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Methodology, data collection, data analysis, review and editing

Abstract

This study was carried out in order to determine how the 3rd grade students of the Department of Elementary Mathematics Education structured their "*if and only if propositions*". The data were obtained by examining the students' answers given to the midterm exam questions and discussing the solutions with the students in the classroom. The study is a case study. As a result of the application, it was found out that the students had difficulty in determining the parts of the hypothesis that are included in "*if and only if*" proposition and therefore dividing the proposition into two "if" proposition. Some students think that the part or parts given as hypothesis should also be proved. When defining propositions, in addition to their "*if and only if* propositions", it is suggested to define new types of proposition in the form of "hypothesis-containing *if and only if* propositions".

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Research Article

An Ontological Study on Proof: If and only If Propositions*

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Abstract

This study was carried out in order to determine how the 3rd grade students of the Department of Elementary Mathematics Education structured their "*if and only if propositions*". The data were obtained by examining the students' answers given to the midterm exam questions and discussing the solutions with the students in the classroom. The study is a case study. As a result of the application, it was found out that the students had difficulty in determining the parts of the hypothesis that are included in "*if and only if*" proposition and therefore dividing the proposition into two "if" proposition. Some students think that the part or parts given as hypothesis should also be proved. When defining propositions, in addition to their "*if and only if* propositions", it is suggested to define new types of proposition in the form of "hypothesis-containing *if and only if* propositions".

Keywords: Theorem, if and only if proposition, cognitive structuring, definition, proof

1. INTRODUCTION

People have ideas about what is going on around them. When they claim that these ideas are true, they form a proposition. People will demand the proof from those who put forward proposition. Evidence is necessary for people to accept the correctness of ideas, that is, for the persuasion of the immediate environment. For this purpose, those who make the claim can give examples, model the situation with the help of materials, conduct experiments, benefit from graphics and various representations (Sevgi & Kartalcı, 2021). However, mathematicians will want a formal proof to accept the truth of a proposition (Polster, 2004). In the world of science, the proof is not just about showing the truth of the claim. The proof also shows why the claim is true and convincing (Hanna, 2000). The proof also has some functions as verification (De Villiers, 1999), explanation, verification of definitions, systematization (Barendregt & Wiedijk, 2005), discovery and communication. Therefore proof is an important tool for mathematicians and for the execution of mathematical science (Knuth, 2002). Proof is included in mathematics curricula and the importance of proof is emphasized (Herbst, 2002).

Proof types be defined and classified in different ways in the literature. For example, considering the purpose of proof, there are four proof types: heuristic, descriptive, and exploratory and (Hanna, 2000; Reis & Renkl, 2002). Hemmi (2010) classifies proof approaches as verification /explanation, induction/deductive, intuitive /formal, proof structure open/proof structure not open, while Tall (1999) classifies proof as enactive proof and formal proof. Harel and Sowder (1998) classify students' proof schemes as "externally based", "empirical", and "analytic" proof. These classifications/definitions are an indicator and product of the efforts of mathematicians and mathematics educators to understand proof. These classifications/definitions are also expressions of

