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ON α-QUASI SHORT MODULES

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Abstract. We introduce and study the concept of α-quasi short modules. Using this concept we extend some of the basic results of α-short modules to α-quasi short modules. We observe that if M is an α-quasi short module then the Noetherian dimension of M is α or $\alpha + 1$ or $\alpha + 2$.

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1. Introduction

Lemonnier [18] has introduced the concept of deviation (resp., codeviation) of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module M_R give the concept of Krull dimension, see [9], [10] and [20] (resp., the concept of dual Krull dimension of M. The dual Krull dimension in [7,8,11,12,13, 14,15,16,17] is called Noetherian dimension and in [5] is called N-dimension. This dimension is called Krull dimension in [21]. The name of dual Krull dimension is also used by some authors, see [1], [2] and [3]). The Noetherian dimension of an R-module M is denoted by $n\text{-dim}\, M$ and by $k\text{-dim}\, M$ we denote the Krull dimension of M. We recall that if an R-module M has Noetherian dimension and α is an ordinal number, then M is called α-atomic if $n\text{-dim}\, M = \alpha$ and $n\text{-dim}\, N < \alpha$, for all proper submodule N of M. An R-module M is called atomic if it is α-atomic for some ordinal α (note, atomic modules are also called conotable, dual critical and N-critical in some other articles; see for example [2], [5] and [19]). We introduced and extensively investigated quasi-Krull dimension and quasi-Noetherian dimension of an R-module M, see [6]. The quasi-Noetherian dimension (resp., quasi-Krull dimension), which is denoted by $qn\text{-dim}\, M$ (resp., $qk\text{-dim}\, M$) is defined to be the codeviation (resp., deviation) of the poset of the non-finitely generated submodules of M. We recall that an R-module M is called α-quasi-atomic, where α is an ordinal, if $qn\text{-dim}\, M = \alpha$ and $qn\text{-dim}\, N < \alpha$ for any proper non-finitely generated submodule N of M. M is said to be quasi-atomic if it is α-quasi-atomic for some α.

Bilhan and Smith have introduced and extensively investigated short modules and almost Noetherian modules, see [4]. Later Davoudian, Karamzadeh and Shirali undertook a systematic study of the concepts of α-short modules and α-almost Noetherian modules, see [8]. We recall that an R-module M is called an α-short module, if for each submodule N of M, either $n\text{-}\dim N \leq \alpha$ or $n\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. We shall call an R-module M to be α-quasi short, if for each non-finitely generated submodule N of M, either $qn\text{-}\dim N \leq \alpha$ or $qn\text{-}\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property. Using this concept, we show that each α-quasi short module M has Noetherian dimension and $\alpha \leq n\text{-}\dim M \leq \alpha + 2$. We also recall that an R-module M is called α-almost Noetherian, if for each proper submodule N of M, $n\text{-}\dim N < \alpha$ and α is the least ordinal number with this property, see [8]. We shall call an R-module M to be α-almost quasi Noetherian if for each proper non-finitely generated submodule N of M, $qn\text{-}\dim N < \alpha$ and α is the least ordinal number with this property. In Section 2, of this paper we investigate some basic properties of α-almost quasi Noetherian and α-quasi short modules. We show that if M is an α-quasi short module (resp., α-almost quasi Noetherian module), then $qn\text{-}\dim M = \alpha$ or $qn\text{-}\dim M = \alpha + 1$ (resp., $qn\text{-}\dim M \leq \alpha$). Thus we observe that if M is an α-quasi short module, then M has Noetherian dimension and $\alpha \leq n\text{-}\dim M \leq \alpha + 2$. In the last section we also investigate some properties of α-almost quasi Noetherian and α-quasi short modules.

2. α-quasi short modules and α-almost quasi Noetherian modules

We recall that an R-module M is called α-almost Noetherian, if for each proper submodule N of M, $n\text{-}\dim N < \alpha$ and α is the least ordinal number with this property. In the following definition we consider a related concept.

Definition 2.1. An R-module M is called *α-almost quasi Noetherian* if for each proper non-finitely generated submodule N of M, $qn\text{-}\dim N < \alpha$ and α is the least ordinal number with this property.

