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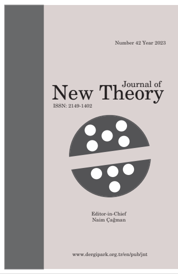

A Note on 4-Dimensional 2-Crossed Modules

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Abstract — The study presents the direct product of two objects in the category of 4-dimensional 2-crossed modules. The structures of the domain, kernel, image, and codomain can be related using isomorphism theorems by defining the kernel and image of a morphism in a category. It then establishes the kernel and image of a morphism in the category of 4-dimensional 2-crossed modules to apply isomorphism theorems. These isomorphism theorems provide a powerful tool to understand the properties of this category. Moreover, isomorphism theorems in 4-dimensional 2-crossed modules allow us to establish connections between different algebraic structures and simplify complicated computations. Lastly, the present research inquires whether additional studies should be conducted.

Keywords *Crossed module, kernel, image, category*

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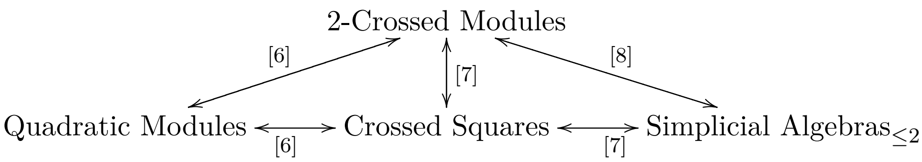
1. Introduction

Crossed modules were first used for groups in Whitehead's work [1] and presented for commutative algebras by Porter in [2]. The idea of crossed modules modeling homotopy 2-types is well-known for becoming useful in a wide range of situations. Conduché [3] presented the idea of 2-crossed modules of groups as an algebraic model of homotopy 3-types. Algebra adaptation of 2-crossed modules is given in [4].

As an algebraic model for homotopy 3-types, Baues [5] established the concept of a quadratic module of groups and provided a relationship from simplicial groups. Actually, a quadratic module structure is a 2-crossed module structure with extra nilpotency conditions. The connection between quadratic and 2-crossed modules was demonstrated in [6]. The relations between the category of 2-crossed modules and related categories such as simplicial groups, quadratic modules, and crossed squares are given in the following diagram:



The 2-truncation of the Moore complex with a simplicial group results in a 2-crossed module. Since the 2-crossed module can be obtained from a simplicial group's Moore complex, it makes sense to consider

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this model while researching algebraic topology from the perspective of simplicial groups. 2-crossed modules have various uses in category theory, including their universal properties, representations, cohomology, and relations with other categorical structures. For more information on 2-crossed modules, see [9–13].

Baues and Bleile developed the idea of 4-dimensional quadratic complexes [14] to examine the presentation of a space X as the mapping cone of a map $\partial(X)$ beneath a space D for the algebraic description of pointed relative CW-complexes with cells in dimension 4. Based on the work of Baues and Bleile, the idea of 4-Dimensional 2-crossed modules was developed in [15] to examine any probable equivalence between homotopy 4-types. Moreover, subobjects and quotient objects in this category are defined in [15].

In this work, we give fundamental properties for a given 4-dimensional 2-crossed module morphism, including the kernel and the image. The isomorphism theorems explain the connection between quotients, homomorphisms, and subobjects. For distinct algebraic structures, there are different iterations of the isomorphism theorem. We also define the direct product to generalize the isomorphism theorems for 4-Dimensional 2-crossed modules.

2. Direct Product of 4-Dimensional 2-Crossed Modules

In this section, we will obtain the direct product of two given 4-dimensional 2-crossed modules. A 4-dimensional 2-crossed module [15] is a complex of algebras

$$\sigma : C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

such that

i. $(C_2, C_1, C_0, \partial_2, \partial_1)$ is a 2-crossed module where $\{-,-\} : C_1 \times C_1 \to C_2$ is the Peiffer lifting,

ii. C_3 is a C_1-module where $\partial_1(C_1)$ acts trivially, and

iii. ∂_3 is a homomorphism of C_1-modules where $\partial_2\partial_3 = 1$.

