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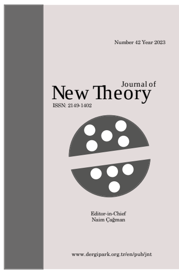

Generalized Cross Product in $(2 + s)$-Dimensional Framed Metric Manifolds with Application to Legendre Curves

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Abstract — This study generalizes the cross product defined in 3-dimensional almost contact metric manifolds and describes a new generalized cross product for $n = 1$ in $(2n + s)$-dimensional framed metric manifolds. Moreover, it studies some of the proposed product's basic properties. It also performs characterizations of the curvature of a Legendre curve on an S-manifold and calculates the curvature of a Legendre curve. Furthermore, it shows that Legendre curves are also biharmonic curves. Next, this study observes that a Legendre curve of osculating order 5 on S-manifolds is imbedded in the 3-dimensional K-contact space. Lastly, the current paper discusses the need for further research.

Keywords *Generalized cross product, S-manifolds, Legendre curves*

Mathematics Subject Classification (2020) 53D10, 53A04

1. Introduction

Yano [1] has led up to the groundwork of S-manifolds and has defined the concept of f-structures in M^{2n+s} manifolds. Almost complex $(s = 0)$ and almost contact $(s = 1)$ structures are examples of f-structures. Goldberg and Yano [2] have defined the concept of the framed f-structures. Moreover, they have suggested a complex structure by the concept of f-structures. Furthermore, they have proposed the concept of framed metric manifolds by examining the normality condition of a metric framed structure. Blair [3] has introduced S-manifolds, generalizing almost complex Kaehler and almost contact Sasakian structures. Sarkar et al. [4] have found the curvature and torsion of Legendre curves in 3-dimensional trans-Sasakian manifolds with respect to the semisymmetric metric connection. Özgür and Güvenç [5] have propounded biharmonic Legendre curves in S-space forms. They have analyzed characterizations of the curvature of the biharmonic Legendre curves in 4 cases.

This paper is organized as follows: Section 2 provides the concept of S-manifolds and some of their basic properties. Section 3 generalizes the new cross-product in a 3-dimensional almost contact metric manifolds defined by Camcı [6] and defines a generalized cross-product in $(2 + s)$-dimensional S-manifolds. Besides, it demonstrates that this cross-product in $\mathbb{R}^4$ is coherent with a triple product [7] by an example. In addition, Section 3 provides the basic properties of the generalized cross-product. Section 4 calculates the curvature of Legendre curves using the generalized cross-product and demonstrated that $(2 + 3)$-dimensional S-manifolds are imbedded in 3-dimensional space. Finally, the need for further research is discussed.

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2. Preliminaries

This section provides the concept of S-manifolds and some of their basic properties.

Definition 2.1. [1] Let (M,g) be a Riemannian manifold with $\dim M = 2n+s$. It is said to be a framed f-structure if f is a tensor field of type $(1,1)$ and rank $2n$ satisfying $f^3+f=0$.

Definition 2.2. [2] Let M^{2n+s} be a manifold with an f-structure of rank $2n$. Then, f-structure is said to be has completed frames if there exists vector fields $\xi_1,\xi_2,\cdots,\xi_s$ on M^{2n+s}, and $\eta_1,\eta_2,\cdots,\eta_s$ are 1-forms, then $\eta_i\circ f=0$, $f\circ\xi_i=0$, and $f^2=-I+\sum_{i=1}^{s}\eta_i\otimes\xi_i$.

Definition 2.3. [8] Let M^{2n+s} be an f-structure with completed frames. The framed f-structure is normal if the tensor field S of type (1,2) given by

$$S=[f,f]+2\sum_{i=1}^{s}d\eta_i\otimes\xi_i$$

vanishes.

Definition 2.4. [2] Let M^{2n+s} be a manifold. Then, M is said to be an S-manifold if the f-structure is normal.

Definition 2.5. [1] Let M^{2n+s} be a Riemannian manifold. The distribution on M spanned by the structure vector fields is denoted by $\mathcal{M}$, and its complementary orthogonal distribution is denoted by D. Consequently, $TM=D\oplus\mathcal{M}$. Moreover, if $X\in D$, then $\eta_i(X)=0$, for any $i\in\{1,2,\cdots,s\}$, and if $X\in\mathcal{M}$, then $fX=0$.

Definition 2.6. [2] Let M^{2n+s} has an f-structure with completed frames. If there exists a Riemannian metric g on M^{2n+s} such that

$$g(X,Y)=g(fX,fY)+\sum_{i=1}^{s}\eta_i(X)\eta_i(Y)$$

and $X,Y\in\chi(M^{2n+s})$, then M^{2n+s} is called that has a metric f-structure.

From now on, the notation ϕ is used instead of f.