It is manifest that if M is an α-almost quasi Noetherian, then each submodule and each factor module of M is β-almost quasi Noetherian for some $\beta \leq \alpha$ (note, see [6, Lemmas 8, 9]).

In view of [6, Lemma 10], we have the next three trivial, but useful facts.

Lemma 2.2. *If M is an α-almost quasi Noetherian module, then M has quasi Noetherian dimension and $qn\text{-}\dim M \leq \alpha$. In particular, $qn\text{-}\dim M = \alpha$ if and only if M is α-quasi atomic.*

Lemma 2.3. *If M is a module with qn-$\dim M = \alpha$, then either M is α-quasi atomic, in which case it is α-almost quasi Noetherian, or it is $\alpha+1$-almost quasi Noetherian.*

Lemma 2.4. *If M is an α-almost quasi Noetherian module, then either M is α-quasi atomic or $\alpha = qn$-$\dim M+1$. In particular, if M is α-almost quasi Noetherian module, where α is a limit ordinal, then M is α-quasi atomic.*

Proposition 2.5. *An R-module M has quasi-Noetherian dimension if and only if M is α-almost quasi Noetherian for some ordinal α.*

In view of Lemma 2.2 and [6, Corollary 5], we have the following result.

Corollary 2.6. *If R-module M is α-almost quasi Noetherian, then M has Noetherian dimension and n-$\dim M \leq \alpha+1$.*

Next we give our definition of α-quasi short modules.

Definition 2.7. An R-module M is called *α-quasi short*, if for each non-finitely generated submodule N of M, either qn-$\dim N \leq \alpha$ or qn-$\dim \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property.

In view of [6, Corollary 3], we have the following results.

Remark 2.8. *If M is an R-module with qn-$\dim M = \alpha$, then M is β-quasi short for some $\beta \leq \alpha$.*

Remark 2.9. *If M is an α-quasi short module, then each submodule and each factor module of M is β-quasi short for some $\beta \leq \alpha$.*

We cite the following result from [6, Lemma 12].

Lemma 2.10. *If M is an R-module and for each non-finitely generated submodule N of M, either N or $\frac{M}{N}$ has quasi Noetherian dimension, then so does M.*

The previous result and Remark 2.8, immediately yield the next result.

Corollary 2.11. *Let M be an α-quasi short module. Then M has quasi Noetherian dimension and $\alpha \leq qn$-$\dim M$.*

The following is now immediate.

Proposition 2.12. *An R-module M has quasi-Noetherian dimension if and only if M is α-quasi short for some ordinal α.*

Proposition 2.13. *If M is an α-quasi short R-module, then either qn-$\dim M = \alpha$ or qn-$\dim M = \alpha + 1$.*

Proof. Clearly in view of Corollary 2.11, we have qn-$\dim M \geq \alpha$. If qn-$\dim M \neq \alpha$, then qn-$\dim M \geq \alpha + 1$. Now let $M_1 \subseteq M_2 \subseteq \dots$ be any ascending chain of non-finitely generated submodules of M. If there exists some k such that qn-$\dim \frac{M}{M_k} \leq \alpha$, then qn-$\dim \frac{M_{i+1}}{M_i} \leq qn$-$\dim \frac{M}{M_i} = qn$-$\dim \frac{M/M_k}{M_i/M_k} \leq qn$-$\dim \frac{M}{M_k} \leq \alpha$ for each $i \geq k$, see [6, Corollary 3]. Otherwise qn-$\dim M_i \leq \alpha$ (M is α-quasi short) for each i, hence qn-$\dim \frac{M_{i+1}}{M_i} \leq qn$-$\dim M_{i+1} \leq \alpha$ for each i. Thus in any case there exists an integer k such that for each $i \geq k$, qn-$\dim \frac{M_{i+1}}{M_i} \leq \alpha$. This shows that qn-$\dim M \leq \alpha + 1$, i.e., qn-$\dim M = \alpha + 1$. □

In view of the previous proposition and [6, Corollary 5] we have the following result.