A morphism between 4-dimensional 2-crossed modules, $f : \sigma_1 \to \sigma_2$, is a commutative diagram

$$\begin{array}{ccccccc} \sigma_1 : C_3 & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 \\ \downarrow{f_3} & & \downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} \\ \sigma_2 : D_3 & \xrightarrow[\delta_3]{} & D_2 & \xrightarrow[\delta_2]{} & D_1 & \xrightarrow[\delta_1]{} & D_0 \end{array}$$

where (f_2, f_1, f_0) is a 2-crossed module morphism and f_3 is an f_0-equivariant homomorphism of modules. We will denote this category with X_2Mod^{4D}.

Let $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ and $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be two objects in X_2Mod^{4D}. We will define the direct product of 4-dimensional 2-crossed modules σ_1 and σ_2. For this, we first define the product of pre-crossed modules ∂_1 and ∂_2.

Proposition 2.1. The algebra homomorphism

$$\begin{array}{rccl} \Phi_1 : & C_1 \times D_1 & \to & C_0 \times D_0 \\ & (c_1, d_1) & \mapsto & \phi_1(c_1, d_1) = (\partial_1(c_1), \delta_1(d_1)) \end{array}$$

is a pre-crossed module of algebras.

PROOF.
For direct product algebras $C_1 \times D_1$ and $C_0 \times D_0$, let the action of $C_0 \times D_0$ on $C_1 \times D_1$ be defined as

$$(c_0, d_0) \cdot (c_1, d_1) = (c_0 \cdot c_1, d_0 \cdot d_1)$$

for $(c_i, d_i) \in C_i \times D_i$ such that $i = 0, 1$. Then, for $(c_0, d_0), (c_0', d_0') \in C_0 \times D_0$ and $(c_1, d_1) \in C_1 \times D_1$, we have

$$\begin{aligned}
[(c_0, d_0) + (c_0', d_0')] \cdot (c_1, d_1) &= (c_0 + c_0', d_0 + d_0') \cdot (c_1, d_1) \\
&= ((c_0 + c_0') \cdot \ c_1, (d_0 + d_0') \cdot d_1) \\
&= (c_0 \cdot \ c_1 + c_0' \cdot \ c_1, d_0 \cdot d_1 + d_0' \cdot d_1) \\
&= (c_0 \cdot \ c_1, d_0 \cdot d_1) + (c_0' \cdot \ c_1, d_0' \cdot d_1) \\
&= (c_0, d_0) \cdot (c_1, d_1) + (c_0', d_0') \cdot (c_1, d_1)
\end{aligned}$$

and

$$\begin{aligned}
[(c_0, d_0)(c_0', d_0')] \cdot (c_1, d_1) &= (c_0 c_0', d_0 d_0') \cdot (c_1, d_1) \\
&= ((c_0 c_0') \cdot \ c_1, (d_0 d_0') \cdot d_1) \\
&= (c_0 \cdot \ (c_0' \cdot \ c_1), d_0 \cdot (d_0' \cdot d_1)) \\
&= (c_0, d_0) \cdot (c_0' \cdot \ c_1, d_0' \cdot d_1) \\
&= (c_0, d_0) \cdot [(c_0', d_0') \cdot (c_1, d_1)]
\end{aligned}$$

Therefore, with this action, $C_1 \times D_1$ is an $(C_0 \times D_0)$-algebra. Moreover, for $(c_0, d_0) \in C_0 \times D_0$ and $(c_1, d_1) \in C_1 \times D_1$,

$$\begin{aligned}
\Phi_1 ((c_0, d_0) \cdot (c_1, d_1)) &= \Phi_1 (c_0 \cdot c_1, d_0 \cdot d_1) \\
&= (\partial_1 (c_0 \cdot c_1), \delta_1 (d_0 \cdot d_1)) \\
&= (c_0 \cdot \partial_1 (c_1), d_0 \delta_1 (d_1)) \\
&= (c_0, d_0) \cdot (\partial_1 (c_1), \delta_1 (d_1)) \\
&= (c_0, d_0) \cdot \Phi_1(c_1, d_1)
\end{aligned}$$

is obtained. Therefore, $\Phi_1 : C_1 \times D_1 \to C_0 \times D_0$ is a pre-crossed module. □

