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The Minus Partial Order on Endomorphism Rings

Tufan Özdin

Department of Mathematics, Faculty of Science and Art, Erzincan Binali Yıldırım University, 24100 Erzincan, Turkey.

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Abstract. Let $S = End(M)$ be the ring of endomorphisms of a right R-module M. In this paper we define the minus parital order for the endomorphism ring of modules. Also, we extend study of minus partial order to the endomorphism ring of a (Rickart) module. Thus, several well-known results concerning minus partial order are generalized.

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1. Introduction

Throughout this article rings are associative with nonzero unity, modules are right modules unless otherwise specified, and morphisms will be written on the left of their arguments. We write $S = End(M)$ for the ring of all endomorphisms of a module M. For a submodule N of M, we use $N \leq M(N < M)$ to mean that N is a submodule of M (respectively, proper submodule), and we write $N \leq^{\oplus} M$ to indicate that N is a direct summand of M. We always use $\mathbb{M}$ to stand for the ring of all $n \times n$ matrices over a ring R. The left annihilator of $\alpha \in S = End(M)$ is denoted by $l_S(\alpha)$. Similarly, the right annihilator of $\alpha \in S = End(M)$ is denoted by $r_S(\alpha)$. The set of idempotents of a ring is denoted by $E(-)$. General background material can be found in [1].

The shorted operators have been introduce in [2] and [3]. The shorted operators are related to electrical network theory, especially computing to a shorted electrical circuit. Also, they used to parallel connections and electrical duality to further algebraic theorems. (see eg. [2, 3, 11]). In [12] and [13], the authors studied the partial order (called the natural partial order) on regular semigroup S. The minus partial order was extensively studied in [5] and [11]. Šemrl defined in [17] the minus partial order on $B(H)$. Let H be a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $A, B \in B(H)$ we write $A \leq B$ if and only if there exist idempotent operators $P, Q \in B(H)$ such that $ImP = \overline{ImA}$, $Ker(A) = Ker(Q)$, $PA = PB$ and $AQ = BQ$. Following Šemrl's approach, the authors introduced in [4] the minus partial order on a ring: Let R be a ring with the unity 1 and $a, b \in R$, then we write $a \leq b$ if there exist idempotent elements $p, q \in R$ such that $l_R(a) = R(1 - p)$, $r_R(a) = (1 - q)R$, $pa = pb$ and $aq = bq$.

In this paper, we will introduce the minus partial order for modules using their endomorphism rings: Let M be a module, $S = End(M)$ with identity 1_M and $\alpha, \beta \in S = End(M)$. Then, we write $\alpha \leq \beta$ if and only if there exist idempotents $p, q \in S$ such that the following hold:

(1) $l_S(\alpha) = S(1 - p)$
(2) $Ker(\alpha) = (1 - q)(M)$
(3) $p\alpha = p\beta$

Email address: tufan.ozdin@hotmail.com; tozdin@erzincan.edu.tr (T. Özdin)

(4) $\alpha q = \beta q$

In Theorem 2.4, we will present another equivalent definition of the minus partial order over endomorphism ring of a module M as: Let α, β be idempotents in $S = End(M)$. Then, $\alpha \leq \beta$ iff $\alpha\beta = \beta\alpha = \alpha$. Among other results, some known results are generalized. For example, we show that the partial order $\leq^-$ is equivalent to the minus partial order $\leq$ on a von Neumann regular ring: For a module M and $\alpha, \beta \in S = End(M)$, if $\alpha + \beta$ is regular, then $\alpha \leq^{\oplus} \alpha + \beta$ iff $\alpha \leq^- \alpha + \beta$, and if S is a von Neumann regular ring with unit 1_S, then $\alpha \leq \beta$ iff $\alpha \leq^- \beta$ (Theorem 2.5).