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the difficulty of proof and teaching proof. In addition, another subject of interest in proof studies is the processes that students in different age groups go through while proving and their ability to prove (Knuth, 2002). One of the reasons for this increase in studies on proof is that it is very difficult to gain proving competence at every level (Jones, 2000). Even at the undergraduate level, the proof is perhaps one of the most difficult issues to understand (Arsac, 2007). No matter how difficult it is to prove and teach proof, it is impossible to discourage mathematicians from their goal of acquiring these competencies in students. But most mathematicians tend to constantly postpone teaching how to make proof to the next class or level of instruction. Research has shown that students should encounter proof activities as early as possible in order to prove proficiencies (Doruk & Kaplan, 2015; Harel & Sowder, 1998; NCTM, 2000; Tall, 2014). In this sense, proof activities are also could be include in the kindergarten period too. In order to form the basis of the proof, classification, matching and comparison concepts are given in the preschool period. Students are expected to be able to use proof methods and techniques both in primary, secondary and high school (NCTM, 2000). Although it is not easy for students to gain the proficiency of proving at the primary level, it is seen that it is not easy to gain them at advanced levels too. For example, let's consider the level of Van Hiele's Understanding of Geometry called "inference about life (order)".This third level coincides with the last years of the second level of primary education and the high school years. It is known that at this level, students can follow the making of the proof, but they cannot prove the proof themselves (Usiskin, 1982). Another factor that reasons the fact that the proof does not get easier as the stages progress is that the expectations for proof increase continuously as the stages increase. As the levels of education progress, the propositions to be proved quickly become more abstract. The propositions transforms into an abstract structure that requires a formal proof, cannot be modeled, cannot be drawn graph-shape, or cannot be tabulated. Undergraduate students are having a higher level of mathematical knowledge and cognitive competencies. However, theorems at this stage are often more abstract and require even more formal proof. So propositions become much more difficult to prove. It is known that undergraduates had difficulty making proof (Doruk, 2019; Oflaz, Bulut & Akcakin, 2016; Sema & Şenol, 2022). There has also been a lot of research on the views of mathematics teacher candidates and teachers towards making proofs (Doruk & Kaplan, 2015; Doruk, Özdemir & Kaplan, 2015; Knuth, 2002; Yopp, 2011). The literature shows that there are various difficulties in making proofs by students. These difficulties are; thinking they can not prove; fear of proving and dislike of proving (Anapa & Şamkar, 2010; De Villiers, 1999a; Gökkurt, Deniz, Akgün & Soylu, 2014; Jones, 2000; Sevgi & Kartalcı, 2021), not understanding the reasons and benefits of proof (De Villiers, 1999), and not knowing how to use proof (De Villiers, 1999; Moore, 1994), their inability to understand the nature of proof, mathematical rules, proof techniques and strategies (Gibson, 1998); their inability to use mathematical language and logical proofs correctly (Moore, 1994). There are various studies too in the literature that deal with the proving process (Harel & Sowder, 1998; Varghese, 2011). There are other studies in the literature that deal with the proof process from different perspectives.

Examination of the proving process can offer opportunities to overcome many of the difficulties of learning and teaching proving. This study was carried out to examine the proof at the university level. In this study, different from the difficulties mentioned above, a proving difficulty that is not encountered in the literature will be discussed. This difficulty is an ontological difficulty, which arises from not defining the "*if and only if*" proposition, which is perhaps the most abstract of the propositions, in sufficient detail. Because this compound proposition contains the "and, or, if" propositions defined before it. In the scope of the study, the effect of the deficiency resulting from the definition of "*if and only if*" propositions on students' ability to prove will be examined by the case study method. This study is a case study, limited to the students and the course where the researcher is conducting the course. With this study, it will be possible to obtain inferences about the teaching of a subject that is perhaps the most difficult to understand for students in advanced mathematics education. And new definitions and naming have been proposed to make the subject more understandable. In addition, this study is thought to be a study that exemplifies the importance of dealing with pure mathematics subjects by field educators. For this purpose, the third-year students of the Department of Elementary Mathematics Teaching were examined how they structured "*if and only if*" proposition. Namely, how do university third-year students make sense of propositions of type theorem1? question will be tried to be answered with this study. As a result of this review, useful information can be offered obtained for students and teachers about how the proposition can be learn and taught.

1.1. Some Concepts Mentioned in the Study

Definition of "Compound proposition": When two propositions are given, these can be combined with links such as *"and (∧)", "or (v)", "if (=>)"*, and "*if and only if (⇔)* " to obtain new propositions. Such propositions are called compound propositions.

Definition of "if and only if proposition": It is defined as $p⇔q = p=>q∧q=>p$

Each of the propositions is indispensable in terms of mathematics and logic. Perhaps the most complex of propositions is "*if and only if*" proposition among the compound propositions.