Definition 2.7. [3] Let M^{2n+s} be an S-manifold. The covariant differentiation ∇ of M^{2n+s} satisfies

$$\nabla_X\xi_i=-\phi X,\quad i\in\{1,2,\cdots,s\}$$

and

$$(\nabla_X\phi)Y=\sum_{i=1}^{s}[g(\phi X,\phi Y)\xi_i+\eta_i(Y)\phi^2X]$$

for all $X,Y\in\chi(M)$. Besides, for all $i\in\{1,2,\cdots,s\}$, $\eta_1\wedge\eta_2\wedge\cdots\wedge\eta_s\wedge(d\eta_i)^n\neq0$ and $\Phi=d\eta_i$ on an S-manifold such that Φ is the fundamental 2-form defined by

$$\Phi(X,Y)=g(X,\phi Y),\quad X,Y\in TM$$

Definition 2.8. [9] A submanifold of an S-manifold is called an integral submanifold if

$$\eta_i(X)=0,\quad i\in\{1,2,\cdots,s\},\ \text{for all } X\in\chi(M)$$

Definition 2.9. [5] A 1-dimensional integral submanifold of an S-manifold $\left(M^{2n+s},\phi,\xi_i,\eta_j,g\right)$, $i,j\in\{1,2,\cdots,s\}$ is called a Legendre curve of M. In other words, a curve $\gamma:I\to M$ is called a Legendre curve if $\eta_i(T)=0$, for all $i\in\{1,2,\cdots,s\}$ such that T is the tangent vector field of γ.

3. Generalized Cross product in $(2+s)$-dimensional Framed Metric Manifolds

This section, firstly, generalizes the cross-product in a 3-dimensional almost contact metric manifolds defined by Camcı [5] and defines a generalized cross-product in $(2+s)$-dimensional S-manifolds.

Definition 3.1. Let A be a matrix of type $s \times (s+1)$. Then,

i. $\tilde{A}_{ijk}$ such that $i,k \in \{1,2,\cdots,s\}$ and $j \in \{i+1,i+2,\cdots,s+1\}$ is the matrix obtained by deleting the i^{th} and j^{th} columns and k^{th} row of the matrix A. Specially, for $s=1$, $\det \tilde{A}_{121}=1$.
ii. $\tilde{\tilde{A}}_{mn}$ such that $m \in \{2,3,\cdots,s\}$ and $n \in \{m+1,m+2,\cdots,s+1\}$ is the matrix obtained by deleting the i^{th} and j^{th} columns of the matrix A and adding the first column of the matrix A to the left of the first column as a new column.

For example, for $s=3$, let

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}.$$

Then,

$$\tilde{A}_{231}=\begin{bmatrix} a_{21} & a_{24} \\ a_{31} & a_{34} \end{bmatrix}$$

and

$$\tilde{\tilde{A}}_{23}=\begin{bmatrix} a_{11} & a_{11} & a_{14} \\ a_{21} & a_{21} & a_{24} \\ a_{31} & a_{31} & a_{34} \end{bmatrix}$$

Definition 3.2. Let $M=\left(M^{2+s},\phi,\xi_i,\eta_j,g\right)$, $i,j \in \{1,2,\cdots,s\}$, be a framed metric manifold in $(2+s)$-dimensional space. We define a generalized cross-product $\times$ by

$$X_1 \times X_2 \times \cdots \times X_{s+1} = \sum_{k=1}^{s}\left(\sum_{i=1}^{s}\sum_{j=i+1}^{s+1}(-1)^{i+j+k}\Phi\left(X_i,X_j\right)\det \tilde{A}_{ijk}\right)\xi_k + \begin{vmatrix} \phi X_1 & \phi X_2 & \cdots & \phi X_{s+1} \\ \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1}) \end{vmatrix} \tag{1}$$

such that $X_1,X_2,\cdots,X_{s+1} \in TM$ and

$$A=\begin{bmatrix} \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1}) \\ \eta_2(X_1) & \eta_2(X_2) & \cdots & \eta_2(X_{s+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1}) \end{bmatrix}$$

Moreover, $\tilde{A}_{ijk}$, $i,k \in \{1,2,\cdots,s\}$ and $j \in \{i+1,i+2,\cdots,s+1\}$, is as in Definition 3.1. For example, for $s=1$, we obtain

$$X_1 \times X_2 = \Phi(X_1,X_2)\xi_1 + \eta_1(X_2)\phi X_1 - \eta_1(X_1)\phi X_2 \tag{2}$$

Equation 2 gives the cross product in 3-dimensional almost contact metric manifolds defined in [6]. For $s=2$, the generalized cross-product is

$$X_1 \times X_2 \times X_3 = \big(\Phi(X_1,X_2)\eta_2(X_3) - \Phi(X_1,X_3)\eta_2(X_2) + \Phi(X_2,X_3)\eta_2(X_1)\big)\xi_1$$
$$+\big(-\Phi(X_1,X_2)\eta_1(X_3) + \Phi(X_1,X_3)\eta_1(X_2) - \Phi(X_2,X_3)\eta_1(X_1)\big)\xi_2$$
$$+\begin{vmatrix} \phi X_1 & \phi X_2 & \phi X_3 \\ \eta_1(X_1) & \eta_1(X_2) & \eta_1(X_3) \\ \eta_2(X_1) & \eta_2(X_2) & \eta_2(X_3) \end{vmatrix}$$

Secondly, it demonstrates that this cross-product in $\mathbb{R}^4$ is coherent with a triple product [7] by an example.