A ring R is called right Rickart if the right annihilator of any single element of R is generated by an idempotent as a right ideal. A left Rickart ring is defined similarly. In [4], the authors proved that this is indeed a partial order when R is a Rickart ring. Let M be an R-module with $S = End(M)$. We say that M is a Rickart module if the right annihilator in M of any single element of S is generated by an idempotent of S; or equivalently, $r_S(\alpha) = Ker(\alpha) \leq^{\oplus} M$ for every $\alpha \in S$; or equivalently, for every $\varphi \in S$, $r_M(\varphi) = Ker(\varphi) = eM$ for some $e^2 = e \in S$. The notion of Rickart module has been recently studied in [9] and also partial order has been studied on Rickart rings in [10, 18, 19]. In [15] and [16] the authors studied regular homomorphism and give some characterizations of regular homomorphism. In the third section, the notion of the module-theoretic version of the minus partial order is extended to endomorphism rings of (Rickart modules) which are Rickart rings. It is shown that if $S = End(M)$ is a Rickart ring and $\alpha, \beta \in S$. Then $\alpha \leq \beta$ iff $\beta - \alpha \leq \beta$ (Corollary 3.5). Recall that $S = End(M)$ is a von Neumann regular ring with unit 1_S, then $\alpha \leq \beta$ iff $\alpha \leq^- \beta$ (Theorem 2.5). We also obtain that if $S = End(M)$ is a Rickart ring and $\alpha, \beta \in S$ then the operator $\leq$ is a partial order in S, then $\alpha \leq \beta$ iff $\beta - \alpha \leq \beta$ (Corollary 3.5).

2. The Minus Partial Order for $End(-)$ of a Module

Let M be a right R-module and $S := End(M)$. We define the minus partial order $\leq$ for $S = End(M)$.

Definition 2.1. Let M be a module, $S = End(M)$ with identity 1_M and $\alpha, \beta \in S = End(M)$. Then, we write $\alpha \leq \beta$ if and only if there exist idempotents $p, q \in S$ such that the followings hold:

(1) $l_S(\alpha) = S(1-p)$
(2) $Ker(\alpha) = (1-q)(M)$
(3) $p\alpha = p\beta$
(4) $\alpha q = \beta q$.

Remark 2.2. By Definition 2.1, one can see that $\alpha = p\alpha$ and $\alpha = \alpha q$.

Lemma 2.3. *Let M be a module, p and q be two idempotents in $S = End(M)$ and $\alpha \in S$. Then,*

(1) $l_S(p) = S(1-p)$
(2) $(1-q)(M) = Ker(q)$
(3) $Ker(l_S(p)) = p(M)$
(4) $l_S(\alpha) = S(1-p) \Leftrightarrow Ker(l_S(\alpha)) = Ker(l_S(p))$.

Proof.
(1) If $u \in l_S(p)$, then $u = u(1-p) \in S(1-p)$. Since $(1-p)p = 0$, we get $S(1-p) \subseteq l_S(p)$.
(2) It is obvious by [1, Lemma 5.6].
(3) The claim is

$$\begin{aligned} Ker(l_S(p)) = Ker(S(1-p)) &= \{m \in M : \gamma(1-p)(m) = 0 \text{ for every } \gamma \in S\} \\ &= \{m \in M : (1-p)(m) = 0\} \\ &= \{m \in M : p(m) = m\} = p(M). \end{aligned}$$

(4) ($\Rightarrow$:) By (1), we have $l_S(\alpha) = S(1-p) = l_S(p)$ so $Ker(l_S(\alpha)) = Ker(l_S(p))$.

