|k(0)|}\left\{|\varphi'(\tau)|-|s'(\tau)|\right\}.$$

Since

$$r'(z)=\frac{1-|k(0)|^2}{\left(1-\overline{k(0)}k(z)\right)^2}k'(z)$$

and

$$|r'(0)|=\frac{|k'(0)|}{1-|k(0)|^2}=\frac{2|c_3-c_2^2|}{1-|c_2|^2},$$

we obtain

$$\frac{2}{1+\frac{2|c_3-c_2^2|}{1-|c_2|^2}}\leq\frac{1+|c_2|}{1-|c_2|}\left\{4|I_p'(\tau)|-1\right\}$$

and

$$\left(\frac{2\left(1-|c_2|\right)^2}{1-|c_2|^2+2|c_3-c_2^2|}+1\right)\frac{1}{4}\leq|I_p'(\tau)|.$$

Conflicts of interest

The authors declare that there are no potential conflicts of interest relevant to this article.

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