Corollary 2.14. *If M is an α-quasi short R-module, then $\alpha \leq n$-$\dim M \leq \alpha + 2$.*

In view of previous corollary every α-quasi short module has Krull dimension, for by a nice result due to Lemonnier, every module has Noetherian dimension if and only if it has Krull dimension, see [18, Corollary 6]. Thus by [20, Lemma 6.2.6], we have the following result.

Proposition 2.15. *Every α-quasi short module has finite uniform dimension.*

Remark 2.16. *An R-module M is -1-quasi short if and only if it is either Noetherian or 1-atomic.*

Proposition 2.17. *Let M be an R-module, with qn-$\dim M = \alpha$, where α is a limit ordinal. Then M is α-quasi short.*

Proof. We know that M is β-quasi short for some $\beta \leq \alpha$. If $\beta < \alpha$, then by Proposition 2.13, qn-$\dim M \leq \beta + 1 < \alpha$, which is a contradiction. Thus M is α-quasi short. □

Proposition 2.18. *Let M be an R-module and qn-$\dim M = \alpha = \beta + 1$. Then M is either α-quasi short or it is β-quasi short.*

Proof. We know that M is γ-quasi short for some $\gamma \leq \alpha$. If $\gamma < \beta$, then by Proposition 2.13, we have qn-$\dim M \leq \gamma + 1 < \beta + 1$, which is impossible. Hence we are done. □

Proposition 2.19. *Let M be an α-quasi atomic R-module, where $\alpha = \beta + 1$, then M is a β-quasi short module.*

Proof. Let N be a non-finitely generated submodule of M, therefore qn-$\dim N < \alpha$. This shows that for some $\beta' \leq \beta$, M is β'-quasi short. If $\beta' < \beta$, then $\beta'+1 \leq \beta < \alpha$. But qn-$\dim M \leq \beta' + 1 \leq \beta < \alpha$, by Proposition 2.13, which is a contradiction. Thus $\beta' = \beta$ and we are done. □

The following remark, which is a trivial consequence of the previous fact, shows that the converse of Proposition 2.17, is not true in general.

Remark 2.20. *Let M be an $\alpha+1$-quasi atomic R-module, where α is a limit ordinal. Then M is an α-quasi short module but qn-$\dim M \neq \alpha$.*

Proposition 2.21. *Let M be an R-module such that qn-$\dim M = \alpha + 1$. Then M is either α-quasi short R-module or there exists a non-finitely generated submodule N of M such that qn-$\dim N = qn$-$\dim \frac{M}{N} = \alpha + 1$.*

Proof. We know that M is α-quasi short or an $\alpha+1$-quasi short R-module, by Proposition 2.18. Let us assume that M is not α-quasi short R-module, hence there exists a non-finitely generated submodule N of M such that qn-$\dim N \geq \alpha+1$ and qn-$\dim \frac{M}{N} \geq \alpha + 1$. This shows that qn-$\dim N = \alpha + 1$ and qn-$\dim \frac{M}{N} = \alpha + 1$ and we are through. □

Proposition 2.22. *Let M be a non-zero α-quasi short R-module. Then either M is β-almost quasi Noetherian for some ordinal $\beta \leq \alpha+1$ or there exists a non-finitely generated submodule N of M with qn-$\dim \frac{M}{N} \leq \alpha$.*

Proof. Suppose that M is not β-almost quasi Noetherian for any $\beta \leq \alpha+1$. This means that there must exist a non-finitely generated submodule N of M such that qn-$\dim N \nleq \alpha$. Inasmuch as M is α-quasi short, we infer that qn-$\dim \frac{M}{N} \leq \alpha$ and we are done. □

Let us cite the next result which is in [15, Theorem 2.9], see also [11, Theorem 3.2].

Theorem 2.23. *For a commutative ring R the following statements are equivalent.*

(1) *Every R-module with finite Noetherian dimension is Noetherian.*
(2) *Every Artinian R-module is Noetherian.*
(3) *Every R-module with Noetherian dimension is both Artinian and Noetherian.*

In view [8, Proposition 2.21], Corollary 2.14 and Corollary 2.6, we have the following result.