$C_1 \times D_1$ acts on $C_2 \times D_2$ and $C_3 \times D_3$ via Φ_1. Define the Peiffer Lifting as

$$\begin{aligned}
\{-,-\}_P : \quad (C_1 \times D_1) \times (C_1 \times D_1) &\to C_1 \times D_1 \\
((c_1, d_1), (c_1', d_1')) &\mapsto \{(c_1, d_1) \otimes (c_1', d_1')\}_P = (\{c_1 \otimes c_1'\}_C, \{d_1 \otimes d_1'\}_D)
\end{aligned}$$

and

$$\begin{aligned}
\Phi_i : \quad (C_i \times D_i) &\to C_{i-1} \times D_{i-1} \\
(c_i, d_i) &\mapsto (\partial_i(c_i), \delta_i(d_i))
\end{aligned}$$

for $i = 2, 3$.

Proposition 2.2. Let $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ and $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be two objects in X_2Mod^{4D}. Then, the direct product of σ_1 and σ_2

$$\sigma_P := (C_3 \times D_3, C_2 \times D_2, C_1 \times D_1, C_0 \times D_0, \Phi_3, \Phi_2, \Phi_1)$$

is an object in X_2Mod^{4D}.

PROOF.
Let $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ and $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be two objects in X_2Mod^{4D}.

Then,

i. PL1. For $(c_1,d_1),(c'_1,d'_1)\in C_1\times D_1$,

$$\begin{aligned}\Phi_2\{(c_1,d_1)\otimes(c'_1,d'_1)\}_P &= \Phi_2\left(\{c_1\otimes c'_1\}_C,\{d_1\otimes d'_1\}_D\right)\\ &= (\partial_2\{c_1\otimes c'_1\}_C,\delta_2\{d_1\otimes d'_1\}_D)\\ &= (c_1c'_1-c_1\cdot\partial_1(c'_1),d_1d'_1-d_1\cdot\delta_1(d'_1))\\ &= (c_1,d_1)(c'_1,d'_1)-(c_1,d_1)\cdot\Phi_1(c'_1,d'_1)\end{aligned}$$

PL2. For $(c_2,d_2),(c'_2,d'_2)\in C_2\times D_2$,

$$\begin{aligned}\{\Phi_2(c_2,d_2)\otimes\Phi_2(c'_2,d'_2)\}_P &= \{(\partial_2\left(c_2\right),\delta_2\left(d_2\right))\otimes(\partial_2\left(c'_2\right),\delta_2\left(d'_2\right))\}_P\\ &= (\{\partial_2\left(c_2\right)\otimes\partial_2\left(c'_2\right)\}_C,\{\delta_2\left(d_2\right)\otimes\delta_2\left(d'_2\right)\}_D)\\ &= (c_2c'_2,d_2d'_2)\\ &= (c_2,d_2)(c'_2,d'_2)\end{aligned}$$

PL3. For $(c_1,d_1),(c'_1,d'_1),(c''_1,d''_1)\in C_1\times D_1$,

$$\begin{aligned}\{(c_1,d_1)\otimes(c'_1,d'_1)(c''_1,d''_1)\}_P &= (\{c_1\otimes c'_1c''_1\}_C,\{d_1\otimes d'_1d''_1\}_D)\\ &= (\{c_1c'_1\otimes c''_1\}_C+\partial_1(c''_1)\cdot\{c_1\otimes c'_1\}_C,\{d_1d'_1\otimes d''_1\}_C+\partial_1(d''_1)\cdot\{d_1\otimes d'_1\}_D)\\ &= \{(c_1,d_1)(c'_1,d'_1)\otimes(c''_1,d''_1)\}_P+\Phi_1\left((c''_1,d''_1)\right)\{(c_1,d_1),(c'_1,d'_1)\}_P\end{aligned}$$

