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A WEIGHTED GOMPERTZ-G FAMILY OF DISTRIBUTIONS FOR RELIABILITY AND LIFETIME DATA ANALYSIS

Joseph Thomas EGHWERIDO[1] and Lawrence Chukwudumebi NZEI[2]

[1]Department of Statistics, Federal University of Petroleum Resources Effurun, Delta State, NIGERIA
[2]Department of Statistics, University of Benin, Edo State, NIGERIA

Abstract. This article is set to push new boundaries with leading-edge innovations in statistical distribution for generating up-to-the-minute contemporary distributions by a mixture of the second record value of the Gompertz distribution and the classical Gompertz model (weighted Gompertz model) using T-X characterization, especially used for two-sided schemes that provide an accurate model. The quantile, ordinary, and complete moments, order statistics, probability, and moments generating functions, entropies, probability weighted moments, Lin's condition random variable, reliability in multicomponent stress strength system, reversed, and moments of residuals life and other reliability characteristics in engineering, actuarial, economics, and environmental technology were derived in their closed form. To investigate and test the flexibility, viability, tractability, and performance of the proposed Weighted Gompertz-G (WGG) generated model, the shapes of some sub-models of the WGG model were examined. The shapes of the sub-models indicated J-shapes, increasing, decreasing, and bathtub hazard rate functions. The maximum likelihood estimation of the WGG-generated model parameters was examined. An illustration with simulation and real-life data analysis indicated that the WGG-generated model provides consistently better goodness-of-fit statistics than some competitive models in the literature.

1. Introduction

Modeling real-life data set requires a distribution that has a true reflection of the character of that data. However, to unravel the interest of some important Poisson scenarios, a parsimonious statistical distribution is required. Hence, new statistical

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[1] eghwerido.joseph@fupre.edu.ng- Corresponding author; 0000-0001-8986-753X;
[2] nzeilawrence@ymail.com; 0000-0002-4441-805X

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models are often introduced to harness salient factors responsive for good decision making.

Oftentimes, change-point models are characterized by abrupt behavioural structures that may be very complicated to handle by the usual classical statistical distributions. These events are but not limited to macroeconomic events characterized by abrupt increases interest rates and inflation. The abrupt behavioural shift might also be the case in extreme events like the storm and rainfall events that have ravaged some countries in recent time. The advent of the novel epidemic COVID 19 is also not exempted. Another example is the lifetime scenario that are subjected to unexpected and rapid shocks. Hence, this study is proposed to deal with such change-point by constructing an appropriate weighted generated distribution called Weighted Gompertz-G (WGG) distribution that can address the differentials. Though the method of generating new distribution is not new, using the weighted generator concept to generate new models is a new approach targeted at change-point problems. Thus, this article will use the weighted Gompertz generator approach to generate new continuous distributions that are more flexible, and viable in their goodness-of-fit test statistics.

The Gompertz model has played a vital role in modeling scenarios that deal with survival times, reliability, human mortality, and actuarial data with exponential increase outcomes. Thus, it has received considerable attention from demographers, economics, and actuaries. This includes [13], and [12] who proposed the shifted Gompertz-G and alpha power Teissier distributions. A flexible alpha power Gompertz distribution was proposed in [14]. [27] emphasised on some applications of the Gompertz distribution in Poisson process. A negative rate of aging parameter with Gompertz distribution was proposed in [22]. [8] proposed the Teissier distribution. [16] proposed the Marshall-Olkin Teissier distribution. The gamma–Gompertz distribution was proposed in [29]. [15] developed the alpha power Marshall-Olkin-G model. [23]developed the Topp-Leone Gompertz distribution with application to glass data. The reliability properties and applications of the alpha power Topp-Leone-G distribution was considered in [17]. However, some researches have been contributed to generating newer classical statistical distributions include [5] and [2] who proposed exponentiated T-X and T-X family of distributions. The type I half-logistic family of distributions proposed by [10]. The beta and generalized gamma-generated distributions by [30]. A tetration distribution developed by [11]. Odd Truncated Inverse Exponential Weibull Exponential by [1]. [24] proposed a New Member from the T-X Family of distribution. A New Odd Log-Logistic Lindley Distribution was proposed in [3]. The Bivariate Lack-of-Memory Distributions was developed in [21]. [20] proposed the U family of distributions. A new estended Weibull distribution was developed in [26]. [18] proposed the alpha power Teissier-G Distribution and its Applications in reliability analysis. Exponentiated Gumbel Weibull Logistic model was developed in [25]. Weighted Weibull-G was introduced by [19].

Let T be a nonnegative random variable with a probability density function (pdf) $f(t)$ such that for a suppose $t > 0$, weight function $w(t) = \beta + exp(\beta t) - 1$,and expectation $E[w(t)] = \frac{\beta\lambda+1}{\lambda}$. Then, [7] defined the pdf and cumulative distribution function (cdf) $F(t)$ of the weighted Gompertz distribution as

$$f(t) = \frac{\beta\lambda^2}{(1+\beta\lambda)}(\beta - 1 + e^{\beta t})e^{(\beta t - \lambda(e^{\beta t}-1)}, \ t > 0 \ \beta \ , \lambda \ > 0, \tag{1}$$

and

$$F(t) = 1 - \left[1 + \frac{\lambda(e^{\beta t} - 1)}{(1+\beta\lambda)}\right]e^{-\lambda(e^{\beta t}-1)}, \ t > 0 \ \beta \ , \lambda \ > 0, \tag{2}$$

with λ and β as the shape and scale parameters.

Modeling abrupt behavioural structure and scenarios has become more complicated as a result of their change-point. Though the method of generating new distribution is not new, using the weighted generator concept to generate new models is a new approach. Hence, this study is motivated to propose a model with a true reflection of the character of the data obtained. Thus, the WGG generated model tends to improve the goodness-of-fit, and the test statistics of the existing distributional models using weighted distribution characterization.

The study aim at introducing a class of generator with the aid of the weighted Gompertz model called the weighted Gompertz generator. This generated model will improve the performance, flexibility and the viability of the goodness-of-fit of the abrupt behaviourial change-point scenarios in lifetime modeling.

2. The Weighted Gompertz-G Distribution

Suppose a nonnegative random variable T is defined on the interval $T \in [m, n]$ for $-\infty < m < n < \infty$ with pdf $r(G(t))$ such that $r(G(t)) = -\log[1 - G(t)]$ is monotonically non-decreasing; $r(G(t))$ is closed in the interval $[m, n]$; and $r(G(t))$ approaches m as t tends to negative infinity, and $r(G(t))$ approaches n as t tends to positive infinity. Thus, by [4] the cdf and the pdf of the WGG generated class of distribution can be expressed as

$$F(t) = 1 - \left[1 + \frac{\lambda[(1 - G(t))^{-\beta} - 1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]} \ t > 0, \ \lambda, \beta > 0, \tag{3}$$

and

$$f(t) = \frac{\lambda^2\beta}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}}g(t)((1 - G(t))^{-\beta} + \beta - 1)e^{-\lambda[(1-G(t))^{-\beta}-1]}, \tag{4}$$

for $t > 0, \ \lambda, \beta > 0$, where $g(t)$, and $G(t)$ are the parents pdf and cdf.

The WGG generated reliability model can be expressed as

$$S_{WGG}(t) = \left[1 + \frac{\lambda[(1 - G(t))^{-\beta} - 1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]} \ t > 0, \ \lambda, \beta > 0. \tag{5}$$

The hazard rate function that corresponds to the WGG generated model is defined as

$$h_{WGG}(t) = \frac{\frac{\lambda^2\beta}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}} g(t)((1-G(t))^{-\beta}+\beta-1)}{\left[1+\frac{\lambda[(1-G(t))^{-\beta}-1]}{1+\beta\lambda}\right]} \quad t>0,\ \lambda,\beta>0. \tag{6}$$

The reversed hazard rate function is obtained as

$$r_{WGG}(t) = \frac{\frac{\lambda^2\beta}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}} g(t)((1-G(t))^{-\beta}+\beta-1)e^{-\lambda[(1-G(t))^{-\beta}-1]}}{1-\left[1+\frac{\lambda[(1-G(t))^{-\beta}-1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]}}, \tag{7}$$

for $t>0,\ \lambda,\beta>0$.

The cumulative hazard rate function of the WGG generated function is give as:

$$H_{WGG}(t) = \log(1+\beta\lambda) - \log([1+\beta\lambda]+\lambda[(1-G(t))^{-\beta}-1]) + \lambda([1-G(t)]^{-\beta}-1). \tag{8}$$

3. The Quantile Function

Quantile is fundamental for the simulation and estimation of a distribution parameter(s). Hence, it is a function that associates the probability distribution function of the WGG generated model of a random variable T such that the probability of the variable being less than or equal to that value equals the probability for a uniform interval $q \in (0,1)$ is defined as

$$t = G^{-1}\left[1-\left[\frac{W_{-1}((q-1)(1+\beta\lambda)e^{(1+\beta\lambda)})-(1+\beta\lambda)}{\lambda}+1\right]^{-\frac{1}{\beta}}\right], \tag{9}$$

where W_{-1} is the Lambert-W or omega function as defined in [13] and [16] such that $W(t) = e^{(W(t))} = t \in [-1,\infty)$.