Proposition 2.24. *The following statements are equivalent for a commutative ring R.*

(1) *Every Artinian R-module is Noetherian.*
(2) *Every m-short module is both Artinian and Noetherian for all integers $m \geq -1$.*
(3) *Every α-short module M is both Artinian and Noetherian for all ordinal α.*
(4) *Every m-almost Noetherian module is both Artinian and Noetherian for all integers $m \geq -1$.*
(5) *Every α-almost Noetherian module is both Artinian and Noetherian for all integers $m \geq -1$.*
(6) *Every m-quasi short module is both Artinian and Noetherian for all integers $m \geq -1$.*
(7) *Every α-quasi short module M is both Artinian and Noetherian for all ordinal α.*
(8) *Every m-almost quasi Noetherian module is both Artinian and Noetherian for all integers $m \geq -1$.*
(9) *Every α-almost quasi Noetherian module M is both Artinian and Noetherian for all ordinal α.*
(10) *No homomorphic image of R can be isomorphic to a dense subring of a complete local domain of Krull dimension 1.*

Finally we conclude this section by providing some examples of α-almost quasi Noetherian (resp., α-quasi short) modules, where α is any ordinal. First, we recall that given any ordinal α there exists an Artinian module M such that $n\text{-dim}\, M = \alpha$, see [15, Example 1]. If α is a limit ordinal number then by [6, Corollary 5], we infer that $qn\text{-dim}\, M = \alpha$. Consequently, we may take M to be an Artinian module with $n\text{-dim}\, M = \alpha$, where α is a limit ordinal number. Hence $qn\text{-dim}\, M = \alpha$ and for any ordinal $\beta \leq \alpha$, we take N to be its β-quasi atomic submodule, see [6, Lemma 15], then by Lemma 2.3, N is β-almost quasi Noetherian. We recall that the only α-almost quasi Noetherian modules, where α is a limit ordinal are α-quasi atomic module, see Lemma 2.4. Therefore to see an example of α-almost quasi Noetherian module which is not α-quasi atomic, the ordinal α must be a non-limit ordinal. Thus we may take M to be a non-quasi atomic module with $qn\text{-dim}\, M = \beta$, where $\alpha = \beta + 1$, hence its follows trivially that M is an α-almost quasi Noetherian. As for examples of α-quasi short modules, one can similarly use the facts that there are Artinian modules with Noetherian dimension equals to α, see [15]. In view of [6, Corollary 5], we infer that $qn\text{-dim}\, M = \alpha$, where α is a limit ordinal number.

By [6, Lemma 15], for each $\beta \le \alpha$ there are β-quasi atomic submodules of M and then apply Propositions 2.17, 2.18, 2.19, to give various examples of α-quasi short modules (for example, by Proposition 2.19, $\alpha + 1$-quasi atomic module is α-quasi short).

3. Properties of α-quasi short modules and α-almost quasi Noetherian modules

In this section some properties of α-quasi short modules over an arbitrary ring R are investigated.

First, in view of Corollaries 2.14, 2.6, and [16, Corollary 1.8] we have the following result.

Proposition 3.1. *If M is an α-quasi short module (resp., α-almost quasi Noetherian module), where α is a countable ordinal, then every submodule of M is countably generated.*

Remark 3.2. *Let M be an R-module and N be a submodule of M such that $qn\text{-}\dim N = \alpha$ and $qn\text{-}\dim \frac{M}{N} = \beta$. If $\sup\{qn\text{-}\dim N, qn\text{-}\dim \frac{M}{N}\} = \gamma$, then $\gamma \le qn\text{-}\dim M \le \gamma + 1$.*

Proof. We know that $n\text{-}\dim N = \alpha$ or $n\text{-}\dim N = \alpha + 1$ and $n\text{-}\dim \frac{M}{N} = \beta$ or $n\text{-}\dim \frac{M}{N} = \beta + 1$, see [6, Corollary 5]. Therefore $n\text{-}\dim M = \sup\{n\text{-}\dim N, n\text{-}\dim \frac{M}{N}\} \le \gamma + 1$. But by [6, Remark 2], we get $qn\text{-}\dim M \le n\text{-}\dim M \le \gamma + 1$. In view of [6, Corollary 3], we get $\gamma \le qn\text{-}\dim M$. This implies that $\gamma \le qn\text{-}\dim M \le \gamma + 1$ and we are done. □

In the following two propositions we investigate the connection between α-short modules and α-quasi short modules.