($\Leftarrow$:) Assume $Ker(l_S(\alpha)) = Ker(l_S(p))$. Then $Ker(l_S(\alpha)) = p(M)$ by (3). Let $\gamma \in l_S(\alpha)$. Since $p(m) \in p(M) = Ker(l_S(\alpha))$ we have $\gamma(p(m)) = 0$ which implies $\gamma = \gamma(1-p) \in S(1-p)$. On the other hand, suppose now $\gamma \in S(1-p) = l_S(p)$. As $\alpha(M) \subseteq Ker(l_S(\alpha)) = Ker(l_S(p)) = p(M)$, we have, for every $\alpha(m) \in \alpha(M)$, there exists $m^* \in M$ such that $\alpha(m) = p(m^*)$. So $\gamma\alpha(m) = \gamma p(m^*) = 0$ by $\gamma p = 0$ which completes the proof. □

Theorem 2.4. *Let M be a module, α, β be idempotents in $S = End(M)$. Then, $\alpha \leq \beta$ iff $\alpha\beta = \beta\alpha = \alpha$.*

Proof. Let α, β be idempotent morphisms in S and $\alpha\beta = \beta\alpha = \alpha$. Then by Lemma 2.3, $l_S(\alpha) = S(1-\alpha)$ and $(1-\beta)(M) = Ker(\beta)$. So $\alpha \leq \beta$ by assumption. On the other hand, suppose $\alpha \leq \beta$. So there exist idempotents $p, q \in S$ as in Definition 2.1. Therefore $\alpha\beta = (p\alpha)\beta = (p\beta)\beta = p\beta = \alpha$ and $\beta\alpha = \beta(\alpha q) = \beta(\beta q) = \beta q = \alpha$. □

Following von Neumann [20], an element a in a ring R is regular if $a = aba$ for some $b \in R$ and R is called a regular ring if every element is regular. Let M and N be right R-modules and $\alpha : M \to N$ be a homomorphism. The homomorphism α is called regular if, there exists a homomorphism $\gamma : N \to M$ such that $\alpha = \alpha\gamma\alpha$ by [8]. According to [7] and [6], for any two elements a, b in a von Neumann regular ring R, the relations $\leq^{\oplus}$ and $\leq^{-}$ on R are defined as follows:

$a \leq^{\oplus} b$ if and only if $bR = aR \oplus (b-a)R$, and called it the direct sum partial order.

$a \leq^{-} b$ if there exists an $x \in R$ such that $ax = bx$ and $xa = xb$, where $axa = a$, and we say that a is less than or equal to b under the minus partial order.

Let M and N be right R-modules. For any $\alpha, \beta \in Hom(M, N)$, we define the partial order $\leq^{\oplus}$ and the minus partial order $\leq^{-}$ as follows:

$$\alpha \leq^{\oplus} \beta \Leftrightarrow \beta S = \alpha S \oplus (\beta - \alpha)S$$

and

$\alpha \leq^{-} \beta$ if there exists a $\gamma \in Hom(N, M)$ such that $\gamma\alpha = \gamma\beta$ and $\alpha\gamma = \beta\gamma$, where $\alpha = \alpha\gamma\alpha$.

Theorem 2.5. *Let M a module and $\alpha, \beta \in S = End(M)$. Then,*

(1) *If $\alpha + \beta$ is regular, then $\alpha \leq^{\oplus} \alpha + \beta$ iff $\alpha \leq^{-} \alpha + \beta$.*
(2) *If S is a von Neumann regular ring then $\alpha \leq \beta$ iff $\alpha \leq^{-} \beta$.*