In particular, the median is obtained when $q = 0.5$.

Theorem 1. *The shape, characteristics, and behaviour of the WGG generated model can be examined by investigating the first and second derivatives of the* log *of the WGG generated pdf model. Thus, for $f'(t) < 0$. Then, then cdf $F(t)$ will be decreasing monotonically for all values of t. The WGG generated model will be bimodal if $f''(t)$ changes its signs from negative to non-negative, viz-a-viz.*

Proof. The $\log f(t)$ is give as

$$\begin{aligned}\log f(t) =& 2\log\lambda + \log\beta - \log(1+\beta\lambda) + \log g(t) - (1+\beta\lambda)\log(1-G(t)) \\ &+ \log([1-G(t)]^{-\beta}+\beta-1) - \lambda([1-G(t)]^{-\beta}-1).\end{aligned}$$

Thus, taking the derivative with respect to the variable,we have

$$\frac{\partial \log f(t)}{\partial t} = \frac{g'(t)}{g(t)} + (1+\beta\lambda)\frac{g(t)}{S(t)} + \frac{\beta g(t) S^{-\beta-1}(t)}{S^{\beta}(t)+\beta-1} - \lambda g(t) S^{-\beta-1}(t),$$

where $S(t) = 1-G(t)$. Hence, $f'(t) < 0$ if $g(t) < 0$.

The second derivative was implemented to determine if the model was bimodal. Thus, the second derivative is given as

$$\begin{aligned}\frac{\partial^2 \log f(t)}{\partial t^2} =& \frac{g''(t)}{g(t)} - \frac{1}{g'(t)} + (1+\beta\lambda)[\frac{g'(t)}{S(t)} + \frac{g^2(t)}{S^2(t)}] - \frac{\lambda g'(t)}{S^{(\beta+1)}(t)} \\ &+ \lambda(1+\beta\lambda)\frac{g^2(t)}{S^{(\beta+2)}(t)} + \frac{\beta g'(t) S^{-\beta-1}(t)}{S^{\beta}(t)+\beta-1} \\ &+\beta(\beta+1)\frac{g^2(t) S^{-(\beta+2)}(t)}{S^{\beta}(t)+\beta-1} + \frac{\beta^2 g^2(t) S^{-2(\beta+1)}(t)}{(S^{\beta}(t)+\beta-1)^2}.\end{aligned}$$

□

4. Order statistics

Order statistics are useful tools to improve the robustness of sampling plans by variables, and shorten test times of Poisson processes.

Let $T_{(1)}, T_{(2)}, T_{(3)}, \dots, T_{(k)}$ be the order statistics for a random variable $T_1, T_2, T_3, \dots, T_k$ with WGG distribution. Then, the WGG density of the u^{th} order statistics is given as

$$f_u(t) = \frac{k!}{(u-1)!(k-u)!} F^{u-1}(t) S^{k-u}(t) f(t) \quad -\infty < t < \infty. \tag{10}$$

However, using the binomial expansion, and noting that $S = 1 - G(t)$, we have the order statistics as

$$\begin{aligned}f_u(t) =& \frac{\beta\lambda^2 S^{-(\beta+1)} k!}{(u-1)!(k-u)!}(S^{-\beta}+\beta-1)\sum_{j=0}^{u-1}(-1)^{u-j-1}\binom{u-1}{j} \\ &\times\left[(1+\beta\lambda)+\lambda(S^{-\beta}-1)\right]^{k+j-u+1} e^{-\lambda(S^{-\beta}-1)(k+j-u+1)}.\end{aligned} \tag{11}$$

The minimum order statistics is obtained when $u = 1$, and the maximum order statistics is obtained when $u = k$ respectively.

4.1. **Record value distributions of the WWG model.**

Let T_i for $i = 1, 2, 3, \dots, k$ be a finite sequence of independently identically distributed random variables with WGG generated cdf $F(t)$ and a record times given as $U(1) = 1$ and $U(k+1) = min\{j > U(k);\ T_j > T_{u(k)}\};\ k \in \mathbb{N}$ with the random variable $T_{u(k)}$ $(k \in \mathbb{N})$ as the upper record values. Then, the pdf of the i upper record value $UR_i = T_{u(k)}$ with a special case of $UR_1 = T_1$ is given as

$$\begin{aligned} f_{UR_i}(t) =& \frac{f(t)}{\Gamma(i)}\{-\log[1-F(t)]\}^{i-1} \\ =& \frac{\lambda^2\beta g(t)((1-G(t))^{-\beta}+\beta-1)e^{-\lambda[(1-G(t))^{-\beta}-1]}}{(1+\beta\lambda)(1-G(t))^{(1+\beta)}\Gamma(i)} \\ &\times\left\{-\log\left[\left[1+\frac{\lambda[(1-G(t))^{-\beta}-1]}{1+\beta\lambda}\right]e^{-\lambda[(1-G(t))^{-\beta}-1]}\right]\right\}^{i-1} \end{aligned} \quad (12)$$

5. Sub-models

Some special sub-models were considered for flexibility, viability, and tractability using the proposed WGG generated model. We present some special cases of the WGG generated family of distributions since it extends several useful distributions in the literature. For all cases listed next, we consider $t, \lambda, \beta > 0$. Especially sub-models with increasing, decreasing shaped data with or without a flat region in modeling. These special sub-models include Burr-XII, Lomax, and Frechet distributions.

5.1. **Weighted Gompertz-G Burr-XII (WGG-B) distribution.** Consider the Burr XII distribution with positive parameters θ and ρ, and cdf and pdf given as $G(t) = 1-(1+t^\theta)^{-\rho}$ and $g(t) = \theta\rho t^{\theta-1}(1+t^\theta)^{-\rho-1}$. Then, inserting these expressions into Equations (3) and (4) gives the WGG-B density function with the cdf and pdf given as

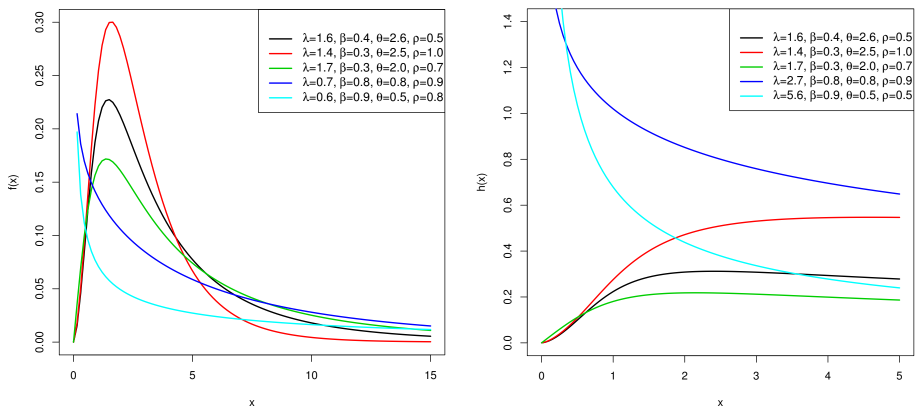
$$F(t) = 1-\left[1+\frac{\lambda[(1+t^\theta)^{\beta\rho}-1]}{1+\beta\lambda}\right]e^{-\lambda[(1+t^\theta)^{\beta\rho}-1]},\ t>0,\ \lambda,\beta,\theta,\rho>0, \quad (13)$$

and

$$\begin{aligned} f(t) =& \frac{\lambda^2\beta(1+t^\theta)^{\rho(1+\beta)}}{(1+\beta\lambda)}((1+t^\theta)^{\beta\rho}+\beta-1)e^{-\lambda[(1+t^\theta)^{\beta\rho}-1]} \\ &\times\theta\rho t^{\theta-1}(1+t^\theta)^{-\rho-1},\ t>0,\ \lambda,\beta,\theta,\rho>0. \end{aligned} \quad (14)$$

Plots of the WGG-B density function for the selected parameter values are displayed in Figure 1a. Figure 1b displays the corresponding hazard rate function (hrfs) for particular values of the parameters. The shapes of the hazard rate function indicated increasing, and decreasing.