Proposition 3.3. *Let M be an α-short R-module. Then M is a β-quasi short module such that $\alpha \in \{\beta, \beta + 1, \beta + 2\}$.*

Proof. Let N be any non-finitely generated submodule of M, then $qn\text{-}\dim N \le n\text{-}\dim N \le \alpha$ or $qn\text{-}\dim \frac{M}{N} \le n\text{-}\dim \frac{M}{N} \le \alpha$, see [6, Remark 2]. This implies that M is β-quasi short for some $\beta \le \alpha$. If M is β-quasi short, then $qn\text{-}\dim M = \beta$ or $qn\text{-}\dim M = \beta + 1$. Hence $\beta \le n\text{-}\dim M \le \beta + 2$, see [6, Corollary 5]. In other hand by [8, Proposition 1.12], we get $\alpha \le n\text{-}\dim M \le \alpha + 1$. Therefore $\beta = \alpha$ or $\alpha = \beta + 1$ or $\alpha = \beta + 2$ (note, we always have $\beta \le \alpha$) and we are done. □

Proposition 3.4. *Let M be a β-quasi short R-module. Then M is an α-short R-module and $\alpha \in \{\beta, \beta + 1, \beta + 2\}$.*

Proof. By Proposition 2.13, $qn\text{-dim}\, M = \beta$ or $qn\text{-dim}\, M = \beta + 1$. This implies that M has Noetherian dimension and $\beta \leq n\text{-dim}\, M \leq \beta + 2$, see [6, Corollary 5]. Thus M is α-short for some ordinal number α, see [8, Remark 1.2.]. By Proposition 3.3, we get $\alpha \in \{\beta, \beta+1, \beta+2\}$ and we are done. □

In view of Propositions 3.3 and 3.4 we have the following result.

Corollary 3.5. *Let M be an R-module and α be a limit ordinal number. Then M is α-short if and only if it is α-quasi short.*

We note that the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ is 0-quasi short.

Proposition 3.6. *Let N be a submodule of an R-module M such that N is α-quasi short and $\frac{M}{N}$ is β-quasi short. Let $\mu = \sup\{\alpha, \beta\}$, then M is γ-quasi short such that $\mu \leq \gamma \leq \mu + 2$.*

Proof. Since N is α-quasi short, thus by Proposition 2.13, $qn\text{-dim}\, N = \alpha$ or $qn\text{-dim}\, N = \alpha+1$. Similarly since $\frac{M}{N}$ is β-quasi short, $qn\text{-dim}\, \frac{M}{N} = \beta$ or $qn\text{-dim}\, \frac{M}{N} = \beta + 1$. Let $\lambda = \sup\{qn\text{-dim}\, N, qn\text{-dim}\, \frac{M}{N}\}$, then $\mu \leq \lambda \leq \mu + 1$. In view of Remark 3.2, we infer that M has quasi Noetherian dimension and $\lambda \leq qn\text{-dim}\, M \leq \lambda + 1$. Therefore $\mu \leq qn\text{-dim}\, M \leq \mu+2$. But by Remark 2.12, M is γ-quasi short for some ordinal number γ and by Proposition 2.13, $\gamma \leq qn\text{-dim}\, M \leq \gamma+1$. This shows that $\mu \leq \gamma \leq \mu + 2$, (note, we always have $\mu \leq \gamma$). □

Using Lemma 2.2, we give the next immediate result which is the counterpart of the previous proposition for α-almost quasi Noetherian modules.