An example of a proposition to be examined in the study:

Types of propositions at the university level are defined in the Abstract Mathematics Lesson. No further or more complex form of a proposition than the "*if and only if*" proposition in later years (in later courses) is formally defined.

Within the scope of this study, students' understanding of their "*if and only if*" proposition will be examined. Thus, the ontological (arising from the definition) adequacy of "*if and only if*" propositions will be critical. The prototype of the theorems to be discussed in this study is Theorem1.

Theorem1: *Let G be a group. A necessary and sufficient condition for a non-empty subset H to be a subgroup of G is that* $ab^{-1} \in H$ *for* $\forall a, b \in H$.

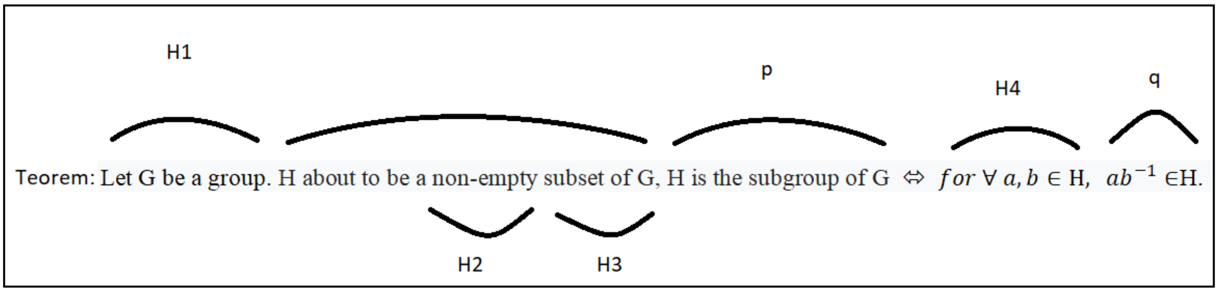

Figure: Theorem1

Explanation of the symbols in Figure:

H1: *It is assumed that G is a group.* This information is given by the authority to the person who will solve the theorem. This information can be used in both directions of proof of the theorem, but it is not wanted to be proved.

H2: It is the assumption that *H is different from the empty set*. This information is a kind of information given by the authority to the person who will solve the theorem. This information can be used in both directions of the theorem's proof, but it is not wanted to be proved.

H3: It is the assumption that *H is a subset of G*. This information is a kind of information given by the authority to the person who will solve the theorem. This information can be used in both directions of the theorem's proof, but it is not wanted to be proved.

H4*:* It denotes that *a and b are given as elements of H*. The authority also provided this information. Although it may seem less powerful than other hypotheses, due to the quantifier "∀" it is actually as important in making proofs as other hypotheses.

***p*:** If taken as *"H is a subgroup of G"* when expressing a hypothesis (during a proof of direction =>); if taken as "*Is H a subgroup of G?"* it means a proposition (during a proof of direction <=).
q: If taken as "$ab^{-1} \in H$" when expressing a hypothesis (during a proof of direction <=); if taken as "Is $ab^{-1} \in H$?" it means a proposition (during a proof of direction =>).

The purpose of this study is to examine how students perceive the "*if and only if*" proposition through the example of theorem1.

2. METHOD

2.1. Research Design

The research is a case study. In this case study research, the researcher is also a teacher. In this study, the researcher deals with a situation that he encountered during his own lecture with a group of students and does not aim to generalize. The results obtained may not be valid for students studying at other universities, in this sense, the case study are a suitable method for this research. The teacher is a researcher, teacher and learner in the process. In the process, it is aimed to examine, criticize and change himself, his students, his students' learning products and the learning environment. The data of this study were collected in 4 stages. These stages:

Stage 1: The researcher began his research with the determination that several of the students had misinterpreted especially the right direction (<=) while answering "if and only if" questions on the midterm exam paper an academic year in which he conducted linear II and introduction to Algebra courses. The researcher then more detailed reviewed and analyzed all the students' exam papers for the Introduction to Algebra course. He found that there were significant problems in the proof of "*if and only if*" theorems of the students.