PL4.a. For $(c_2,d_2)\in C_2\times D_2$ and $(c_1,d_1)\in C_1\times D_1$,

$$\begin{aligned}\{\Phi_2\left(c_2,d_2\right)\otimes(c_1,d_1)\}_P &= \{(\partial_2(c_2),\delta_2(d_2))\otimes(c_1,d_1)\}_P\\ &= (\{(\partial_2(c_2)\otimes c_1)\}_C,(\delta_2(d_2)\otimes d_1)\}_D)\\ &= (c_2\cdot c_1-\partial_2(c_2)\cdot c_1,d_2\cdot d_1-\delta_2(d_2)\cdot d_1)\\ &= (c_2,d_2)\cdot(c_1,d_1)-\Phi_2\left(c_2,d_2\right)\cdot(c_1,d_1)\end{aligned}$$

b. For $(c_2,d_2)\in C_2\times D_2$ and $(c_1,d_1)\in C_1\times D_1$,

$$\begin{aligned}\{(c_1,d_1)\otimes\Phi_2\left(c_2,d_2\right)\}_P &= \{(c_1,d_1)\otimes(\partial_2(c_2),\delta_2(d_2))\}_P\\ &= (\{c_1\otimes\partial_2(c_2)\}_C,\{d_1\otimes\delta_2(d_2)\}_D)\\ &= \{c_1\otimes\partial_2(c_2)\}_C,\{d_1\otimes\delta_2(d_2)\}_D\\ &= (c_1\cdot c_2,d_1\cdot d_2)\\ &= (c_1,d_1)\cdot(c_2,d_2)\end{aligned}$$

PL5. For $(c_1,d_1),(c'_1,d'_1)\in C_1\times D_1$ and $(c_0,d_0)\in C_0\times D_0$,

$$\begin{aligned}\{(c_1,d_1)\otimes(c'_1,d'_1)\}_P\cdot(c_0,d_0) &= (\{c_1\otimes c'_1\}_C,\{d_1\otimes d'_1\}_D)\cdot(c_0,d_0)\\ &= (\{c_1\otimes c'_1\}_C\cdot(c_0,d_0),\{d_1\otimes d'_1\}_D)\\ &= (\{c_1\cdot c_0\otimes c'_1\cdot d_0\}_C,\{d_1\otimes d'_1\}_D)\\ &= \{(c_1\cdot c_0,d_1)\otimes(c'_1\cdot d_0,d'_1)\}_P\\ &= \{(c_1,d_1)\cdot(c_0,d_0)\otimes(c'_1,d'_1)\}_P\end{aligned}$$

and

$$\begin{aligned}\{(c_1,d_1)\otimes(c_1',d_1')\}_P\cdot(c_0,d_0) &= (\{c_1\otimes c_1'\}_C,\{d_1\otimes d_1'\}_D)\cdot(c_0,d_0)\\ &= (\{c_1\otimes c_1'\}_C,\{d_1\otimes d_1'\}_D\cdot(c_0,d_0))\\ &= (\{c_1\otimes c_1'\}_C,\{d_1\cdot c_0\otimes d_1'\cdot d_0\}_D)\\ &= \{(c_1,d_1\cdot c_0)\otimes(c_1',d_1'\cdot d_0)\}_P\\ &= \{(c_1,d_1)\otimes(c_1',d_1')\cdot(c_0,d_0)\}_P\end{aligned}$$

ii. $C_2\times D_2$ acts on $C_3\times D_3$ via $C_1\times D_1$ trivially. Therefore, $C_3\times D_3$ is a $C_2\times D_2$-module.

iii. For $(c_3,d_3)\in C_3\times D_3$,

$$\begin{aligned}\Phi_2\Phi_3(c_3,d_3) &= \Phi_2\left(\partial_3\left(c_3\right),\delta_3\left(d_3\right)\right)\\ &= \left(\partial_2\partial_3\left(c_3\right),\delta_2\delta_3\left(d_3\right)\right)\\ &= (1_{C_1},1_{D_1})\\ &= 1_{C_1\times D_1}\end{aligned}$$

is obtained. □

3. Kernel and Image of a Morphism in X_2Mod^{4D}

In this section, we give the notions of the kernel and image of a morphism in the category X_2Mod^{4D}. Throughout this section, we will consider the morphism

$$f=(f_3,f_2,f_1,f_0):\sigma_1:=(C_3,C_2,C_1,C_0,\partial_3,\partial_2,\partial_1)\to\sigma_2:=(D_3,D_2,D_1,D_0,\delta_3,\delta_2,\delta_1)$$

in X_2Mod^{4D}.