Proposition 3.7. *Let N be a submodule of an R-module M such that N is α-almost quasi Noetherian and $\frac{M}{N}$ is β-almost quasi Noetherian. Let $\mu = \sup\{\alpha, \beta\}$, then M is γ-almost quasi Noetherian such that $\mu \leq \gamma \leq \mu + 2$.*

Corollary 3.8. *Let R be a ring. If M_1 is an α_1-quasi short (resp., α_1-almost quasi Noetherian) R-module and M_2 is an α_2-quasi short (resp., α_2-almost quasi Noetherian) R-module and let $\alpha = \sup\{\alpha_1, \alpha_2\}$. Then $M_1 \oplus M_2$ is μ-quasi short (resp., μ-almost quasi Noetherian) for some ordinal number μ such that $\alpha \leq \mu \leq \alpha + 2$.*

Example 3.9. *If $M_1 = M_2 = \mathbb{Z}$, then M_1 and M_2 are -1-quasi short (resp., -1-almost quasi Noetherian) $\mathbb{Z}$-modules such that $M_1 \oplus M_2$ is also -1-quasi short (resp., -1-almost quasi Noetherian). Now let $M_1 = M_2 = \mathbb{Z}_{p^\infty}$. In this case the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is -1-quasi short (resp., -1-almost quasi Noetherian), but the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}p^\infty$ is 0-quasi short (resp., 0-almost quasi Noetherian).*

Proposition 3.10. *Let R be a ring and M be a nonzero α-quasi short module, which is not a quasi atomic module, then M contains a non-finitely generated submodule L such that $qn\text{-}\dim \frac{M}{L} \leq \alpha$.*

Proof. Since M is not quasi atomic, we infer that there exists a non-finitely generated submodule $L \subsetneq M$, such that $qn\text{-}\dim L = qn\text{-}\dim M$. We know that $qn\text{-}\dim M = \alpha$ or $qn\text{-}\dim M = \alpha+1$, by Proposition 2.13. If $qn\text{-}\dim M = \alpha$ it is clear that $qn\text{-}\dim \frac{M}{L} \leq \alpha$. Hence we may suppose that $qn\text{-}\dim L = qn\text{-}\dim M = \alpha + 1$. If $qn\text{-}\dim \frac{M}{L} = \alpha + 1$, then M is γ-quasi short module for some $\gamma \geq \alpha + 1$, which is a contradiction. Consequently, $qn\text{-}\dim \frac{M}{L} \leq \alpha$ and we are done. □

The following example gives a module satisfying the condition of Proposition 3.10.

Example 3.11. *Let $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ and $L = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$. By the comment which follows* [6, Remark 2]*, we infer that $qn\text{-}\dim L_{\mathbb{Z}} = 1$. Therefore $qn\text{-}\dim M_{\mathbb{Z}} = 1$, see* [6, Lemma 8]. *Thus M is not quasi atomic. But $\frac{M}{L} \simeq \mathbb{Z}_{p^\infty}$, thus $qn\text{-}\dim \frac{M}{L} = 0$, see* [6, Remark 1]. *Clearly M is a 0-quasi short module.*

Theorem 3.12. *Let α be an ordinal number and M be an R-module. If every proper non-finitely generated submodule of M is γ-quasi short for some ordinal number $\gamma \leq \alpha$. Then $qn\text{-}\dim M \leq \alpha + 2$, in particular, M is μ-short for some ordinal $\mu \leq \alpha + 2$.*

Proof. Let $N \subsetneq M$ be any non-finitely generated submodule. Since N is γ-quasi short for some ordinal number $\gamma \leq \alpha$, we infer that $qn\text{-}\dim N \leq \gamma + 1 \leq \alpha + 1$, by Proposition 2.13. This immediately implies that $qn\text{-}\dim M \leq \alpha + 2$, see [6, Lemma 10]. The final part is now evident. □

The next result is the dual of Theorem 3.12.