Stage 2: While preparing the lecture notes, the researcher determined theorems of the theorem1 type. And he designed the course in such a way that he can examine how they translate the theorem into "if" statements during the proof of these theorems.

Stage 3: After the researcher has studied the subjects, proved the basic concepts and propositions, he wrote the theorem1 type propositions on the board: (1) asked the students what the type of proposition was. (2) Asked them to convert the theorem into two "if" propositions. (3) He observed how they fragmented the propositions and interviewed 3-4 students about the event. (4) One of the students who thought of breaking down as $p \Leftrightarrow q$ was asked to write their answer on the board. (5) It was discussed whether the answer was correct in the class. (6) the students were asked whether they could prove *H4* acceptance while solving => direction and whether they could prove *H1, H2, H3* acceptance during the proof of the <= direction. (7) The class was asked to interpret the student's answer and other possible answers were written on the board.

Stage 4: The situation was discussed with the students, thus aimed to change the students' understanding.

2.2. Participant Group

The Department of Elementary Mathematics Education, 2012-2013 academic year, third grade students constitute the participant group of the study. In the first phase of the study, the midterm exam papers of 86 students who took the Introduction to Algebra course were examined. These students are students of two classes: the evening class program and the daytime class program. The second and third phase of the study was conducted with 47 students and the fourth phase with 44 students. The difference in the number of students is due to the attendance status of the students at the time the relevant stage of the study was carried out.

The participants of are study is the students to whom the researcher teaches. In the first stage, the exam papers of a total of 86 students in two classes were handled. In the second and third stages, the study was carried out with 47-44 students who attended the class that week.

2.3. Data Collection Tools

The data of the first stage of this study were obtained from the answers given by the students in the midterm exam to the question expressed in figure 1. The data of the first stage of this study were obtained from the answers given by the students in the midterm exam to the question expressed in figure 1. The data from the second and third stages were obtained by asking the students the theorems selected by the researcher, structurally similar to the one in figure 1. That is, in these two stages, the observations, the interview and the answers in the students' notebooks constitute the data of the study. The data of the fourth stage of the study were obtained by solving the same question used in the first stage on the blackboard in the classroom.

The validity of research is closely related to the extent to which the measurement tool reflects the subject to be measured. In this study, the same theorem was used in the 1st and 4th stages. In the 2nd and 3rd stages, structurally identical theorems were used. This is an element that increases the validity of the research.

The reliability of the research is directly related to the ability to get the same results from the measurements again and again. Similar four questions were used in the study. The rapport of the answers in the exam papers on the proof of the theorem with the data obtained from the class indicates the reliability of the study. In other words, data triangulation was provided by document analysis, interview-group interviews and observations.

2.4. Data Analysis

The answers of the students in the midterm exam papers and the answers in the classroom applications were analyzed according to the information in Figure 1, where the theorem1 proposition is explained.

Similarly, the data recorded in the third and fourth stages were analyzed by determining how the students made sense of the expressions H1, H2, H3, H4, p, q.

Problem Status

When we examine theorem1, the proposition is an "*if and only if*" proposition. During the proof, the proposition must be transformed into two "if" propositions. If we write the right direction, (=>) symbolized as to be proved; If we write the right direction, (=>): *If a non-empty subset H of a group of G is a subgroup of G "Is* $ab^{-1} \in H$ *for* $\forall a, b \in H$*?"*

Proof of the right direction of the theorem (=>): before examining the proof, examine fig.1.

In the **right** direction of the theorem, students need to accept that *G is a group (H1), H is a non-empty set (H2), H is a subset of G (H3), H is a subgroup and* $a, b \in H$ (acceptance of *p*' derived from the proposition *p*) and to answer the question *"Is* $ab^{-1} \in H$*?"* by using this information.