Example 3.3. For the subspace $V = \{(x_1, x_2, 0,0) \mid x_1, x_2 \in \mathbb{R}\}$ of the 4-dimensional Euclidean space $\mathbb{R}^4(x_1,x_2,x_3,x_4)$, the projection function $\pi(x_1,x_2,x_3,x_4) = (x_1,x_2,0,0)$, and the almost complex plane $J(x_1,x_2,x_3,x_4) = (x_2,-x_1,-x_4,x_3)$, $\big(\mathbb{R}^4(x_1,x_2,x_3,x_4), \phi, \xi_i, \eta_j, g\big)$, $i,j \in \{1,2\}$, is a framed metric manifold such that $\phi = J \circ \pi$, $\eta_1 = dx_3$, $\eta_2 = dx_4$, $\xi_1 = (0,0,1,0)$, $\xi_2 = (0,0,0,1)$, and g is the standard Euclidean metric. As a result, $X_1 \times X_2 \times X_3 = X_1 \wedge X_2 \wedge X_3$ such that $X_1 \wedge X_2 \wedge X_3$ is the triple product in $\mathbb{R}^4$ provided in [7].

Finally, this section provides some of the basic properties of the generalized cross-product.

Theorem 3.4. Let $M = \big(M^{2+s}, \phi, \xi_i, \eta_j, g\big)$, $i,j \in \{1,2,\cdots,s\}$ be a framed metric manifold in $(2+s)$-dimensional space. Then, for all $X_1, X_2, \cdots, X_{s+1} \in TM$, the generalized cross-product has the following properties:

i. The generalized cross-product is bilinear and antisymmetric.
ii. $X_1 \times X_2 \times \cdots \times X_{s+1}$ is perpendicular to each of $X_1, X_2, \cdots, X_{s+1}$.
iii. $\phi X = \xi_1 \times \xi_2 \times \cdots \times \xi_s \times X$.

Proof.

i. The proof is straightforward from the fundamental 2-form Φ and the determinant function's bilinearity.

ii. We need to show that $g(X_1 \times X_2 \times \cdots \times X_{s+1}, X_t) = 0$, for $t \in \{1,2,\cdots,s+1\}$. For $t = 1$, from Equation 1, we obtain

$$g(X_1 \times X_2 \times \cdots \times X_{s+1}, X_1) = \sum_{k=1}^{s}\left(\sum_{i=1}^{s}\sum_{j=i+1}^{s+1}(-1)^{i+j+k}\Phi(X_i,X_j)\det\tilde{A}_{ijk}\right)\eta_k(X_1) + \begin{vmatrix} 0 & g(\phi X_2, X_1) & \cdots & g(\phi X_{s+1}, X_1) \\ \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1}) \end{vmatrix} \tag{3}$$

Thus,

$$\begin{vmatrix} 0 & g(\phi X_2, X_1) & \cdots & g(\phi X_{s+1}, X_1) \\ \eta_1(X_1) & \eta_1(X_2) & \cdots & \eta_1(X_{s+1}) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_s(X_1) & \eta_s(X_2) & \cdots & \eta_s(X_{s+1}) \end{vmatrix} = \sum_{k=1}^{s}\left(\sum_{j=2}^{s+1}(-1)^{j+k}\Phi(X_1,X_j)\det\tilde{A}_{1jk}\right)\eta_k(X_1) \tag{4}$$

If we substitute Equation 4 for Equation 3, we get

$$g(X_1 \times X_2 \times X_3 \times \cdots \times X_{s+1}, X_1) = \sum_{i=2}^{s}\sum_{j=i+1}^{s+1}(-1)^{i+j}\Phi(X_i,X_j)\det\tilde{\tilde{A}}_{ij}$$

Here, the $\tilde{\tilde{A}}_{ij}$, $i \in \{2,3,\cdots,s\}$ and $j \in \{i+1, i+2, \cdots, s+1\}$, is as in Definition 3.1. Thus, $\det \tilde{\tilde{A}}_{ij} = 0$. Therefore,

$$g(X_1 \times X_2 \times X_3 \times \cdots \times X_{s+1}, X_1) = 0$$

Similarly, it is proved that $g(X_1 \times X_2 \times X_3 \times \cdots \times X_{s+1}, X_t) = 0$, for $t \in \{2,3,\cdots,s+1\}$.