5.2. **Weighted Gompertz-G Lomax (WGG-L) distribution.** Consider the Lomax distribution with positive shape parameters θ and scale parameter ρ, and cdf and pdf given as $G(t) = 1-(1+\frac{t}{\rho})^{-\rho}$ and $g(t) = \frac{\theta}{\rho}[1+\frac{t}{\rho}]^{-(\theta+1)}$. Then, inserting these expressions into Equations (3) and (4) gives the WGG-L density function with the cdf and pdf given as



(A) WGG-B density

(B) WGG-B hazard rate function

FIGURE 1. The plots WGG-B model for selected values of parameters

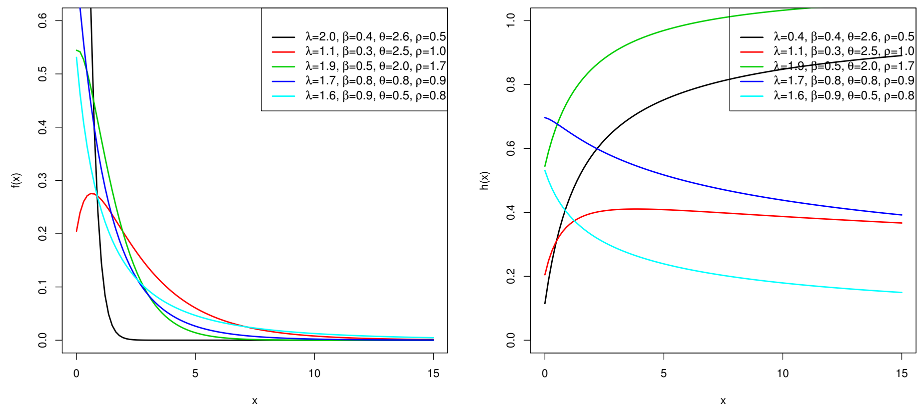
$$F(t) = 1 - \left[1 + \frac{\lambda[(1+\frac{t}{\rho})^{\beta\rho} - 1]}{1+\beta\lambda}\right] e^{-\lambda[(1+\frac{t}{\rho})^{\beta\rho}-1]},\ t > 0,\ \lambda, \beta, \theta, \rho > 0, \qquad (15)$$

and

$$\begin{aligned} f(t) =& \frac{\lambda^2\beta(1+\frac{t}{\rho})^{\rho(1+\beta)}}{(1+\beta\lambda)}((1+\frac{t}{\rho})^{\beta\rho} + \beta - 1)e^{-\lambda[(1+\frac{t}{\rho})^{\beta\rho}-1]}, \\ &\times\frac{\theta}{\rho}[1+\frac{t}{\rho}]^{-(\theta+1)},\ t > 0,\ \lambda, \beta, \theta, \rho > 0. \end{aligned} \qquad (16)$$

Plots of the WGG-L density function for the selected parameter values are displayed in Figure 2a. Figure 2b displays the corresponding hrfs for some particular values of the parameters. The shapes of the hazard rate function indicated increasing, and decreasing.

5.3. **Weighted Gompertz-G Frechet (WGG-F) distribution.** Consider the Frechet distribution with positive shape parameters θ and scale parameter ρ, and cdf and pdf given as $G(t) = e^{-(\frac{\theta}{t})^{\rho}}$ and $g(t) = \rho\theta^{\rho}t^{-\rho-1}e^{-(\frac{\theta}{t})^{\rho}}$. Then, inserting these expressions into Equations (3) and (4) gives the WGG-F density function with the cdf and pdf given as



(A) WGG-L density (B) WGG-L hazard rate function

FIGURE 2. The plots WGG-L model for selected values of parameters

$$F(t) = 1 - \left[1 + \frac{\lambda[(1 - e^{-(\frac{\theta}{t})^{\rho}})^{-\beta} - 1]}{1 + \beta\lambda}\right] e^{-\lambda[(1-e^{-(\frac{\theta}{t})^{\rho}})^{-\beta}-1]},\ t > 0,\ \lambda, \beta, \theta, \rho > 0, \tag{17}$$

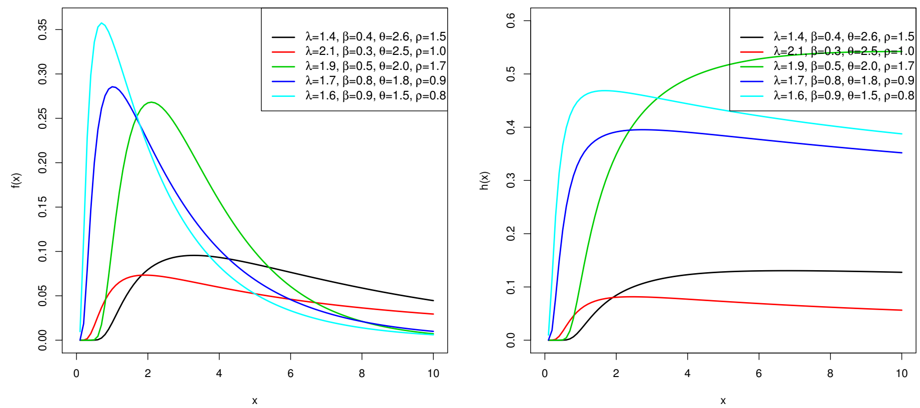
and

$$\begin{aligned} f(t) =& \frac{\lambda^2 \beta}{(1 + \beta\lambda)(1 - e^{-(\frac{\theta}{t})^{\rho}})^{(1+\beta)}}((1 - e^{-(\frac{\theta}{t})^{\rho}})^{-\beta} + \beta - 1)e^{-\lambda[(1-e^{-(\frac{\theta}{t})^{\rho}})^{-\beta}-1]} \\ &\times \rho\theta^{\rho} t^{-\rho-1} e^{-(\frac{\theta}{t})^{\rho}},\ t > 0,\ \lambda, \beta, \theta, \rho > 0. \end{aligned} \tag{18}$$

Plots of the WGG-F density function for the selected parameter values are displayed in Figure 3a. Figure 3b displays the corresponding hrfs for some particular values of the parameters. The shapes of the hazard rate function indicated an increase.

6. Mathematical Expression

To examine the productivity of the WGG generated model, mathematical expansion of the pdf and cdf is carried out. The exponential term in (3) and (4) can



(A) WGG-F density

(B) WGG-F hazard rate function

FIGURE 3. The plots WGG-F model for selected values of parameters

be expressed as

$$e^{-\lambda((1-G(t))^{-\beta}-1)} = \sum_{w=0}^{\infty} \frac{(-1)^w \lambda^w ((1-G(t))^{-\beta}-1)^w}{w!}.$$

Also, by binomial expansion, we have

$$((1-G(t))^{-\beta}-1)^w = \sum_{u=0}^{w} (-1)^{w-u} \binom{w}{u} (1-G(t))^{-u\beta}.$$

Hence, the WGG generated pdf can be expressed as power function as

$$f(t) = \sum_{i,w=0}^{\infty} \sum_{u=0}^{w} \alpha \mu_{(i,w,u)} g(t) G^i(t), \tag{19}$$

where

$$\alpha = \frac{\Gamma(u\beta+2\beta+i+1)}{\Gamma(u\beta+2\beta+1)} + (\beta-1)\frac{\Gamma(u\beta+\beta+i+1)}{\Gamma(u\beta+\beta+1)},$$

and

$$\mu_{(i,w,u)} = (-1)^{2w-u+i} \binom{w}{u} \frac{\lambda^{w+2}}{i!w!} \frac{\beta}{(1+\beta\lambda)},$$

where $\Gamma(\cdot)$ is a gamma function.

7. Statistical Properties

The viability and performance of the proposed model will be investigated by examining some general statistical properties of the WGG generated model in this section.

Oftentimes, the expectation, variance, and moments of random variables can be obtained from some characteristics of the distribution function. Some of these functions are the probability generating function and the moment generating function.

Lin's condition random variable The Lin's function for a pdf f of a random variable T with a support $t > 0$ is defined as

$$L_{f(t)} = -t\frac{f'(t)}{f(t)} = -t\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\alpha\mu_{(i,w,u)}\frac{ig^2(t)G^{i-1}(t) + g'(t)G^i(t)}{g(t)G^i(t)}.$$

Incomplete moments

The incomplete moments of the WGG generated model allow the shape of the moments of WGG generated distribution, which is of interest for many areas, including econometrics, finance, and reliability, to be visible.

The k^{th} incomplete moment, say $\tau_k(t)$ of the WGG generated moment is given as

$$\tau_k(y) = \sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\alpha\mu_{(i,w,u)}\eta_{k,i}(y),$$

where $\eta_{k,i} = \int_0^y t^k g(t)G^i(t)dt$.