Proof.
(1) The claim follows from [14, Theorem 10].
(2) ($:\Rightarrow$) Let p, q be idempotents of S as in Definition 2.1. Since S is von Neumann regular there exists $\gamma \in S$ such that $\alpha\gamma\alpha = \alpha$. Let $\tau = q\gamma p$. Then, we have $\alpha\tau\alpha = \alpha(q\gamma p)\alpha = \alpha\gamma\alpha = \alpha$, $\alpha\tau = \alpha q\gamma p = \beta q\gamma p = \beta\tau$, $\tau\alpha = q\gamma p\alpha = q\gamma p\beta = \tau\beta$. Hence $\alpha \leq^{-} \beta$.
($\Leftarrow:$) Suppose $\alpha \leq^{-} \beta$. So there exists $\gamma \in S$ such that $\alpha\gamma\alpha = \alpha$, $\alpha\gamma = \beta\gamma$, $\gamma\alpha = \gamma\beta$. Let $p = \alpha\gamma$ and $q = \gamma\alpha$. Clearly, p is an idempotent in S and $1 - p \in l_S(\alpha)$. If $\eta \in l_S(\alpha)$, then $\eta p = \eta(\alpha\gamma) = 0$ so $\eta = \eta(1-p) \in S(1-p)$. Moreover, $p\alpha = \alpha\gamma\alpha = \alpha\gamma\beta = p\beta$. Similarly, q is an idempotent in S and $(1-q)(m) \in Ker(\alpha)$. If $m \in Ker(\alpha)$ then $\alpha(m) = 0$ which implies $\gamma(\alpha(m)) = q(m) = 0$. So for $m \in Ker(\alpha)$, $m = m - q(m) = (1-q)(m) \in (1-q)(M)$. Now, it is easy to see that $\alpha q = \beta q$ as above. □

Let 1_S be the identity of S then for the orthogonal idempotents of S there exists a decomposition of the identity 1_S, i.e., $1_S = e_1 + \cdots + e_n$. Let $1_S = e_1 + \cdots + e_n$ and $1_S = f_1 + \cdots + f_n$ be two decomposition of the identity of a ring S. Then, for any $\alpha \in S$, we have

$$\alpha = 1_S \alpha 1_S = (e_1 + \ldots + e_n)\alpha(f_1 + \ldots + f_n) = \sum_{i,j=1}^{n} e_i \alpha f_j,$$

and by the above

$$S = \bigoplus_{i,j=1}^{n} e_i S f_j.$$

Let $\alpha_{ij} = e_i \alpha f_j$ then one can write α as $e \times f$ matrix:

$$\begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{pmatrix}.$$

Theorem 2.6. *Let M be a module and let $\alpha, \beta \in S = End(M)$. Then, $\alpha \leq \beta$ if and only if there exist idempotents $p, q \in S$ such that the following hold:*

(1) $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $\beta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}$*. In fact you can easily see that* $\alpha_1 = \alpha$ *and* $\beta_1 = \beta(1-q)$*;*
(2) If $v \in Sp$ *and* $v\alpha_1 = 0$ *then* $v = 0$*;*
(3) If $v(m) \in q(M)$ *and* $\alpha_1 v(m) = 0$ *for any* $m \in M$ *and* $v \in S$ *then* $v = 0$*.*

Proof. Suppose that $\alpha, \beta \in S$ and $p, q \in S$ are idempotents as in Definition 2.1. We know that $\alpha = p\alpha = \alpha q = p\alpha q$, so for decompositions $1_S = p + (1-p)$ and $1_S = q + (1-q)$ one has $p \times q$ matrix

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let

$$\beta = \begin{pmatrix} \beta_4 & \beta_2 \\ \beta_3 & \beta_1 \end{pmatrix}$$

be $p \times q$ matrix. We get

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_4 - \alpha_1 & \beta_2 \\ \beta_3 & \beta_1 \end{pmatrix} = \begin{pmatrix} p(\beta_4 - \alpha_1) & p\beta_2 \\ 0 & 0 \end{pmatrix} = 0$$

by writing $p(\beta - \alpha) = 0$ in matrix form. Therefore, $p(\beta_4 - \alpha_1) = 0$ and $p\beta_2 = 0$. Since $p\alpha_1 = \alpha_1$, $p\beta_4 = \beta_4$, and $p\beta_2 = \beta_2$, we get $\alpha_1 = \beta_4$ and $\beta_2 = 0$. Analogously, from $(\beta - \alpha)q = 0$ we get $\beta_3 = 0$. So we have the statement (1). For the statement (2), suppose $v \in Sp$ and $v\alpha_1 = 0$. From (1), we $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}$ so $v\alpha = 0$ which implies $v \in l_S(\alpha) = S(1-p) = l_S(p)$, that is $v = vp = 0$. For the statement (3), suppose $v(m) \in q(M)$ and $\alpha_1 v(m) = 0$, we have $\alpha v(m) = 0$ by (1). So $v(m) \in Ker(\alpha) = (1-q)(M) = Ker(q)$ which gives $v(m) \in Ker(q) \cap q(M) = 0$.