iii. If $\xi_i = Y_i$, for all $i \in \{1,2,\cdots,s\}$, and $X = Y_{s+1}$, for all $X, \xi_1, \xi_2, \cdots, \xi_s \in TM$, from Equation 1,

$$\xi_1 \times \xi_2 \times \ldots \times \xi_s \times X = Y_1 \times Y_2 \times \ldots \times Y_{s+1} = \sum_{k=1}^{s} \left(\sum_{i=1}^{s} \sum_{j=i+1}^{s+1} (-1)^{i+j+k} \Phi(Y_i, Y_j) \det \tilde{A}_{ijk} \right) \xi_k + \begin{vmatrix} \phi Y_1 & \phi Y_2 & \cdots & \phi Y_{s+1} \\ \eta_1(Y_1) & \eta_1(Y_2) & \cdots & \eta_1(Y_{s+1}) \\ \vdots & \vdots & \vdots & \vdots \\ \eta_s(Y_1) & \eta_s(Y_2) & \cdots & \eta_s(Y_{s+1}) \end{vmatrix} \quad (5)$$

From Equation 5, $\Phi(Y_i, Y_j) = 0$, $i, j \in \{1,2,\cdots,s+1\}$. Then, if we substitute the expressions $\phi Y_i = \phi \xi_i = 0$ and $Y_{s+1} = X$ in Equation 5, we get $\xi_1 \times \xi_2 \times \ldots \times \xi_s \times X = \phi X$. □

4. Legendre Curves in S-manifolds

Let $\gamma: I \to M$ be a unit-speed curve in an n-dimensional Riemannian manifold (M, g) and $k_1, k_2, \cdots, k_r$ be positive functions on I. If there is an orthonormal basis $\{V_1, V_2, \cdots, V_r\}$ along γ that satisfies the following Frenet equations, γ is called a Frenet curve of osculating order r:

$$V_1 = \gamma'$$

$$\nabla_{V_1} V_1 = k_1 V_2$$

$$\nabla_{V_1} V_2 = -k_1 V_1 + k_2 V_3$$

$$\vdots$$

$$\nabla_{V_1} V_r = -k_{r-1} V_{r-1}$$

Theorem 4.1. Let $M = (M^{2+s}, \phi, \xi_i, \eta_j, g)$, $i, j \in \{1,2,\cdots,n\}$ be an S-manifold and $\gamma: I \to M$ be a Legendre curve of osculating order $(2+s)$. Then, for $\varepsilon = \pm 1$, the following equations are obtained:

$$V_2 = \varepsilon \phi V_1$$

$$V_3 = \frac{\varepsilon}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_s$$

$$k_2 = \sqrt{s}$$

and

$$k_3 = 0$$

Proof.

For the function $\sigma_{ij}: I \to \mathbb{R}$ defined by $\sigma_{ij}(s) = g(V_i, \xi_j)$, for $i \in \{1,2,\cdots,s+2\}$ and $j \in \{1,2,\cdots,s\}$,

$$\xi_j = \sum_{i=1}^{s+2} \sigma_{ij} V_i, \quad j \in \{1,2,\cdots,s\} \quad (6)$$

Let γ be a Legendre curve. Then,

$$\sigma_{11} = \sigma_{12} = \cdots = \sigma_{1s} = 0 \tag{7}$$

Moreover, from Theorem 3.4,

$$\phi V_1 = (-1)^{s+1} \begin{vmatrix} V_2 & V_3 & \cdots & V_{s+1} & V_{s+2} \\ \sigma_{21} & \sigma_{31} & \cdots & \sigma_{(s+1)1} & \sigma_{(s+2)1} \\ \sigma_{22} & \sigma_{32} & \cdots & \sigma_{(s+1)2} & \sigma_{(s+2)2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_{2s} & \sigma_{3s} & \cdots & \sigma_{(s+1)s} & \sigma_{(s+2)s} \end{vmatrix} \tag{8}$$

If we take the derivative from $\sigma_{11} = 0 = g(V_1, \xi_1)$, then

$$\sigma_{21} = \sigma_{22} = \sigma_{23} = \cdots = \sigma_{2s} = 0 \tag{9}$$

From Equations 8-10,

$$\phi V_1 = (-1)^{s+1} \begin{vmatrix} \sigma_{31} & \cdots & \sigma_{(s+1)1} & \sigma_{(s+2)1} \\ \sigma_{32} & \cdots & \sigma_{(s+1)2} & \sigma_{(s+2)2} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{3s} & \cdots & \sigma_{(s+1)s} & \sigma_{(s+2)s} \end{vmatrix} V_2 \tag{10}$$

If we substitute Equations 7 and 9 in Equation 6, then

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_s \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{31} & \sigma_{41} & \cdots & \sigma_{(s+2)1} \\ \sigma_{32} & \sigma_{42} & \cdots & \sigma_{(s+2)2} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{3s} & \sigma_{42} & \cdots & \sigma_{(s+2)s} \end{bmatrix}}_{B} \begin{bmatrix} V_3 \\ V_4 \\ \vdots \\ V_{s+2} \end{bmatrix} \tag{11}$$