Let's answer the question "*Is* $ab^{-1} \in H$ *for* $\forall a, b \in H$*?"*

Let's take the elements *a and b from the set H* (*by the proposition H4*) since *b* is an element of *H*, we know that *H* also contains the element b^{-1}. *If H is a subgroup of G, also, H is a group and in a group*, each element has its opposite (we used the expression *p* when making this inference). Now we have elements *a* and b^{-1}. Moreover, both are elements of *H*. If we process these two elements according to the process of *H*, we get (ab^{-1}). Is this new element *H*'s element? Yes, its element. Because, if *H* is a subgroup of *G*, *H* is a group and has the property of closure so, ab^{-1} is the element of *H*. Then the proof ends.

In fact, in the **right** direction of the theorem (=>), the proposition can be proved without using every given information (*H4* and *p*' statements).

In order to complete the proof, it must be proved in the left direction symbolized as (<=).

Proof of the second direction of the theorem (<=)**:** before examining the proof, examine fig.1.

They need to answer the question "*If a non-empty subset H of a group G provides the condition* $\forall a, b \in H \Rightarrow ab^{-1} \in H$*, is H the subgroup of G?"*

In order for *H* to be a subgroup of *G*, tasks specified in the following stages must be fulfilled:

1: Whether *H* is different from the empty set should be examined. It is given as *H2* information.
2: *H* must be a subset of the *G*. It is given as *H3* information
3: H must have closure property.
4: Elements of *H* must be provided associative property.
5: The identity element must also be an element of the set *H*.
6: Every element in the set *H* must have an inverse.

Proof (<=) : Let's take the elements *a and b* from the set *H* (by the proposition *H4*). The information $ab^{-1} \in H$ for $\forall a, b \in H$ is given to us (from proposition *q*). For each element of *H*, this property is provided, so I can take it as $b=a$. Thus, it becomes $aa^{-1} \in H$. So $aa^{-1} = e \in H$ is obtained (The answer to the fifth question was obtained by the proposition *H4* and *q*).

After this stage, if I take $a=e$ and $b=a$ then we get $ea^{-1} \in H$ for $\forall a \in H$. So the inverse of every element will also be in *H* (the answer to the sixth question was obtained by the proposition *q*').

The associative property will be provided since every element of *H* will be an element of *G* (*H3*) and *G* is a group (*H1*).

We have obtained by answering the sixth question that b^{-1} will also be the element of *H* for the element *B*. If we take the elements *a* and b^{-1} instead of *a* and *b* in the expression *q* provided for each element, we get that $a(b^{-1})^{-1}= ab$ is the element of *H*. In other words, closure property will also be provided (by the proposition *q*' and 6th).

As can be seen, the information *H1, H3, H4* and *q*' (hypotheses) must be used to make (<=) the direction of the proof. In other words, students need to know that *G* is a group and that *H* is a different set from the empty set given to them as information. It is also necessary to recognize that *H* is given to them as a subset of *G* and that for $\forall a, b \in H$ the ab^{-1} is the element of *H*, and they examine whether *H* is a subset of *G*.

In light of the above explanations, it is seen that the proposition is not a standard proposition in the form of $p \Leftrightarrow q$ (as defined in abstract mathematics books). Because the proposition is not fragmented in the form of *p=>q* and *q=>p*. Otherwise, the fragmentation would be as follows:

(=>): If a non-empty subset *H* of a group of *G* is a subgroup of *G*, is $ab^{-1} \in H$ for $\forall a, b \in H$?

(<=): If $ab^{-1} \in H$ *for* $\forall a,b \in H$, is a non-empty subset of group *G* a subgroup of *G*?