Probability generating function

This is a useful mechanism for characterizing the distribution of the random variable T with the WGG generated model. It can succinctly be used to describe the sequence of the probability of the random variable T with the WGG distribution. Hence, a random variable T with a WGG distribution has the probability generating function defined as

$$\begin{aligned} P(z) &= \sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\int_0^{\infty} z^t\alpha\mu_{(i,w,u)}g(t)G^i(t)dt \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{(\log z)^a\alpha\mu_{(i,w,u)}}{a!}\int_0^{\infty} t^a g(t)G^i(t)dt \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{(\log z)^a\alpha\mu_{(i,w,u)}}{a!}p(z), \end{aligned} \tag{20}$$

where

$$p(z) = \int_0^{\infty} t^a g(t)G^i(t)dt \ |z| \leq 1.$$

Moment generating function

The probability density function of the random variable T can be identified using the moment generating function instrument. This is, however, possible since the moment generating function is a non-negative integral of measurable function. Thus, for a random variable T with a WGG distribution, the moment generating function is given as

$$\begin{aligned} M_T(z) &= \sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\int_0^{\infty} e^{zt}\alpha\mu_{(i,w,u)}g(t)G^i(t)dt \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{z^a\alpha\mu_{(i,w,u)}}{a!}\int_0^{\infty} t^a g(t)G^i(t)dt \qquad (21) \\ &= \sum_{i,w,a=0}^{\infty}\sum_{u=0}^{w}\frac{z^a\alpha\mu_{(i,w,u)}}{a!}p(z) \end{aligned}$$

Probability weighted moments

One of the widely used characteristics of a distribution is called L-moments or probability weighted moments. This characteristic is used in hydrology to estimate the parameters of flood distributions. This might be because it is less sensitive to outliers, lower sampling variability, and fast convergence to asymptotic normality. The shape of the WGG generated probability distribution can also be summarized using the L-moments. Thus, L-moments are defined as:

$$P_{wm}(w,v) = \int_0^{\infty} t^w F^v(t)f(t)dt. \qquad (22)$$

However, F^v can be expressed as

$$F^v = \sum_{i,w=0}^{\infty}\sum_{u=0}^{w+p}\sum_{p=0}^{v}(-1)^{2w+v+i-u}\binom{v}{p}\binom{w+p}{u}\frac{\lambda^{w+p}p^w\Gamma(k\beta+i)}{w!i!\Gamma(k\beta)(1+\beta\lambda)^p}G^i(t)$$

where $\Gamma(\cdot)$ is a gamma function. Hence, L-moments is given as

$$P_{wm}(w,v) = \sum_{i,w=0}^{\infty}\sum_{u=0}^{w+p}\sum_{p=0}^{v}R_{(i,w,u,p)}\alpha\mu_{(i,w,u)}T_i \qquad (23)$$

where

$$T_i = \int_0^{\infty} t^w g(t)G^{2i}(t)dt$$

and

$$R_{(i,w,u,p)} = (-1)^{2w+v+i-u}\binom{v}{p}\binom{w+p}{u}\frac{\lambda^{w+p}p^w\Gamma(k\beta+i)}{w!i!\Gamma(k\beta)(1+\beta\lambda)^p}.$$

Entropies

The heterogeneity or impurity of the target variable of Poisson process can be measured by the amount of uncertainty associated in the value of a random variable.

Thus, the Shannon entropy of WGG generated random variable T is defined as

$$S_e(T) = E\Big[-\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\Big(\log\mu_{(i,w,u)} + \log\mu + \log g(t) + i\log G(t)\Big)\Big] \tag{24}$$

The Renyi entropy is a measure that increasingly weighs all WGG generated random events with nonzero probability. As θ approaches zero, the WGG generated Renyi entropy is given as

$$R_\theta = \frac{1}{(1-\theta)}\log\int_0^\infty f^\theta(t)dt \ \theta > 0, \theta \neq 0. \tag{25}$$

This implies

$$\begin{aligned} R_\theta =& \frac{1}{(1-\theta)}\log\int_0^\infty\Big(\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\mu_{(i,w,u)}\alpha g(t)G^i(t)\Big)^\theta dt \\ =& \frac{1}{(1-\theta)}\log\Big[\Big(\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\mu_{(i,w,u)}\alpha\Big)^\theta\int_0^\infty g(t)^\theta G^i(t)^\theta\Big]dt \\ =& \frac{1}{(1-\theta)}\log\Big[\Big(\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\mu_{(i,w,u)}\alpha D_i\Big)^\theta\Big], \end{aligned} \tag{26}$$

where

$$D_i = \int_0^\infty g(t)G^i(t)dt. \ i = 1, 2, 3, \ldots$$

Moment of the residual In reliability theory, and life testing scenarios, the additional lifetime a process or a product that a component or chain has survived up to time t is called the vitality function or residual life function or truncated moment. It can also be used to obtain the distribution function $F(t)$. Thus, the k^{th} moment of the residual life defined as $M_{rs}(x) = E[(T-x)^k \mid T \geq t]$. Hence, it is expressed as

$$\begin{aligned} M_{rs}(x) =& \frac{1}{1-F(x)}\int_x^\infty (T-x)^k f(t)dt = \frac{1}{1-F(x)}\sum_{a=1}^{k}(-1)^{k-a}x^{k-a}\int_x^\infty t^a f(t)dt \\ =& \frac{\alpha}{1-F(x)}\sum_{i,w=0}^{\infty}\sum_{u=0}^{w}\sum_{a=1}^{k}(-1)^{k-a}x^{k-a}\mu_{(i,w,u)}\Im_i, \end{aligned} \tag{27}$$

where

$$\Im_i = \int_x^\infty t^a g(t)G^i(t)dt.$$

Theorem 2. *Let T be a random variable with a WGG generated probability distribution function $F(t)$. Let $S(t) = 1 - F(t)$ and $M_k(y) = E[(T-y)^k \mid T > y],\ y \geq 0$.*

Then,

$$\frac{M_k'(y) + kM_k(y)}{M_k(y)} = \frac{M_{k-1}'(y) + (k-1)M_{k-1}(y)}{M_{k-1}(y)} \text{ or}$$

equivalently,

$$M_{k-1}'(y) = -(k-1)M_{k-2}'(y) + \frac{M_k'(y)}{M_k(y)}M_{k-1}(y) + \frac{kM_{k-1}^2(y)}{M_k(y)}.$$

Proof. Let

$$M_k(y) = \frac{1}{S(y)} \int_y^\infty k(t-y)^{k-1} S(t)dt.$$

Then,

$$\log M_k(y) = \log \int_y^\infty k(t-y)^{k-1} S(t)dt - \log S(y).$$

Thus, differentiating with respect to y, we have

$$\frac{M_k'(y)}{M_k(y)} = \frac{\int_y^\infty -k(k-1)(t-y)^{k-2}S(t)dt}{\int_y^\infty k(t-y)^{k-1}S(t)dt} - \frac{S'(y)}{S(y)} = \frac{-kM_{k-1}(y)}{M_k(y)} - \frac{S'(y)}{S(y)}.$$

Hence,

$$\frac{M_k'(y) + kM_{k-1}(y)}{M_k(y)} = -\frac{S'(y)}{S(y)} = \frac{M_{k-1}'(y) + (k-2)M_{k-2}(y)}{M_{k-1}(y)}.$$

□

8. Parameter Estimation

It is intuitive to note that the parameters of the WGG generated model are descriptive measures of the entire population that determine the shape and location of the curve on the plot of the WGG generated distribution. Hence, for a better forecasting and regression analysis of the proposed WGG model to be efficient, there is a need to obtain the parameter estimates of the WGG generated model. Thus, in this section, the parameters of the WGG generated model are estimated using the maximum likelihood estimation (MLE) method.

8.1. **Maximum Likelihood.** Let $\mathbf{T} = (T_1, T_2, \ldots, T_k)$ be a random sample obtained from the WGG generated distribution with unknown parameter vector $\Theta = (\beta, \lambda, \psi)^T$. Let $\mathbf{t} = (t_1, t_2, \ldots, t_k)$ be a sample value of a random sample $\mathbf{T}$. Then, we can obtain the log-likelihood as

$$
\begin{aligned}
\ell =&2k\log\lambda + k\log\beta + \sum_{a=1}^{k}\log g(t_a,\psi) - k\log(1+\beta\lambda)\\
&-(1+\beta\lambda)\sum_{a=1}^{k}\log(1-G(t_a,\psi)) + \sum_{a=1}^{k}\log((1-G(t_a,\psi))^{-\beta}+\beta-1)\\
&-\sum_{a=1}^{k}\lambda((1-G(t_a,\psi))^{-\beta}-1).
\end{aligned} \tag{28}
$$