Since $\{\xi_1, \xi_2, \ldots, \xi_s\}$ and $\{V_3, \ldots, V_{s+2}\}$ are orthonormal systems, B is an orthonormal matrix. In this case, $|B| = \pm 1$ where $|B|$ is the determinant of the matrix B. From Equation 10, for $\varepsilon = \pm 1$,

$$\phi V_1 = \varepsilon V_2 \tag{12}$$

From Equation 12,

$$\phi V_2 = \varepsilon V_1 \tag{13}$$

If we take the derivative from $\sigma_{2i} = g(V_2, \xi_i) = 0$, $i \in \{1,2,\cdots,s\}$,

$$k_2 \sigma_{3k} = g(V_2, \phi V_1), \quad k \in \{1,2,\cdots,s\} \tag{14}$$

From Equation 8,

$$g(V_2, \phi V_1) = 1 \tag{15}$$

Then, from Equations 14 and 15,

$$k_2\sigma_{3k} = 1, \quad k \in \{1,2,\cdots,s\}$$

In this case,

$$\sigma_{3k} = \frac{1}{k_2}, \quad k \in \{1,2,\cdots,s\} \tag{16}$$

Since B is an orthonormal matrix,

$$\sum_{k=1}^{s} (\sigma_{3k})^2 = 1 \tag{17}$$

From Equations 16 and 17, for $\varepsilon = \pm 1$,

$$\sigma_{3k} = \frac{\varepsilon}{\sqrt{s}}, \quad k \in \{1,2,\cdots,s\} \tag{18}$$

and

$$k_2 = \sqrt{s}$$

If we take the derivative from Equation 17, then

$$-k_2 g(V_2, \xi_1) + k_3 g(V_4, \xi_1) + g(V_3, \phi V_1) = 0 \tag{19}$$

From Equations 9 and 12, Equation 19 becomes

$$k_3\sigma_{41} = 0 \tag{20}$$

Similarly,

$$k_3\sigma_{42} = k_3\sigma_{43} = \cdots = k_3\sigma_{4s} = 0 \tag{21}$$

As the matrix B is orthonormal,

$$(\sigma_{41})^2 + (\sigma_{42})^2 + \cdots + (\sigma_{4s})^2 = 1 \tag{22}$$

From Equations 20-22,

$$k_3 = 0$$

Since B is an orthonormal matrix, we can write Equation 11 as follows:

$$\begin{bmatrix} V_3 \\ V_4 \\ \vdots \\ V_{s+2} \end{bmatrix} = B^T \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_s \end{bmatrix} \tag{23}$$

From Equations 11 and 18,

$$B^T = \begin{bmatrix} \frac{\varepsilon}{\sqrt{s}} & \frac{\varepsilon}{\sqrt{s}} & \cdots & \frac{\varepsilon}{\sqrt{s}} \\ \sigma_{41} & \sigma_{42} & \cdots & \sigma_{4s} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{(s+2)1} & \sigma_{(s+2)2} & \cdots & \sigma_{(s+2s,)s} \end{bmatrix} \tag{24}$$

and from Equations 23 and 24,

$$V_3 = \frac{\varepsilon}{\sqrt{s}} \sum_{\alpha=1}^{s} \xi_\alpha$$

is obtained. □

Remark 4.2. In [5], Özgür and Güvenç assumed that $V_2 = \phi V_1$ and obtained the same results as in Theorem 4.1 for biharmonic Legendre curves. Thus, Legendre curves are also biharmonic.

In [10], Hasegawa et al. have introduced $\mathbb{R}^{2+s}(-3s)$ space as follows: Let the coordinate functions' set of $M = \mathbb{R}^{2+s}$ be $\{x, y, z_1, \cdots, z_s\}$. In this space,

$$\xi_i = 2\frac{\partial}{\partial z_i}, \quad i \in \{1,2,\cdots,s\}$$

$$\eta_j = \frac{1}{2}\left(dz_j - y dx\right), \quad j \in \{1,2,\cdots,s\} \tag{25}$$

and

$$g = \frac{1}{4}(dx^2 + dy^2) + \sum_{j=1}^{s} \eta_j \otimes \eta_j$$

with

$$X = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + \sum_{i=1}^{s} \xi_i \frac{\partial}{\partial z_i} \in \chi(M)$$

Thus, $(M = \mathbb{R}^{2+s}, \phi, \xi_i, \eta_j, g)$ is an S-space form with the constant φ-sectional curvature $c = -3s$. Besides, $\{e, \phi e, \xi_1, \xi_2, \cdots, \xi_s\}$ is an orthonormal basis on $\mathbb{R}^{2+s}(-3s)$ such that

$$e_1 = e = 2\frac{\partial}{\partial y}, \qquad e_2 = \phi e = 2\left(\frac{\partial}{\partial x} + y \sum_{i=1}^{s} \xi_i\right), \qquad \text{and} \qquad \xi_i = 2\frac{\partial}{\partial z_i}, \quad i \in \{1,2,\cdots,s\}$$