Proposition 3.1. The object

$$\begin{array}{ccccccc} & & \ker f_1\times\ker f_1 & & & & \\ & & \Big\downarrow{\scriptstyle \{-,-\}_{\ker}} & & & & \\ \ker f_3 & \xrightarrow[\overline{\partial_3}]{} & \ker f_2 & \xrightarrow[\overline{\partial_2}]{} & \ker f_1 & \xrightarrow[\overline{\partial_1}]{} & \ker f_0\end{array}$$

in X_2Mod^{4D} is an ideal of σ_1 where $\overline{\partial_i}$ are restrictions, for $i=0,1,2$.

PROOF.
$\ker f_i$ is an ideal of C_i, for $i=0,1,2,3$. The commutativity of the following diagram

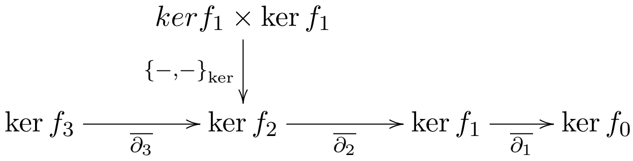
$$\begin{array}{ccccccc}\ker f_3 & \xrightarrow{\partial_3|_{\ker f_3}} & \ker f_2 & \xrightarrow{\partial_2|_{\ker f_2}} & \ker f_1 & \xrightarrow{\partial_1|_{ker f_1}} & \ker f_0\\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow\\ C_3 & \xrightarrow[\partial_3]{} & C_2 & \xrightarrow[\partial_2]{} & C_1 & \xrightarrow[\partial_1]{} & C_0\end{array}$$

implies $\partial_i(\ker f_{i+1})\subset\ker f_i$, for $i=1,2,3$. The axioms of an ideal in X_2Mod^{4D} can be calculated using the definition of a kernel, i.e.,

$$f_2(x_0\cdot c_1)=f_1(x_0)\cdot f_2(c_1)=0\cdot f_2(c_1)=0 \text{ implies that } x_0\cdot c_1\in\ker f_2$$

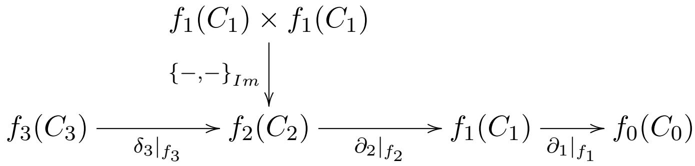
for $x_0\in\ker f_0$ and $c_1\in C_1$. □

Definition 3.2. The ideal



is called the kernel of the morphism f in X_2Mod^{4D} .

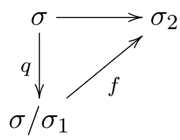
Definition 3.3. The subobject



of σ_2 is called the image of the morphism f in X_2Mod^{4D} .

4. Universal Property in X_2Mod^{4D}

In this section, we show the existence and uniqueness of a morphism $f : \sigma/\sigma_1 \to \sigma_2$ in X_2Mod^{4D} that makes the following diagram



commutative where σ_1 is an ideal of σ_1 and σ_2 is another object in X_2Mod^{4D}.

Theorem 4.1. [15] Let

$$\sigma_1 : D_3 \xrightarrow{\delta_3} D_2 \xrightarrow{\delta_2} D_1 \xrightarrow{\delta_1} D_0$$

be an ideal of

$$\sigma : C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

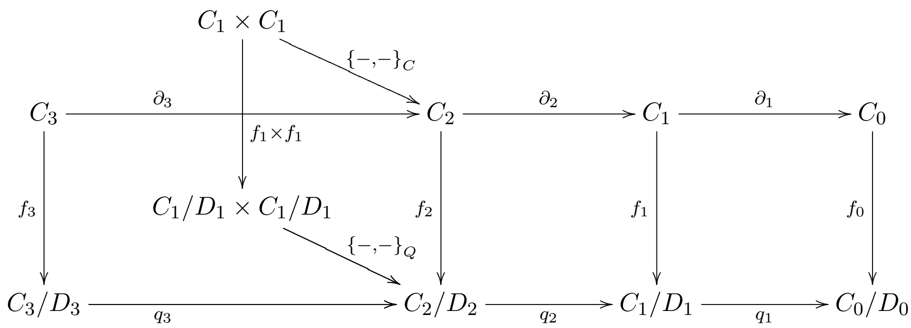
in X_2Mod^{4D}. Then,

$$\sigma/\sigma_1 : C_3/D_3 \xrightarrow{q_3} C_2/D_2 \xrightarrow{q_2} C_1/D_1 \xrightarrow{q_1} C_0/D_0$$

is an object in X_2Mod^{4D}.