The parameters of the WGG generated model are obtained by taking the first partial derivative of the log-likelihood of the WGG model with respect to each of the parameters and equate to zero. Thus, we have

$$
\frac{\partial\ell}{\partial\lambda} = \frac{2k}{\lambda} - \frac{k\beta}{1+\beta\lambda} - \beta\sum_{a=1}^{k}\log(1-G(t_a,\psi)) - \sum_{a=1}^{k}((1-G(t_a,\psi))^{-\beta}-1) = 0, \tag{29}
$$

$$
\begin{aligned}
\frac{\partial\ell}{\partial\psi} =&\sum_{a=1}^{k}\frac{g'(t_a,\psi)}{g(t_a,\psi)} + \beta\sum_{a=1}^{k}\frac{g(t_a,\psi)(1-G(t_a,\psi))^{-\beta-1}}{((1-G(t_a,\psi))^{-\beta}+\beta-1)}\\
&+(1+\beta\lambda)\sum_{a=1}^{k}\frac{g(t_a,\psi)}{1-G(t_a,\psi)} - \beta\lambda\sum_{a=1}^{k}g(t_a,\psi)(1-G(t_a,\psi))^{-\beta-1} = 0,
\end{aligned} \tag{30}
$$

and

$$
\begin{aligned}
\frac{\partial\ell}{\partial\beta} =&\frac{k}{\beta} - \lambda\sum_{a=1}^{k}\log(1-G(t_a,\psi)) + \sum_{a=1}^{k}\frac{(1-G(t_a,\psi))^{-\beta}\log(1-G(t_a,\psi))}{(1-G(t_a,\psi))^{-\beta}+\beta-1}\\
&-\frac{k\lambda}{1+\beta\lambda} - \lambda\sum_{a=1}^{k}(1-G(t_a,\psi))^{-\beta}\log(1-G(t_a,\psi)) = 0.
\end{aligned} \tag{31}
$$

However, the solutions to the nonlinear equations (29), (30), and (31) are obtained in closed form using numerical methods. These numerical methods are beyond the scope of this article.

9. Applications

The viability, tractability, and performance of the WGG generated model is examined by first performing a Monte Carlo simulation of some sub-models of the proposed model. The real-life applications of some of the sub-models of the proposed model were investigated and compared to some competitive-related models in the literature. The WGG sub-models were compared with some existing models based on their mean squared errors in the simulation cases and goodness-of-fit test statistics in life applications.

9.1. **Simulation study.** A Monte Carlo simulation was carried out to test the flexibility and efficiency of the proposed distribution. The simulation was achieved using the quantile function in (9) to generate random data for the proposed model with $0 < q < 1$ for various values of $\lambda = 1.0, \beta = 1.0, \theta = 0.2$ and $\rho = 1.0$ for the Burr XII sub-model. $\lambda = 0.9, \beta = 2.3, \theta = 0.1$ and $\rho = 0.01$ for the Lomax sub-model, and $\lambda = 0.1, \beta = 0.1, \theta = 0.3$ and $\rho = 0.8$ for the Frechet sub-model for 1000 replicated trials.

The sample size n are taken as $n = 5, 10, 20, 50, 100, 150, 200,$ 250,300, $350, 400, 450,$ and 500 The simulation studied the mean estimated (ME), biases, and mean squared errors (MSE). The result of the simulation is as shown in Table 1. In Table one, we observed that the biases converge to zero as sample sizes increase. The estimated mean also converges to the true value as the sample sizes increases. The mean square errors converge to zero.

The bias is obtained for $(W = \lambda, \beta, \theta, \rho)$ as

$$B\hat{ias}_W = \frac{1}{1000} \sum_{i=1}^{1000} \left(\hat{W}_i - W \right).$$

Also, the MSE is obtained as

$$M\hat{S}E_W = \frac{1}{1000} \sum_{i=1}^{1000} \left(\hat{W}_i - W \right)^2.$$

9.2. **Life applications.** In most cases of statistical modeling, the interest is to estimate the model parameters and evaluate their test statistics goodness-of-fit. Thus, in this section, the viability, tractability, and effectiveness of the proposed model is investigated with the illustration of real-life data sets. The measures of the test statistics' goodness-of-fit were examined with some existing neighbourhood models in the literature. These models in the literature include, but are not limited to, the class of Weibull, Gompertz, Kumaraswamy, and Frechet distributions. The test statistics considered include the Akaike information criterion (AIC), Anderson-Darling (A), Cramer-von Mises (W), Kolmogrov-Smirnov (KS), and p-value (p-val). The larger the p-value and the smaller the test statistics the better the model fits the data.

9.2.1. *Obesity Data.* The first data consist of 22 obesity among children and adolescents aged 12-19 by selected characteristics: United States, selected between 2015 - 2018 as reported by [9]. The data are available in https://www.cdc.gov/nchs/hus /contents.htm-Table-027. The data were measured based on height and weight. The data are as follows:

18.9,15.1,23.1,9.8,25.7,26.9,19.8,16.0,19.2,12.0,28.0,29.2,
17.9,14.2,27.0,7.4,23.3,24.6,23.9,21.7,18.4,10.6.

The descriptive statistics of the data are given in Table 2.

TABLE 1. The mean estimates (ME), biases and mean squared errors (MSE) for λ, β, θ and ρ with WGG generated sub-models

Distribution	Parameters	n	ME	Bias	MSE
Burr XII		05	1.0602 0.8288 0.1762 1.0033	0.3603 0.3288 0.8306 0.7033	0.3597 0.1811 0.6907 0.6271
		10	1.1340 0.8115 0.1718 0.9558	0.3592 0.3157 0.2286 0.6779	0.3544 0.1581 0.6873 0.5358
		20	1.1951 0.7708 0.1694 0.9466	0.3592 0.3157 0.1285 0.6740	0.3474 0.1303 0.1871 0.5326
	$\lambda = 1.0$	50	1.1601 0.7416 0.1715 0.9556	0.2284 0.1115 0.1283 0.0667	0.3349 0.1255 0.0370 0.5226
		100	1.0615 0.7558 0.1724 0.9740	0.1244 0.0192 0.1283 0.0648	0.1345 0.0240 0.0269 0.4176
	$\beta = 1.0$	150	1.0219 0.7796 0.1714 0.9648	0.1159 0.0178 0.1282 0.0582	0.0400 0.0221 0.0269 0.3150
		200	0.9716 0.7961 0.1717 0.9779	0.1089 0.0130 0.0278 0.0575	0.0331 0.0211 0.0261 0.2107
	$\theta = 0.2$	250	0.9756 0.7893 0.1729 0.9582	0.0181 0.0101 0.0176 0.0418	0.0317 0.0180 0.0256 0.1012
		300	0.9397 0.8078 0.1722 0.9667	0.0028 0.0098 0.0173 0.0406	0.0259 0.0164 0.0152 0.0196
	$\rho = 1.0$	350	0.9408 0.8157 0.1717 0.9575	0.0021 0.0096 0.0101 0.0466	0.0158 0.0149 0.0148 0.0190
		400	0.9408 0.8157 0.1717 0.9575	0.0010 0.0078 0.0062 0.0298	0.0028 0.0142 0.0134 0.0106
		450	0.9841 0.8092 0.1727 0.9269	0.0001 0.0058 0.0058 0.0269	0.0009 0.0084 0.0127 0.0094
		500	0.9911 0.8030 0.1742 0.9152	0.0001 0.0016 0.0038 0.0152	0.0006 0.0063 0.0101 0.0092
Lomax		05	1.0055 2.1054 0.0956 0.0143	0.7055 0.4735 1.3044 0.0918	0.7714 0.4765 1.3050 0.0919
		10	0.9631 2.1852 0.0959 0.0109	0.6631 0.4724 1.3041 0.0917	0.7078 0.4740 1.3046 0.0917
		20	0.9085 2.2429 0.0963 0.0098	0.6085 0.4718 1.3037 0.0916	0.6287 0.4727 1.3042 0.0917
	$\lambda = 0.9$	50	0.8801 2.2735 0.0951 0.0084	0.6071 0.1010 1.3049 0.0416	0.1091 0.1720 0.1043 0.0216
		100	0.8806 2.2724 0.0958 0.0082	0.0232 0.0208 0.0142 0.0391	0.0254 0.0420 0.0138 0.0116
	$\beta = 2.3$	150	0.8870 2.2708 0.0977 0.0083	0.0196 0.0192 0.0123 0.0313	0.0238 0.0300 0.0125 0.0113
		200	0.8870 2.2718 0.0991 0.0084	0.0133 0.0184 0.0109 0.0212	0.0147 0.0292 0.0114 0.0112
	$\theta = 0.1$	250	0.8896 2.2710 0.0987 0.0084	0.0132 0.0181 0.0093 0.0211	0.0136 0.0198 0.0111 0.0111
		300	0.8962 2.2692 0.1010 0.0087	0.0096 0.0173 0.0082 0.0210	0.0122 0.0180 0.0092 0.0110
	$\rho = 0.01$	350	0.8993 2.2684 0.1020 0.0088	0.0070 0.0159 0.0080 0.0199	0.0100 0.0166 0.0082 0.0109
		400	0.9016 2.2681 0.1028 0.0089	0.0030 0.0129 0.0072 0.0195	0.0097 0.0121 0.0073 0.0098
		450	0.9002 2.2673 0.1014 0.0090	0.0006 0.0057 0.0066 0.0191	0.0055 0.0089 0.0067 0.0094
		500	0.9001 2.2659 0.1005 0.0091	0.0001 0.0054 0.0055 0.0157	0.0038 0.0075 0.0056 0.0069
Frechet		05	0.0488 0.3418 0.0439 1.0031	0.1515 0.4418 0.1193 0.8031	0.1978 0.6305 0.1201 0.9035
		10	0.0485 0.3282 0.0472 0.9886	0.1512 0.4282 0.1191 0.7886	0.1899 0.6233 0.1195 0.8846
		20	0.0556 0.2775 0.0467 0.9320	0.1444 0.3775 0.1185 0.7320	0.1740 0.5772 0.1193 0.8254
	$\lambda = 0.1$	50	0.0830 0.1931 0.0453 0.8610	0.1170 0.2931 0.1183 0.0610	0.1355 0.2597 0.1189 0.1302
		100	0.0973 0.1332 0.0384 0.8298	0.1027 0.2332 0.1175 0.0435	0.0227 0.0289 0.0188 0.0141
	$\beta = 0.1$	150	0.1020 0.1129 0.0352 0.8287	0.0980 0.2129 0.1174 0.0423	0.0205 0.0269 0.0182 0.0113
		200	0.1054 0.1054 0.0334 0.8234	0.0946 0.2024 0.1166 0.0402	0.0191 0.0225 0.0179 0.0104
	$\theta = 0.3$	250	0.1069 0.1046 0.0326 0.8244	0.0935 0.1976 0.1148 0.0368	0.0177 0.0138 0.0176 0.0094
		300	0.1084 0.1036 0.0325 0.8352	0.0916 0.2036 0.1116 0.0352	0.0154 0.0131 0.0167 0.0083
	$\lambda = 0.8$	350	0.1102 0.1024 0.0317 0.8068	0.0898 0.2024 0.1061 0.0298	0.0124 0.0110 0.0160 0.0074
		400	0.1041 0.1016 0.0315 0.8012	0.0889 0.2036 0.1047 0.0287	0.0113 0.0102 0.0153 0.0066
		450	0.1021 0.1014 0.0309 0.8003	0.0879 0.2037 0.1033 0.0244	0.0100 0.0099 0.0137 0.0053
		500	0.1006 0.1007 0.0307 0.8005	0.0874 0.2034 0.1028 0.0234	0.0092 0.0091 0.0127 0.0029