According to this basis, the Levi-Civita connection is calculated as

$$\nabla_e e = \nabla_{\phi e} \phi e = 0$$

$$\nabla_e \phi e = \sum_{i=1}^{s} \xi_i \tag{26}$$

$$\nabla_{\phi e} e = -\sum_{i=1}^{s} \xi_i$$
$$\nabla_e \xi_i = \nabla_{\xi_i} e = -\phi e \tag{26}$$

and

$$\nabla_{\phi e} \xi_i = \nabla_{\xi_i} \phi e = e$$

We will examine Legendre curves in $\mathbb{R}^{2+s}(-3s)$. Let $\gamma: I \to \mathbb{R}^{2+s}(-3s)$ be a Legendre curve. Let

$$\gamma(t) = \big(x(t), y(t), z_1(t), \dots, z_s(t)\big)$$

such that t is the arc-length parameter. If the tangent vector field of γ is V_1, then $\eta_j(V_1) = 0$, $j \in \{1,2,\cdots,s\}$ since γ is a Legendre curve. From Equation 25,

$$z_1'(t) = z_2'(t) = \cdots = z_s'(t) = y(t)x'(t)$$

If $z_i(t) = f(t)$, then

$$\gamma(t) = (x(t), y(t), f(t) + c_1, f(t) + c_2, \cdots, f(t) + c_s)$$

If the tangent vector field of the curve γ in terms of basis $\{e, \phi e, \xi_1, \xi_2, \dots, \xi_s\}$ is as follows:

$$V_1 = \frac{1}{2}(y'e + x'\phi e)$$

Since γ is a unit speed curve,

$$(x')^2 + (y')^2 = 4$$

Hence, we have the following example:

Example 4.3. $\gamma: I \to \mathbb{R}^{2+s}(-3s), \gamma(t) = (x(t), y(t), f(t) + c_1, f(t) + c_2, \dots, f(t) + c_s)$ is a unit speed Legendre curve such that

$$x(t) = \frac{2}{c}\cos\theta(t) + x_0$$
$$y(t) = \frac{2}{c}\sin\theta(t) + y_0$$
$$f(t) = \frac{1}{c^2}\sin 2\theta + \frac{2y_0}{c}\cos\theta - \frac{2}{c}t$$

and

$$\theta(t) = ct + c_0$$

The tangent vector field of γ in terms of basis $\{e, \phi e, \xi_1, \xi_2, \dots, \xi_s\}$ is $V_1 = \cos\theta\, e - \sin\theta\, \phi e$. From 26,

$$\nabla_{V_1} e = \sin\theta \sum_{i=1}^{s} \xi_i$$
$$\nabla_{V_1} \phi e = \cos\theta \sum_{i=1}^{s} \xi_i$$
$$\nabla_{V_1} \xi_i = -\sin\theta\, e - \cos\theta\, \phi e, \quad i \in \{1,2,\cdots,s\}$$
$$\nabla_{V_1} V_1 = -c\sin\theta\, e - c\cos\theta\, \phi e$$

and

$$k_1 = c$$

Then, $V_2 = -(\sin\theta\, e + \cos\theta\, \phi e)$ and $\phi V_1 = \cos\theta\, \phi e + \sin\theta\, e = -V_2$. If

$$E_1 = \gamma' = V_1$$

$$E_2 = \nabla_{V_1} V_1$$

and

$$E_3 = \nabla_{V_1}(\nabla_{V_1} V_1) - \frac{\langle \nabla_{V_1}(\nabla_{V_1} V_1), \nabla_{V_1} V_1 \rangle}{\langle \nabla_{V_1} V_1, \nabla_{V_1} V_1 \rangle} \nabla_{V_1} V_1 - \frac{\langle \nabla_{V_1}(\nabla_{V_1} V_1), V_1 \rangle}{\langle V_1, V_1 \rangle} V_1$$

then

$$V_3 = \frac{E_3}{\|E_3\|}$$

and

$$\nabla_{V_1}(\nabla_{V_1} V_1) = -c^2 \cos\theta\, e + c^2 \sin\theta\, \phi e - c \sum_{i=1}^{s} \xi_i$$