In fact, this fragmentation of the proposition may not be a problem for a mathematician who specializes in proof. However, for a non-specialized theorem solver (for a student who learns the proof), this can be a big problem. Because we see that there are 4 components within the theorem that we can express as acceptance (see. Fig1). The expressions *H1, H2, H3 and H4* are the acceptance that will be used by the students during the proofs (statements whose accuracy has been given by the authority-to be considered true). Also, there are two other expressions that we will symbolize as *p* and *q*. These two statements are propositions in the theorem (claims in need of proof). But when this "*if and only if*" proposition is fragmented into two "*if*" propositions, *p* turns into an assumption for the direction (=>), we call it *p*' and *q* is a proposition; for the direction (<=), *q* turns into an assumption, we call it *q*' and *p* is a proposition.

But if the students are not aware of what we can call the hypothesis (acceptance) in the fragmentation of the proposition; then the students may also try to prove the propositions given to them. This will cause the propositions to become incomprehensible, difficult and impossible, and there will be no proof.

Let us try to explain again the problem expressed in this paragraph through theorem1.

If students fragment the **right** direction of the theorem (=>) as *if a non-empty subset of H of the group G is a subgroup of G, "Is* $ab^{-1} \in H$ *for* $\forall a, b \in H$*?";* if they fragment the second direction of the theorem (<=) *as if "*$ab^{-1} \in H$ *for* $\forall a,b \in H$*", "Is G a group?", "Is it a subset of non-empty G?",* *"Is it a subgroup of G?"*; or while proving the second direction, instead of asking only "Is *H the subgroup of G?*" question, if they are trying to answer the questions such as *"Is G a group?", "Is G different from the empty set?", "Is H a subset of G?"*, then the students start to solve the second direction of the proposition by asking questions that cannot be answered. As a result, they cannot solve the theorem. In this sense, it is necessary to examine how students make sense of theorems of type theorem1. "How do university third-year students make sense of propositions of type theorem1?" question will be tried to be answered with this study.

3. FINDINGS

The findings of this study are given in 4 titles in accordance with the 4 stages carried out.

Table 1. Findings on stage 1:

		n	**Percent (%)**
No Answer		57	66
Only answer the direction (=>) correctly	No attempt to make the <= direction	5	14
	Tried attempt to make the <= direction but no result	7	
Try attempt to make the both direction		13	15
Solve the theorem correctly		4	5
Total		**86**	**100**

At this stage, the written papers of the 86 students who took the midterm exam were examined. 57 (66%) of the 86 students did not answer the question on the paper. 12 (14%) students were only able to answer the direction (=>) correctly. Of these students, 5 (5.8%) students have never attempted to make the <= direction (there is no wording that has been slandered or deleted on the paper) while 7 (8.1%) students scribbled and (or) tried to delete their answers but did not reach the result. 13 (15%) students tried to do both directions but did not reach the correct results. Only 4 (5%) students were able to solve the theorem correctly.

In other words, while 70 (81%) students could not solve theorem1 at all, 12+4 (18.6%) students were able to solve the direction=>. Only 4 (4.7%) students were able to solve the <= direction. Each student who solved the <= direction (4 people) was able to prove the => direction as well. At this stage, the students ' perceptions of theorem1 could not be determined exactly because they usually devote their time to more familiar-simple questions. However, the researcher found that there were some problems mentioned in the proof of the theorem, especially in the case of the problem in the <= direction. The second stage was designed to understand the prevalence extent and causes of distress. How the students did the proof was examined by handing over the papers many times. In order to see the accuracy of the information obtained in the later stages of the study, the written papers were re-examined many times.

Findings on stage 2:

At this stage, a theorem similar to theorem1 was written on the board and students were asked to say what the type of theorem was. This process took 2 weeks and 6 hours of lessons. The students ' perceptions of 3 similar theorems were examined during the process.