TABLE 2. The Descriptive statistics of obesity among children and adolescents data set to 2 decimal points

Mean	Median	σ	IQR	Variance	Kurtosis	Skewness	25%	75%	99%
19.67	19.50	6.30	9.10	39.66	-1.12	-0.29	15.33	24.43	28.95

We observed from Table 2 that the a negative kurtosis and skewness were obtained. This implies that the distribution of the obesity data is flatter than a normal curve with the same mean and standard deviation. Hence, the data are left skewed.

Table 3 shows the test statistics of the goodness-of-fit measure of comparison adopted for comprehensive comparison.

Table 3: The goodness-of-fit measure of obesity among children and adolescents data set (standard errors in parentheses)

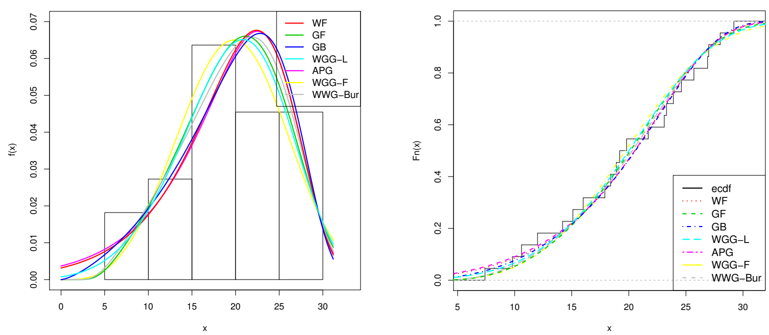
Distribution	p-value	AIC	KS	W	A	Estimates
WGG-B	0.9997	97.9173	0.0219	0.0012	0.0528	$\hat{\lambda} = 0.2201(0.1022)$
						$\hat{\beta} = 1.2315(0.0898)$
						$\hat{\rho} = 0.0872(0.0098)$
						$\hat{\theta} = 0.2124(0.0252)$
WGG-L	0.9390	102.3145	0.1071	0.0278	0.2021	$\hat{\lambda} = 0.0075(0.0020)$
						$\hat{\beta} = 1.2912(0.0125)$
						$\hat{\rho} = 0.1142(0.0967)$
						$\hat{\theta} = 0.8494(0.3254)$
WGG-F	0.9324	109.8906	0.2095	0.0452	0.3183	$\hat{\lambda} = 0.0022(0.0004)$
						$\hat{\beta} = 2.4857(0.8351)$
						$\hat{\rho} = 0.8792(0.2743)$
						$\hat{\theta} = 1.0516(0.2778)$
KB	0.7640	153.2259	0.1356	0.0811	0.5349	$\hat{\alpha} = 33.4661(17.9125)$
						$\hat{\beta} = 47.4488(46.2083)$
						$\hat{\rho} = 0.0331(1.8429)$
						$\hat{\theta} = 21.8947(7.1942)$
KL	0.6961	154.7281	0.1443	0.1006	0.6475	$\hat{\alpha} = 14.5201(14.8943)$
						$\hat{\beta} = 1.3267(1.7158)$
						$\hat{\rho} = 0.0079(0.0030)$
						$\hat{\theta} = 20.3753(16.9103)$
KF	0.7788	152.0259	0.1336	0.0642	0.4347	$\hat{\alpha} = 5.1639(6.2917)$
						$\hat{\beta} = 166.4803(246.7566)$
						$\hat{\rho} = 0.6187(0.1771)$
						$\hat{\theta} = 20.9740(34.9274)$
KW	0.6844	147.3603	0.0915	0.0194	0.1346	$\hat{\alpha} = 0.2099(0.2886)$
						$\hat{\beta} = 1.1818(1.3944)$
						$\hat{\rho} = 0.0356(0.0084)$
						$\hat{\theta} = 12.5159(17.5384)$
APG	0.6866	147.2333	0.0901	0.0296	0.2029	$\hat{\alpha} = 1.8905(2.7086)$
						$\hat{\beta} = 0.0051(0.0037)$
						$\hat{\rho} = 0.1627(0.0287)$
GB	0.2511	156.8906	0.2095	0.0452	0.3183	$\hat{\alpha} = 0.0022(0.0004)$
						$\hat{\beta} = 2.4857(0.8351)$

Table 3 – *Continued from previous page*

Distribution	p-value	AIC	KS	W	A	Estimates
						$\hat{\rho} = 0.8792(0.2743)$
						$\hat{\theta} = 1.0516(0.2778)$
GF	0.5457	148.5009	0.1055	0.0249	0.1808	$\hat{\alpha} = 0.7959(3.4859)$
						$\hat{\beta} = 6.7388(13.2331)$
						$\hat{\rho} = 1.0350(0.8096)$
						$\hat{\theta} = 27.6166(54.6914)$
GL	0.5390	149.3145	0.1071	0.0278	0.2021	$\hat{\alpha} = 0.0075(0.0080)$
						$\hat{\beta} = 6.2912(4.8123)$
						$\hat{\rho} = 0.1142(0.0967)$
						$\hat{\theta} = 0.8494(0.7254)$
WL	0.4568	146.4643	0.1024	0.0245	0.1779	$\hat{\alpha} = 1.0199(2.1532)$
						$\hat{\beta} = 7.5745(3.4940)$
						$\hat{\rho} = 31.1218(20.4511)$
						$\hat{\theta} = 5.2402(0.7263)$
WF	0.8703	149.7794	0.1203	0.0348	0.2507	$\hat{\alpha} = 0.0413(0.2325)$
						$\hat{\beta} = 7.7834(2.0028)$
						$\hat{\rho} = 7.7834(8.9362)$
						$\hat{\theta} = 3.4925(4.9272)$
WB	0.7543	149.9324	0.1228	0.0371	0.2652	$\hat{\alpha} = 0.0073(0.0079)$
						$\hat{\beta} = 6.9216(3.2352)$
						$\hat{\rho} = 0.3107(0.5863)$
						$\hat{\theta} = 1.1514(2.4617)$
GE	0.4868	147.0767	0.0900	0.0284	0.1958	$\hat{\alpha} = 0.0093(0.0111)$
						$\hat{\beta} = 0.5355(0.6937)$
						$\hat{\rho} = 0.3373(0.4140)$
GW	0.6824	148.0124	0.0926	0.0226	0.1569	$\hat{\alpha} = 0.0335(0.1008)$
						$\hat{\beta} = 0.0745(0.1903)$
						$\hat{\rho} = 0.1381(0.1830)$
						$\hat{\theta} = 2.4173(0.3935)$
TF	0.5892	149.5043	0.0883	0.0293	0.2053	$\hat{\alpha} = 0.0086(0.0091)$
						$\hat{\beta} = 0.3939(2.0093)$
						$\hat{\rho} = 0.6124(1.6391)$
						$\hat{\theta} = -0.0118(0.0251)$

Figure 5 shows the empirical histogram and cdfs of the obesity real-life data applications.