Then,

$$\langle \nabla_{V_1}(\nabla_{V_1} V_1), \nabla_{V_1} V_1 \rangle = 0$$

$$\langle \nabla_{V_1}(\nabla_{V_1} V_1), V_1 \rangle = -c^2$$

$$E_3 = -c \sum_{i=1}^{s} \xi_i$$

and

$$V_3 = \frac{-\sum_{i=1}^{s} \xi_i}{\sqrt{s}}$$

From $k_2 = \langle \nabla_{V_1} V_2, V_3 \rangle$ and

$$\nabla_{V_1} V_2 = -\left(c\cos\theta\, e - c\sin\theta\, \phi e + \sum_{i=1}^{s} \xi_i \right)$$

we obtain $k_2 = \sqrt{s}$. Similarly, since

$$\nabla_{V_1} V_3 = \sqrt{s}(\sin\theta\, e + \cos\theta\, \phi e)$$

and

$$\nabla_{V_1} V_3 = -k_2 V_2 + k_3 V_4$$

we obtain $k_3 V_4 = 0$. Thus, $k_3 = 0$. □

Let $\alpha: I \to M$ be a unit-speed curve in a 4-dimensional Riemannian manifold (M, g). The Frenet vectors of the curve α are

$$V_1 = \alpha', \quad V_2 = \frac{\alpha''}{\|\alpha''\|}, \quad V_4 = -\frac{\alpha' \times \alpha'' \times \alpha'''}{\|\alpha' \times \alpha'' \times \alpha'''\|}, \quad \text{and} \quad V_3 = V_4 \times V_1 \times V_2 \tag{27}$$

and the system $\{V_1, V_2, V_3, V_4\}$ is an orthonormal system in 4-dimensional space [11]. From Equation 27,

$$V_4 = -V_1 \times V_2 \times V_3 \tag{28}$$

is obtained.

Theorem 4.4. Let $M = \left(M^{2+2}, \phi, \xi_i, \eta_j, g\right)$, $i, j \in \{1,2\}$, be an S-manifold and $\gamma: I \to M^{2+2}$ be a Legendre curve of osculating order 4. Then, $V_4 = \frac{\varepsilon}{\sqrt{2}}(\xi_1 - \xi_2)$.

Proof.

From Equations 1 and 28,

$$V_4 = \left(g(V_1, \phi V_2)\eta_2(V_3) - g(V_1, \phi V_3)\eta_2(V_2) + g(V_2, \phi V_3)\eta_2(V_1)\right)\xi_1$$

$$+\left(-g(V_1, \phi V_2)\eta_1(V_3) + g(V_1, \phi V_3)\eta_1(V_2) - g(V_2, \phi V_3)\eta_1(V_1)\right)\xi_2 + \begin{vmatrix} \phi V_1 & \phi V_2 & \phi V_3 \\ \eta_1(V_1) & \eta_1(V_2) & \eta_1(V_3) \\ \eta_2(V_1) & \eta_2(V_2) & \eta_2(V_3) \end{vmatrix} \tag{29}$$

From Equations 9, 12, 13, 18, 28, and 29,

$$V_4 = \frac{\varepsilon}{\sqrt{2}}(\xi_1 - \xi_2)$$

is obtained. □

Example 4.5. Let $c_1, c_2 \in \mathbb{R}$. Then, the curve $\gamma: I \to \mathbb{R}^{2+2}(-6)$ defined by

$$\gamma(t) = \left(2\ln\left|\sqrt{1+t^2} + t\right|, 2\sqrt{1+t^2}, 4t + c_1, 4t + c_2\right)$$

is a unit speed Legendre curve. The tangent vector field of γ in terms of basis $\{e, \phi e, \xi_1, \xi_2\}$ is

$$V_1 = \frac{t}{\sqrt{1+t^2}}e + \frac{1}{\sqrt{1+t^2}}\phi e$$

From Equation 26,

$$\nabla_{V_1} e = -\frac{1}{\sqrt{1+t^2}}(\xi_1 + \xi_2)$$

$$\nabla_{V_1} \phi e = \frac{t}{\sqrt{1+t^2}}(\xi_1 + \xi_2)$$

$$\nabla_{V_1} \xi_i = \frac{1}{\sqrt{1+t^2}}(e - t\phi e), \quad i \in \{1,2\}$$

$$\nabla_{V_1} V_1 = \frac{1}{(1+t^2)^{3/2}}e - \frac{t}{(1+t^2)^{3/2}}\phi e$$

and

$$k_1 = \frac{1}{1+t^2}$$

then

$$V_2 = \frac{1}{\sqrt{1+t^2}}e - \frac{t}{\sqrt{1+t^2}}\phi \text{ and } \phi V_1 = -V_2$$