For the converse, let the statements (1) − (3) hold for idempotents $p, q \in S$. It is easy to see that $p(\alpha - \beta) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\beta_1 \end{pmatrix} = 0$ and $(\alpha - \beta)q = \begin{pmatrix} 0 & 0 \\ 0 & -\beta_1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} = 0$. So statements (3) and (4) of Definition 2.1 are verified. Now, we prove $l_S(\alpha) = S(1-p)$. If $\gamma \in S(1-p)$ then one can write γ as $q \times p$ martix:

$$\gamma = \begin{pmatrix} 0 & \gamma_2 \\ 0 & \gamma_4 \end{pmatrix}.$$

Clearly $\gamma\alpha = 0$, that is $\gamma \in l_S(\alpha)$. For the reverse inclusion, let $\omega \in l_S(\alpha)$ then for $\omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}$,

$$\omega\alpha = \begin{pmatrix} \omega_1\alpha_1 & 0 \\ \omega_3\alpha_1 & 0 \end{pmatrix} = 0.$$

So, we get $\omega_1\alpha_1 = 0$ and $\omega_3\alpha_1 = 0$. But $\omega_1, \omega_3 \in Sp$ so $\omega_1 = \omega_3 = 0$ by (2). Hence, $\omega = \begin{pmatrix} 0 & \omega_2 \\ 0 & \omega_4 \end{pmatrix} \in S(1-p)$. Now we prove $Ker(\alpha) = (1-q)(M)$. For $p \times q$ matrix $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q \times q$ matrix $1 - q = \begin{pmatrix} 0 & 0 \\ 0 & 1-q \end{pmatrix}$ direct calculation gives the equality. More precisely, let $m \in Ker(\alpha)$, then $0 = \alpha(m) = \alpha(q(m)) = \alpha_1(q(m))$. But $q = 0$ by (3) which gives $Ker(\alpha) \subseteq (1-q)(M)$. For the reverse inclusion let $m \in (1-q)(M)$ that is $m = (1-q)(m*)$. Then, $\alpha(m) = \alpha((1-q)(m*)) = (\alpha(1-q))(m*) = 0$ by calculation of matrix representation of α and $1-q$. So we are done. □

3. The Minus Partial Order in Endomorphism Rings of Rickart Modules

According to [9], a right R-module M with $S = End_R(M)$ is called Rickart if the right annihilator in M of any single element of S is generated by an idempotent of S. Equivalently, for every $\varphi \in S$, $r_M(\varphi) = Ker(\varphi) = eM$ for some $e^2 = e \in S$.

Let us also denote:

$$LP(\alpha) = \{p \in E(S) : l_S(\alpha) = S(1-p)\},$$
$$RP(\alpha) = \{q \in E(S) : Ker(\alpha) = (1-q)M\}.$$

By Lemma 2.3, we get

$$LP(\alpha) = \{p \in E(S) : l_S(\alpha) = l_S(p)\},$$
$$RP(\alpha) = \{q \in E(S) : Ker(\alpha) = Ker(q)\}.$$

We also note that by [9] if M is a Rickart module, then S is a right Rickart ring. Thus, we see that if M is a Rickart module, then $RP(\alpha)$ is nonempty.

Lemma 3.1. *Let $\alpha \in S$, $p \in LP(\alpha)$ and $q \in RP(\alpha)$. Then,*
(1) $LP(\alpha) = \{\begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix} : p_1 \in pS(1-p)\}$,
(2) $RP(\alpha) = \{\begin{pmatrix} q & 0 \\ q_1 & 0 \end{pmatrix} : q_1 \in (1-q)Sq\}$.