Theorem 4.2. Let $\sigma_2 := (D_3, D_2, D_1, D_0, \delta_3, \delta_2, \delta_1)$ be an ideal of $\sigma_1 := (C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1)$ in X_2Mod^{4D}. If $\beta : \sigma_1 \to \sigma_3 := (E_3, E_2, E_1, E_0, \eta_3, \eta_2, \eta_1)$ is a morphism, then there exists a unique morphism $\alpha : \sigma_1/\sigma_2 \to \sigma_3$ in X_2Mod^{4D}.

Proof.
If σ_2 is an ideal of σ_1, then



(1)

there exists $f_{i+1} : C_i \to C_i/D_i$, for $i = 0, 1, 2, 3$. Since, for $c_0 \in C_0$ and $d_i \in D_i$, we have

$$f_{i+1}(c_0 \cdot d_0) = (c_0 \cdot d_i) + D_i = (c_0 + D_i) \cdot (d_i + D_i) = f_0(c_0) \cdot f_i(d_i), \quad i = 0, 1, 2, 3$$

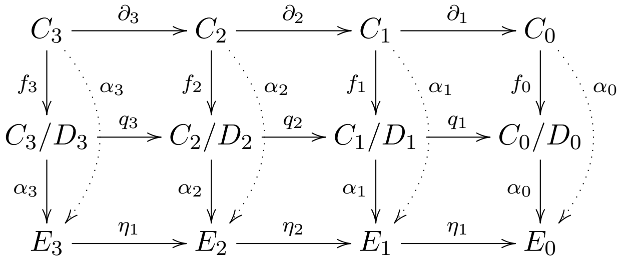
and Diagram 1 is commutative. Moreover, $f = (f_1, f_2, f_3, f_4)$ is a morphism in X_2Mod^{4D}. Since, for all $c + D_i, c' + D_i \in C_i/D_i$,

$$\begin{aligned} c + D_i = c' + D_i &\Rightarrow c - c' \in D_i \\ &\Rightarrow \beta_i(c - c') \in \beta_i(D_i) \\ &\Rightarrow \beta_i(c) - \beta_i(c') \in \beta_i(D_i) \\ &\Rightarrow D_i \subset Ker\beta_i \\ &\Rightarrow \beta_i(D_i) = 1 \end{aligned}$$

$\alpha_i : C_i/D_i \to E_i$ $(i = 1, 2, 3, 4)$ are well defined. Furthermore, for $i = 1, 2, 3$,

$$q_i\alpha_{i+1}(c_i + D_i) = q_i\beta_{i+1}(c_i) = \beta_i(\partial_i(c_i)) = \alpha_i(\partial_i(c_i) + D_i) = \alpha_i q_i(c_i + D_i)$$

As a result, α is the unique morphism in X_2Mod^{4D}, making the following diagram



commutative. □

5. Conclusion

In this work, we defined the direct products in the category of 4-Dimensional 2-crossed modules. We provide the kernel and image of a morphism in this category in order to adapt the isomorphism theorems. Given a morphism $f = (f_3, f_2, f_1, f_0) : \sigma \to \sigma'$ in X_2Mod^{4D}, the mappings $f_i^* : \sigma/\ker f \to f(\sigma)$ defined as $f_i^*(x + \ker f_i) = f_i(x)$, for $i = 0, 1, 2, 3$, are isomorphisms in this category. Using the isomorphism theorem on groups, it can be easily shown that f^* is an isomorphism. As a result, we get

$$\sigma/\ker f \cong f(\sigma)$$

In future studies, isomorphism theorems can be given for this category. Moreover, using semi-direct product groupoid adaptations can be obtained.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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