Thus,

$$\nabla_{V_1}V_2 = -\frac{t}{(1+t^2)^{3/2}}e - \frac{1}{(1+t^2)^{3/2}}\phi e - (\xi_1+\xi_2)$$

and

$$\nabla_{V_1}V_2 = -k_1V_1 + k_2V_3$$

Therefore, we obtain

$$k_2 = \sqrt{2} \text{ and } V_3 = -\frac{\xi_1+\xi_2}{\sqrt{2}}$$

Since $\nabla_{V_1}V_3 = -\frac{\sqrt{2}}{\sqrt{1+t^2}}(e+t\phi e)$ and $\nabla_{V_1}V_3 = -k_2V_2 + k_3V_4$, we obtain $k_3 = 0$. From Equation 28,

$$V_4 = \begin{vmatrix} e & \phi e & \xi_1 & \xi_2 \\ \frac{1}{\sqrt{1+t^2}} & -\frac{t}{\sqrt{1+t^2}} & 0 & 0 \\ \frac{t}{\sqrt{1+t^2}} & \frac{1}{\sqrt{1+t^2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}(\xi_2-\xi_1)$$

is obtained. □

Theorem 4.6. Let $M = \left(M^{2+3},\phi,\xi_i,\eta_j,g\right)$, $i,j\in\{1,2,3\}$, be an S-manifold and $\gamma: I\to M^{2+3}$ be a Legendre curve of osculating order 5. γ is imbedded in the 3-dimensional K-contact space.

PROOF.

Let

$$U_1 = \cos\theta\, V_4 - \sin\theta\, V_5$$
$$U_2 = \sin\theta\, V_4 + \cos\theta\, V_5 \tag{30}$$

such that

$$\theta(s) = \int k_4 ds \tag{31}$$

From Equations 30 and 31,

$$\nabla_{V_1}U_1 = 0$$

and

$$\nabla_{V_1}U_2 = 0$$

Therefore, U_1 and U_2 are constant. From Equation 30, $\{V_1,V_2,V_3,U_1,U_2\}$ is an orthonormal basis. For the functions

$$f_i : I\to\mathbb{R}$$

$$s\to f_i(s) = \langle\gamma(s)-\gamma(0),U_i\rangle$$

such that $i\in\{1,2\}$ from Equations 30 and 31, we get, for all $s\in I$,

$$f_i'(s) = \langle V_1,U_i\rangle = 0$$

$$f_i(s) = c\in\mathbb{R}$$

and

$$f_i(0) = \langle \gamma(0) - \gamma(0), U_i \rangle = c = 0$$

Hence, for all $s \in I$,

$$f_i(s) = 0, \quad i \in \{1,2\}$$

Then, $\gamma(s) - \gamma(0) \in \mathrm{Sp}\{V_1, V_2, V_3\}$. Let $\omega = \{X \in \chi(M) : \ g(X, U_1) = 0 \wedge g(X, U_2) = 0\}$. Since

$$\omega = \mathrm{Sp}\{V_1, V_2, V_3\} \text{ and } \gamma(s) - \gamma(0) \in \omega$$

For the function

$$\pi : \chi(M) \to \omega$$

$$X \quad \to \pi(X) = \bar{X} + \frac{\lambda_1 + \lambda_2 + \lambda_3}{3}\xi$$

such that $\bar{X} \in D = \{X \in \chi(M) : \ \eta_i(X) = 0, \forall i \in \{1,2,3\}\}$, $X = \bar{X} + \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3$, and $\xi = \frac{\xi_1+\xi_2+\xi_3}{3}$, if we get $\eta = \eta_1 + \eta_2 + \eta_3$, $\tilde{\phi} = \phi$, and $\tilde{g} = 3g$, then $(w^3, \tilde{\phi}, \xi, \eta, \tilde{g})$ is a K-contact space. Since $\tilde{g} = 3g$, then $\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k$ and $\tilde{\nabla} = \nabla$. Because $d\eta = d\eta_1 + d\eta_2 + d\eta_3 = 3d\eta_1 = 3\Phi = 3g = \tilde{g}$, then $\tilde{\Phi} = d\eta$. Moreover,

$$\tilde{\nabla}_{\pi X}\xi = \nabla_{\pi X}\left(\frac{\xi_1 + \xi_2 + \xi_3}{3}\right) = \frac{1}{3}(\nabla_{\pi X}\xi_1 + \nabla_{\pi X}\xi_2 + \nabla_{\pi X}\xi_3) = \frac{-3}{3}\phi(\pi X) = -\phi(\pi X) = -\tilde{\phi}(\pi X)$$

As

$$\eta(V_1) = (\eta_1 + \eta_2 + \eta_3)(V_1) = \eta_1(V_1) + \eta_2(V_1) + \eta_3(V_1) = 0$$

The curve γ is also a Legendre curve at ω. □

5. Conclusion

This study generalized the cross product defined in 3-dimensional almost contact metric manifolds and defined a new generalized cross product in $(2n+s)$-dimensional framed metric manifolds such that $n = 1$. Moreover, it characterized the curvatures of Legendre curves in S-manifolds. Moreover, this study proved that Legendre curves are biharmonic. Besides, it demonstrated that $(2+3)$-dimensional S-manifolds are imbedded in 3-dimensional space. In future studies, researchers can investigate Slant curves in S-manifolds using the generalized cross product herein.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's doctoral dissertation supervised by the second author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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