Proof. Suppose that $p \in LP(\alpha)$ and $q \in RP(\alpha)$. Then, $(1-p)\alpha = 0$ and $\alpha(1-q) = 0$. Let $p' = \begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix}$ as a $p \times p$ matrix. We get $p'^2 = p'$. By using $1 = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$ and $(1-p') \in l_S(\alpha)$, we obtain

$$(1-p')\alpha = \begin{pmatrix} 0 & -p_1 \\ 0 & 1-p \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\omega = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix} \in l_S(\alpha)$ is a $p \times p$ matrix, then $\begin{pmatrix} \omega_1 & \omega_2 \\ \omega_3 & \omega_4 \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and $\omega_1\alpha_1 = 0$ $(\omega_1\alpha = 0)$, $\omega_3\alpha_1 = 0$ $(\omega_3\alpha = 0)$. Thus, $\omega_1 = \omega_1 p = 0$ and $\omega_3 = \omega_3 p = 0$. Also suppose that $\omega_1 = 0$ and $\omega_3 = 0$. So $\omega\alpha = 0$. From $\begin{pmatrix} 0 & \omega_2 \\ 0 & \omega_4 \end{pmatrix}\begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix} = 0$, we get $\omega \in l_S(p')$, so $l_S(\alpha) \subseteq l_S(p')$. By using $(1-p')\alpha = 0$ and Lemma 2.3, $l_S(p') = S(1-p') \subseteq l_S(\alpha)$. Then, $l_S(\alpha) = S(1-p')$ and $p' \in LP(\alpha)$.

Assume that $p' = \begin{pmatrix} p_2 & p_1 \\ p_3 & p_4 \end{pmatrix} \in LP(\alpha)$ be a $p \times p$ matrix. Also we have $l_S(\alpha) = S(1-p') = l_S(p')$ and $l_S(\alpha) = l_S(p)$. Thus,

$$0 = (1-p')p = \begin{pmatrix} p-p_2 & -p_1 \\ -p_3 & 1-p-p_4 \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} p-p_2 & 0 \\ -p_3 & 0 \end{pmatrix}$$

and $p_2 = p$ and $p_3 = 0$. Considering $l_S(\alpha) \subseteq l_S(p')$, we get

$$0 = \begin{pmatrix} 0 & \omega_2 \\ 0 & \omega_4 \end{pmatrix}\begin{pmatrix} p & p_1 \\ 0 & p_4 \end{pmatrix} = \begin{pmatrix} 0 & \omega_2 p_4 \\ 0 & \omega_4 p_4 \end{pmatrix}$$

and $\omega_4 p_4 = 0$ from $\omega_4 \in (1-p)S(1-p)$. If $\omega_4 = 1-p$, we get $p_4 = 0$. Hence, $p' = \begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix}$, this completes the proof. Similarly, we can obtain the statement (2). □

Corollary 3.2. *Let $\alpha \leq \beta$ for $\alpha, \beta \in S$ and let $p, q \in E(S)$ be the corresponding idempotents. Then,*

$$\{p' \in LP(\alpha) : \alpha = p'\beta\} = \{\begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix} : p_1 \in pS(1-p) \text{ and } p_1\beta_1 = 0\}$$

and

$$\{q' \in RP(\alpha) : \alpha = \beta q'\} = \{\begin{pmatrix} q & 0 \\ q_1 & 0 \end{pmatrix} : q_1 \in (1-q)Sq \text{ and } \beta_1 q_1 = 0\},$$

where β_1 is as in Theorem 2.6.