9.2.2. *Precipitations in Karachi city, Pakistan Data.* The second data examined comprises 59 annual maximum precipitations in Karachi city, Pakistan, for the



(A) Empirical densities

(B) Empirical cdfs

FIGURE 4. The Empirical densities and cdfs of obesity among children and adolescents data set

years 1950-2009 as used in [6]. The precipitation records help water management studies and flood defense systems to predict floods and droughts. The precipitation data also help to minimize the risk of large hydraulic structures. The values of the data are:

11.8, 6.5, 54.9, 39.9, 16.8, 30.2, 38.4, 76.9, 73.4, 117.6, 157.7, 148.6, 11.4, 5.6, 63.6, 62.4, 85, 256.3, 24.9, 148.6, 160.5, 131.3, 77, 155.2, 217.2, 105.5, 166.8, 157.9, 73.6, 291.4, , 30, 270.4, 160, 96.3, 185.7, 429.3, 184.9, 262.5, 80.6, 138.2, 28, 39.3, 210.3, 315.7, 107.7, 33.3, 302.6, 159.1, 78.7, 33.2, 52.2, 92.7,150.4, 43.7, 68.3, 20.8, 179.4, 245.7, 19.5.

The descriptive statistics of the data are given in Table 4.

TABLE 4. The Descriptive statistics of annual maximum precipitations in Karachi city, Pakistan data set to 2 decimal points

Mean	Median	σ	IQR	Variance	Kurtosis	Skewness	25%	75%	99%
118.40	92.70	93.21	120.65	8688.99	0.64	0.99	39.60	160.25	363.41

We observed from Table 4 that the a positive kurtosis and skewness indicated that distribution is peaked and possesses thick tails, and most values are clustered around the left tail of the distribution while the right tail of the distribution is longer.

Table 5: The goodness-of-fit measure of maximum precipitations in Karachi city, Pakistan data set (standard errors in parentheses)

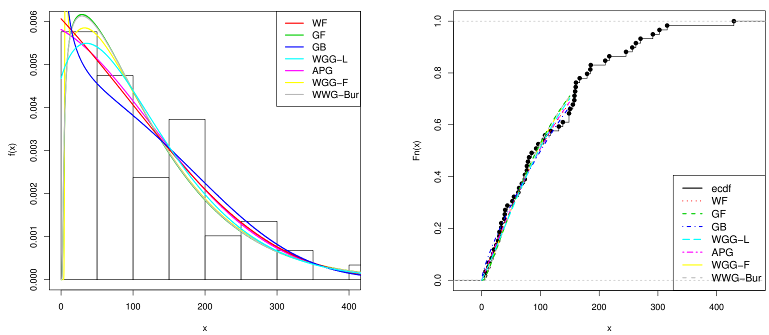
Distribution	p-value	AIC	KS	W	A	Estimates
WGG-B	0.9470	383.7050	0.0961	0.0454	0.2577	$\hat{\lambda} = 0.0951(0.1763)$
						$\hat{\beta} = 1.2401(0.1875)$
						$\hat{\rho} = 1.7186(0.2943)$
						$\hat{\theta} = 1.9784(0.6677)$
WGG-L	0.9376	391.7440	0.0968	0.0461	0.2616	$\hat{\lambda} = 0.1689(0.0778)$
						$\hat{\beta} = 1.2239(0.0682)$
						$\hat{\rho} = 1.0199(0.0332)$
						$\hat{\theta} = 1.5536(0.0941)$
WGG-F	0.8617	401.1031	0.0989	0.0688	0.3093	$\hat{\lambda} = 1.4858(0.4944)$
						$\hat{\beta} = 1.2023(0.7353)$
						$\hat{\rho} = 1.1434(0.1052)$
						$\hat{\theta} = 1.3496(0.9372)$
KB	0.2911	691.8905	0.1276	0.1372	0.8463	$\hat{\alpha} = 8.3342(2.2157)$
						$\hat{\beta} = 56.1819(92.7683)$
						$\hat{\rho} = 0.0182(0.0000)$
						$\hat{\theta} = 11.1408(1.0780)$
KL	0.4207	687.9069	0.1145	0.0848	0.4997	$\hat{\alpha} = 1.7166(0.2951)$
						$\hat{\beta} = 3.3847(2.8572)$
						$\hat{\rho} = 0.0040(0.0010)$
						$\hat{\theta} = 1.5341(1.0421)$
KF	0.3786	687.6918	0.1185	0.0883	0.5257	$\hat{\alpha} = 6.8464(2.1692)$
						$\hat{\beta} = 161.821(229.22)$
						$\hat{\rho} = 0.2188(0.0564)$
						$\hat{\theta} = 30.025(31.898)$
KW	0.7467	684.7171	0.0883	0.0467	0.2692	$\hat{\alpha} = 0.8755(0.4893)$
						$\hat{\beta} = 0.5662(0.6176)$
						$\hat{\rho} = 0.0112(0.0098)$
						$\hat{\theta} = 1.3454(0.3905)$
APG	0.8959	682.9092	0.0748	0.0438	0.2641	$\hat{\alpha} = 1.5772(2.1911)$
						$\hat{\beta} = 0.0073(0.0040)$
						$\hat{\rho} = 0.0023(0.0022)$
GB	0.6326	684.8519	0.0972	0.0491	0.2803	$\hat{\alpha} = 0.0075(0.0045)$
						$\hat{\beta} = 2.7856(1.9958)$
						$\hat{\rho} = 0.3543(0.3103)$
						$\hat{\theta} = 1.2401(0.9676)$

Table 5 – *Continued from previous page*

Distribution	p-value	AIC	KS	W	A	Estimates
GF	0.6774	683.6124	0.0937	0.0435	0.2460	$\hat{\alpha} = 0.1587(0.4962)$
						$\hat{\beta} = 1.8235(2.5702)$
						$\hat{\rho} = 0.7248(0.7488)$
						$\hat{\theta} = 22.8034(61.9942)$
GL	0.7704	685.1274	0.0864	0.0488	0.2849	$\hat{\alpha} = 0.1380(2.0119)$
						$\hat{\beta} = 1.7962(38.1464)$
						$\hat{\rho} = 0.0437(0.3829)$
						$\hat{\theta} = 0.7748(16.8750)$
WF	0.7042	681.9136	0.0916	0.0434	0.2462	$\hat{\alpha} = 0.0358(0.0180)$
						$\hat{\beta} = 0.2947(0.1467)$
						$\hat{\rho} = 4.1927(2.0935)$
						$\hat{\theta} = 8.6209(1.7749)$
WB	0.5757	684.6917	0.1016	0.0519	0.2954	$\hat{\alpha} = 0.0073(0.0049)$
						$\hat{\beta} = 2.4377(1.0282)$
						$\hat{\rho} = 0.4476(0.4885)$
						$\hat{\theta} = 0.9901(1.3096)$
WL	0.1985	751.7122	0.1398	0.0730	0.4267	$\hat{\alpha} = 3.6920(0.7601)$
						$\hat{\beta} = 0.0923(0.0253)$
						$\hat{\rho} = 0.6424(0.0600)$
						$\hat{\theta} = 0.1421(0.5247)$
GE	0.3220	682.9042	0.0717	0.0420	0.2562	$\hat{\alpha} = 0.0857(0.0178)$
						$\hat{\beta} = 0.0438(0.0473)$
						$\hat{\rho} = 0.0707(0.6964)$
GW	0.4824	687.6262	0.0604	0.0582	0.3764	$\hat{\alpha} = 0.0341(0.0082)$
						$\hat{\beta} = 0.0787(0.0160)$
						$\hat{\rho} = 0.3342(0.0000)$
						$\hat{\theta} = 0.7105(0.0072)$
TF	0.3617	701.1031	0.1201	0.2688	1.6393	$\hat{\alpha} = 28.4858(29.4944)$
						$\hat{\beta} = 31.2023(13.7353)$
						$\hat{\rho} = 1.1434(0.1052)$
						$\hat{\theta} = 0.9372(4.2815)$

Figure 6 shows the empirical histogram and cdfs of the obesity real-life data applications.