Theorem 3.13. *Let M be a nonzero R-module and α be an ordinal number. Let for each proper non-finitely generated submodule N of M, $\frac{M}{N}$ be γ-quasi short for some ordinal number $\gamma \leq \alpha$. Then $qn\text{-}\dim M \leq \alpha + 2$, in particular, M is μ-short for some ordinal $\mu \leq \alpha + 2$.*

Proof. Let N be any proper non-finitely generated submodule of M, then $\frac{M}{N}$ is γ-quasi short for some ordinal number $\gamma \leq \alpha$. In view of Proposition 2.13, we infer that $qn\text{-}\dim \frac{M}{N} \leq \gamma + 1 \leq \alpha + 1$. Therefore $qn\text{-}\dim M \leq \sup\{qn\text{-}\dim \frac{M}{N} : N \text{ is nonfinitely generated submodule of } M\} + 1 \leq \alpha + 2$, see [6, Lemma 11]. The final part is now evident. □

The next immediate result is the counterparts of Theorems 3.12, 3.13, for α-almost quasi Noetherian modules.

Proposition 3.14. *Let M be an R-module and α be an ordinal number. If each proper non-finitely generated submodule N of M (resp., for each proper non-finitely generated submodule N of M, $\frac{M}{N}$) is γ-almost quasi Noetherian with $\gamma \leq \alpha$, then $qn\text{-}\dim M \leq \alpha + 1$ and M is an μ-almost quasi Noetherian module with $\mu \leq \alpha + 2$ (resp., $qn\text{-}\dim M \leq \alpha + 1$ and M is an μ-almost quasi Noetherian module with $\mu \leq \alpha + 2$).*

The following result is evident. We give the proof for the sake of completeness.

Proposition 3.15. *If M has finite Goldie dimension, then*

$$qn\text{-}\dim M \leq \sup\{qn\text{-}\dim \frac{M}{E} + 1 : E \subset_e M \ and\ E\ is\ non\text{-}finitely\ generated\}$$

if either side exists.

Proof. Let $\alpha = \sup\{qn\text{-}\dim \frac{M}{E} : E$ is essential and non-finitely generated$\}$, then it sufficient to show that $qn\text{-}\dim M$ exists and $qn\text{-}\dim M \leq \alpha$. Now let $N_1 \subset N_2 \subset \cdots \subset N_i \subset \ldots$ be an infinite ascending chain of non-finitely generated submodule of M, then by our assumption there exists some integer k such that N_i is essential in N_{i+1} for all $i \geq k$ (note, M has finite Goldie dimension). This means that there exists a submodule P of M such that $N_i \oplus P$ is essential in M for all $i \geq k$. It is clear that for each i, $N_i \oplus P$ is a non-finitely generated submodule of M (note, if $N_i \oplus P$ is finitely generated, then N_i is finitely generated which is a contradiction). But $\frac{N_{i+1}}{N_i} \simeq \frac{N_{i+1} \oplus P}{N_i \oplus P}$ for all $i \geq k$. In view of [6, Lemma 8], we infer that $qn\text{-}\dim \frac{N_{i+1}}{N_i} = qn\text{-}\dim \frac{N_{i+1} \oplus P}{N_i \oplus P} \leq qn\text{-}\dim \frac{M}{N_i \oplus P} < \alpha$ for each $i \geq k$ and hence $qn\text{-}\dim M \leq \alpha$. □

Proposition 3.16. *Let R be a semiprime ring. If the right R-module R is α-quasi short, then $qn\text{-}\dim R = \alpha$ or $qn\text{-}\dim \frac{R}{E} \leq \alpha$ for each non-finitely generated essential right ideal E of R.*

Proof. Suppose that there exists an essential non-finitely generated right ideal E' of R such that $qn\text{-}\dim \frac{R}{E'} \nleq \alpha$. Since R is α-quasi short, we infer that $qn\text{-}\dim E' \leq \alpha$. In view of Corollary 2.14, R has Noetherian dimension. Therefore R is a right Goldie ring, see [10, Corollary 3.4]. Hence there exists a regular element c in E', which implies that $qn\text{-}\dim R = qn\text{-}\dim cR \leq qn\text{-}\dim E'_R \leq \alpha$. Consequently, we must have $qn\text{-}\dim R = \alpha$, by Proposition 2.13. □

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