Proof. From $\alpha \leq \beta$, we have

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}.$$

If $p' = \begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix}$ with $p_1 \in pS(1-p)$ and $p_1\beta_1 = 0$ and by Lemma 3.1, then $p' \in LP(\alpha)$. We get

$$p'\beta = \begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} = \begin{pmatrix} p\alpha_1 & p_1\beta_1 \\ 0 & 0 \end{pmatrix}.$$

Since $\alpha_1 = p\alpha q$, we have $p\alpha_1 = \alpha_1$. Then,

$$p'\beta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix} = \alpha.$$

For the converse, assume that $p' \in LP(\alpha)$ and $\alpha = p'\beta$. From Lemma 3.1, we have $p' = \begin{pmatrix} p & p_1 \\ 0 & 0 \end{pmatrix}$. Thus, we write

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} = p'\beta = \begin{pmatrix} \alpha_1 & p_1\beta_1 \\ 0 & 0 \end{pmatrix}.$$

Hence, $p_1\beta_1 = 0$. □

We proceed to obtain an equivalent condition of the minus partial order in endomorphism rings of Rickart modules.

Theorem 3.3. *Let S be a Rickart ring and $\alpha, \beta \in S$. Then, $\alpha \leq^- \beta$ iff α, β have the matrix form as follows:*

$$\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}_{p\times q}, \quad \beta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}_{p\times q},$$

where $p, q \in E(S)$.

Proof. Let $\alpha, \beta \in S$ and $\alpha \leq^- \beta$. There exist $p, q \in E(S)$ such that $\alpha = p\beta = \beta q$. Then, $p\alpha = \alpha$ and $\alpha = \alpha q$. Thus, $(1-p)\alpha = 0$ and $\alpha(1-q) = 0$ which implies that $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}_{p\times q}$ where $\alpha_1 = p\alpha q$. Let $\begin{pmatrix} \beta_4 & \beta_2 \\ \beta_3 & \beta_1 \end{pmatrix}_{p\times q}$. Then,

$$\begin{aligned} \beta_4 &= p\beta q = p\alpha q = \alpha_1, \\ \beta_3 &= (1-p)\beta q = (1-p)p\beta = 0, \\ \beta_2 &= p\beta(1-q) = \beta q(1-q) = 0. \end{aligned}$$

Thus, $\beta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}_{p\times q}$.

For the converse, let α and β have the stated above matrix forms. From $p\alpha_1 = pp\alpha q = \alpha_1 = p\alpha qq = \alpha_1 q$, then

$$\begin{aligned} p\beta &= \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}_{p\times p}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}_{p\times q} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} = \alpha, \\ \beta q &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}_{p\times q}\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}_{q\times q} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} = \alpha. \end{aligned}$$

So, $\alpha \leq^- \beta$. □

Recall that R is a right Rickart ring with only two idempotents 0 and 1 iff R is a domain (see [9, Remark 4.10]).

Theorem 3.4. *Let S be Rickart ring and $\alpha, \beta \in S$. Then, $\alpha \leq \beta$ if and only if there exists $1_S = e_1 + e_2 + e_3$ and $1_S = f_1 + f_2 + f_3$ such that the following conditions hold*

(1) $\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as $e \times f$ matrix,
(2) If $v \in Se_1$ and $v\alpha_1 = 0$, then $v = 0$,
(3) If $v \in f_1S$ and $\alpha_1 v = 0$, then $v = 0$,
(4) If $v \in Se_2$ and $v\beta_1 = 0$, then $v = 0$,
(5) If $v \in f_2S$ and $\beta_1 v = 0$, then $v = 0$.

Proof. ($\Leftarrow$:) This implication follows from Theorem 2.6.