9.3. **Discussion.** In Tables 3 and 5, we observed that the p-values of the WGG generated models are the highest with the lowest AIC test statistic in Burr XII, Lomax, and Frechet sub-models. Hence, the WGG model has provided a better alternative to making statistical distributions more flexible, and viable compared



(A) Empirical densities

(B) Empirical cdfs

FIGURE 5. The Empirical densities and cdfs of maximum precipitations in Karachi city, Pakistan data set

to the model generated by Gompertz, Weibull, Kumaraswamy, and Alpha power models.

10. Conclusion

Intuitively, a two-parameter weighted Gompertz-G generated distribution was examined and introduced by making use of a weighted Gompertz and the T-X characterizations. The newly developed model has found its uses in cases where two-sided abrupt changes schemes occurred in applications. The WGG model has provided a better alternative to making statistical distributions more flexible, and viable compared to the model generated by Gompertz, Weibull, Kumaraswamy, and Alpha power models. The statistical properties and estimations of the model parameters were obtained. The viability and flexibility of the WGG-generated model were demonstrated by illustration of a simulation and real data sets using their goodness-of-fit statistics. The outcomes of the WGG-generated test statistics indicated a better viable, tractable, flexible, and parsimonious generator compared to some competitive models in the literature. Hence, it can be used as a better alternative in reliability theory and extreme value theory.

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References

[1] Abd-AL-Motalib, R. S., Abed AL-Kadim, K., The odd truncated inverse exponential Weibull exponential distribution, *Journal of Positive School Psychology* , 6(2) (2022), 5361-5375.

[2] Aljarrah, M. A., Lee, C., Famoye, F., On generating T-X family of distributions using quantile functions, *Journal of Statistical Distributions and Applications*, 1(1) (2014), 1-17. https://doi.org/10.1186/2195-5832-1-2

[3] Alizadeh, M., Altun, E., Ozel, G., Afshari, M., Eftekharian, A., A new odd log-logistic Lindley distribution with properties and applications, *Sankhya A*, 81 (2019), 323-346. https://doi.org/10.1007/s13171-018-0142-x

[4] Alzaatreh, A., Famoye, F., Lee, C., A new method for generating families of continuous distributions, *Metron*, 71 (2013), 63-79. https://doi.org/10.1007/s40300-013-0007-y

[5] Alzaghal, A., Famoye, F., Lee, C., Exponentiated T-X family of distributions with some applications, *International Journal of Statistics and Probability*, 2(3) (2013), 31–49. https://doi.org/10.5539/ijsp.v2n3p31

[6] Arshad, M. Z., Iqbal, M. Z., Mutairi, A. A., A comprehensive review of datasets for statistical research in probability and quality control, *Journal of Mathematical and Computational Science*, 11(3) (2021), 3663-3728. https://doi.org/10.28919/jmcs/5692

[7] Bakouch, S. H., Ahmed, M. T., El-Bar, A., A new weighted Gompertz distribution with applications to reliability data, *Applications of Mathematics*, 3 (2017), 269–296. https://doi.org/10.21136/AM.2017.0277-16

[8] Bemmaor, A. C., Modelling the Diffusion of New Durable Goods: Word-of-mouth Effect Versus Consumer Heterogeneity: In G. Laurent, G.L. Lilien, B. Pras eds., Research 98Traditions in Marketing. Boston, Kluwer Academic Publishers, 1, 201-229, 1992. https://doi.org/10.1007/978-94-011-1402-8-6

[9] CDC, Obesity among children and adolescelent between the aged 2- 19 years by selected characteristics, https://www.cdc.gov/nchs/hus/contents.htm-Table-027, (2020).

[10] Cordeiro, G. M., Alizadeh, M., Diniz Marinho, P. R., The type I half-logistic family of distributions, *Journal of Statistical Computation and Simulation*, 86(4) (2016), 707-728. https://doi.org/10.1080/00949655.2015.1031233

[11] Dijoux, Y., Construction of the tetration distribution based on the continuous iteration of the exponential-minus-one function, *Applied Stochastic Models in Business and Industry*, (2020), 1-26. https://doi.org/10.1002/asmb.2538

[12] Eghwerido, J. T., The alpha power Teissier distribution and its applications, *Afr. Stat.*, 16(2) (2021), 2731-2745. http://dx.doi.org/10.16929/as/2021.2731.181

[13] Eghwerido, J. T., Agu, F. I., The shifted Gompertz-G family of distributions: properties, and applications, *Mathematica Slovaca*, 71(5) (2021a), 1291-1308. http://dx.doi.org/10.1515/ms-2021-0053

[14] Eghwerido, J. T., Nzei, L. C, Agu, F. I., Alpha power Gompertz distribution: properties and applications, *Sankhya A - The Indian Journal of Statistics*, 83(1) (2021b), 449-475. http://dx.doi.org/10.1007/s13171-020-00198-0

[15] Eghwerido, J. T., Oguntunde, P. E., Agu, F. I., The alpha power Marshall-Olkin-G family of distributions: properties, and applications, *Sankhya A*, 85 (2023), 172-197. http://dx.doi.org/10.1007/s13171-020.00235-y

[16] Eghwerido, J. T., The Marshall-Olkin Teissier generated model for lifetime data, *Journal of the Belarusian State University: Mathematics and Informatics*, 1 (2022), 46-65. https://doi.org/10.33581/2520-6508-2022-1-46-65

[17] Eghwerido, J. T., Agu, F. I., The statistical properties and applications of the alpha power Topp-Leone-G distribution, *Heliyon*, 8(6) (2020a), 1-10. https://doi.org/10.1016/j.heliyon.2022.e09775

[18] Eghwerido, J. T., Agu, F. I., The alpha power Muth-G distribution and its applications in survival and reliability analyses, *Mathematica Slovaca*, 73(6) (2023), 1-18.

[19] Hassana, A. S., Shawkia, A. W., Muhammeda, A. H., Weighted Weibull-G family of distributions: theory and application in the analysis of renewable energy sources,*Journal of Positive School Psychology*, 6(3) (2022), 9201-9216.

[20] Jamal, F., Chesneau, C., Saboor, A., Aslam, M., Tahir, M. H., Mashwani, W. K., The U family of distributions: properties and applications, *Mathematica Slovaca*, 72(1) (2022), 217-240. https://doi.org/10.1515/ms-2022-0015

[21] Lin, G. D., Dou, X., Kuriki, S., The bivariate lack-of-memory distributions, *Sankhya A*, 81 (2019), 273-297. https://doi.org/10.1007/s13171-017-0119-1

[22] Marshall, A., Olkin, I., Life Distributions. Structure of Nonparametric, Semiparametric, and Parametric Families. Springer Series in Statistics, Springer, New York, 2007.

[23] Nzei, L. C., Eghwerido, J. T., Ekhosuehi, N., Topp-Leone Gompertz distribution: properties and application, *Journal of Data Science*, 18(4) (2020), 782-794. https://doi.org/10.6339/JDS.202010-18(4)-0012

[24] Osatohanmwen, P., Oyegue, F. O., Ogbonmwan, S. M., A new member from the T-X family of distributions: the Gumbel-Burr XII distribution and its properties, *Sankhya A*, 81 (2019), 298-322. https://doi.org/10.1007/s13171-017-0110-x

[25] Osatohanmwen, P., Efe-Eyefia, E., Oyegue, F. O., Osemwenkhae, J. E., Ogbonmwan, S. M., Afere, B. A., The exponentiated Gumbel-Weibull(logistic) distributions with application to Nigeria's COVID-19 infections data, *Annal of Data Science*, 9 (2022), 909-943. https://doi.org/10.1007/s40745-022-00373-0

[26] Peng, X., Yan, Z., Estimation and application for a new extended Weibull distribution, *Reliab. Eng. Syst. Saf.*, 121 (2014), 34-42. https://doi.org/10.1016/j.ress.2013.07.007

[27] Pollard, J. H., Valkovics, E. J., The Gompertz distribution and its applications, *Genus*, 48 (1992), 15-28. https://doi.org/10.2307/29789100

[28] Rastogi, M. K., Merovci, F., Bayesian estimation for parameters and reliability characteristic of the Weibull Rayleigh distribution, *Journal of King Saud University-Science*, 30 (2018), 472-478. https://doi.org/10.1016/j.jksus.2017.05.008

[29] Shama,M. S., Dey, S., Altun, E., Afify, A. Z., The gamma–Gompertz distribution: theory and applications, *Mathematics and Computers in Simulation*, 193 (2022), 689-712. https://doi.org/10.1016/j.matcom.2021.10.024

[30] Zografos, K., Balakrishnan, N., On families of beta and generalized gamma-generated distributions and associated inference, *Statistical Methodology*, 6(4) (2009), 344–362. https://doi.org/10.1016/j.stamet.2008.12.003