($\Rightarrow$:) Let $\alpha \leq \beta$. Considering Theorem 2.6, we have $\alpha = \begin{pmatrix} \alpha_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}_{p \times q}$ as $p \times q$ matrix, if $v \in S\,p$ and $v\alpha_1 = 0$ then $v = 0$, if $v(m) \in q(M)$ and $\alpha_1 v(m) = 0$ for any $m \in M$ then $v = 0$. Since S is Rickart ring, there exists $r, s \in E(R)$ such that $l_S(\beta_1) = S(1 - r) = l_S(r)$ and $r_M(\beta_1) = Ker(\beta_1) = (1 - s)(M)$. From $\beta_1 \in (1 - p)S(1 - q)$, we write $p\beta_1 = 0$. Then, $p \in l_S(\beta_1) = l_S(r)$, so $pr = 0$. Suppose that $r' = r - rp(1 - r) = r - rp = r(1 - p)$. We get $r'^2 = (r - rp)(r - rp) = r - rp = r(1 - p) = r'$, so $r' \in E(S)$. On the other hand, if $\omega\beta_1 = 0$, then $\omega \in l_S(\beta_1) = l_S(r)$ and $\omega r = 0$. So $\omega r' = 0$. Then, we have $(1 - r')\beta_1 = 0$ from $\beta_1 = r\beta_1$. Thus, $l_S(\beta_1) = l_S(r') = S(1 - r')$. Also if $pr = 0$, then $p \in l_S(r) = l_S(\beta_1) = l_S(r')$ and $pr' = 0$. Then, $r'p = r(1 - p)p = 0$. Set $e_1 = p$, $e_2 = r'$ and $e_3 = 1 - p - r'$. Thus, $1_S = e_1 + e_2 + e_3$ is decomposition of the identity of the ring S and by $l_S(\beta_1) = l_S(r') = S(1 - r')$, we conclude that $\omega\beta_1 = 0$ implies $\omega = 0$ when $\omega \in S e_2$.

Moreover, set $f_1 = q$, $f_2 = (1 - q)s$ and $f_3 = 1 - f_1 - f_2$. We get $1_S = f_1 + f_2 + f_3$ decomposition of the identity of the ring S. Thus, we obtain the statement (5). Since $e_1 = p$ and $f_1 = q$, the statement (2) and (3) are satisfied from Theorem 2.6. We have $e_2\beta_1 f_2 = r(1-p)\beta_1(1-q)s = r\beta_1 s = \beta_1 s = \beta_1$ since $l_S(\beta_1) = S(1-r)$ and $Ker(\beta_1) = (1-s)(M)$. This show that the statement (1) holds. □

Moreover, from Theorem 3.4 it follows that the statement (1) – (5) are equivalent to $e_1 \in LP(\alpha)$, $e_2 \in LP(\beta - \alpha)$, $f_1 \in RP(\alpha)$ and $f_2 \in RP(\beta - \alpha)$.

Corollary 3.5. *Let S be Rickart ring and $\alpha, \beta \in S$. Then, $\alpha \leq \beta$ iff $\beta - \alpha \leq \beta$.*

Theorem 3.6. *Suppose that S is Rickart ring and $\alpha, \beta \in S$. Then, $\alpha \leq \beta$ if and only if there exists idempotents $e_1 \in LP(\alpha)$, $e_2 \in LP(\beta - \alpha)$, $f_1 \in RP(\alpha)$ and $f_2 \in RP(\beta - \alpha)$ such that $e_1 e_2 = 0$ and $f_2 f_1 = 0$.*

Proof. ($\Rightarrow$:) Suppose that $e_1 \in LP(\alpha)$, $e_2 \in LP(\beta - \alpha)$ and $e_1 e_2 = 0$. Then, $(1 - e_1)\alpha = 0$ and $(1 - e_1)(\beta - \alpha) = 0$, so $e_1\beta = e_1\alpha + e_1(\beta - \alpha) = \alpha + e_1 e_2(\beta - \alpha) = \alpha$. Also assume that $f_1 \in RP(\alpha)$, $f_2 \in RP(\beta - \alpha)$ and $f_2 f_1 = 0$. We write $\alpha f_1 = \alpha$ and $\beta f_1 = \alpha f_1 + (\beta - \alpha) f_1 = \alpha + (\beta - \alpha) f_2 f_1$. Thus, we get $\alpha \leq \beta \in S$.

($\Leftarrow$:) It follows from Theorem 3.4. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author confirms sole responsibility for the following: study conception, data collection, analysis and interpretation of results, and manuscript preparation. I confirm that the data supporting the findings of this study are available within the article.

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