Findings from asking for first similar theorem: First, the theorem was written on the blackboard board. When the students were asked about the type of theorem, all of the students who were in the course at the time (the application was conducted with about 47 students in one class) stated that the proposition was an "*if and only if*" type proposition (students were asked to write the situation in their notebooks). In the sequel, he was asked how to resolve the proposition. Students have

written that the proposition should be converted into two “if” propositions. When it is said to break the theorem, a large number of students (more than 30) have written and broken the proposition as the k⇔r proposition. About 10 students suggest;
After writing *H1+H2+H3* to the first line they break the proposition to the next bottom line as
p = > H4 + q and,
H4 + q=>p Let us call this structure the possible structure to be true.

The researcher, who wants to focus on the <= dimension, approached 4 of the students who thought to propose as *p⇔q* and fragmented the proposition as *H4 + q = >H1+H2+H3 + p*. He asked “Are we going to prove *p* while doing this direction of the proof?", "Are we going to prove *H1?"*, “Are we going to prove *H2*?", “Are we going to prove *H3*?". Two of the students answered, “*I don't know*". The third student although stated that the answer to *H2* and *H3* should be sought in the first place, *H1* was already known, but after a few seconds the researcher walked away and said, “*Excuse me, teacher, I couldn't be sure*.” The fourth student said that answers to *H2* and *H3* should be sought, that in order to prove that *H* is a subgroup, it should be examined whether it is different from the empty set and that *H* is a subset. When the researcher asked “*Are you sure?*" he turned his notebook back a few pages, opening the conditions of being subsets and showing the conditions. As the last word, “*teacher, I do not know you wrote like this”* and he smiled.

Findings from asking for second similar theorem: The application was similar to that of the first application. All of the students have said that the type is an "*if and only if*" proposition. They have written the => and <= directions. Similarly, four students were interviewed who thought and fragmented the proposition as *p⇔q*. The researcher asked them “Are we going to prove *p*?”, “Are we going to prove *H1*?”, “Are we going to prove *H2*?”, “Are we going to prove *H3*?”. The first student answered as “*We will prove p, I do not know the others*”. While the second student first said that answers to *H1, H2, and H3* should be sought, after a little hesitation, he said: “I think we would not search answer for *H1*”. Then he has sequenced the questions, “teacher, you asked a similar question in the last lesson, is there a problem? Why did you ask again? What is the truth?”

Another important point here is that although the researcher has solved the theorem correctly after the first application, students generally do not have an increase in awareness about the proof of the theorem, the fragmentation of the theorem and the questions to be answered.

Findings from asking for third similar theorem: The application was similar to that of the first and second application. All of the students have said that the type is an "*if and only if*" proposition. Most of the students have written the => and <= directions. Some of the students (about 10 people) refused to do so. These students’ behavior may have been due to the fact that they did not see a problem with their actions in the second practice. In other words, the students reacted to this application of the researcher, which, according to them, they could not make sense. 5 students have broken down the proposition correctly. At this stage, 4 students were interviewed who fragmented the proposition in the form of *p⇔q*. Three of these students said “*I do not know*” while one of them answered, “*may or may not*".

The researcher thinks that at the end of the second application, the student who reacted as “*why did you ask similar questions again*” was effective in reacting to the other students. As a result, the applications made up to this point indicate that the students have problems in perceiving theorems of the theorem1 type. At this point, Stage 2 has been terminated.

Findings on Stage 3:

After the third similar theorem was asked, theorem1 was written on the board and the students were noted by saying “Shall we solve the exam question again”. Since the exam grade was not announced, the students turned to the theorem in a related way. They were asked to perform the same operations again for theorem1, and a student who broke the theorem in the form k⇔r wrote on the board the => and <= directions of the theorem. When he was asked which questions he would seek while proving his <= direction, "*I don't know*" he answered. Upon this, the proposition was written on the board as in Figure 1 and it was asked which of these statements should be answered during the proof. The answers are as follows:

The proposition p must be answered: 44 students (100%, all of those who were in class that day)

H1 must be answered: 5 students (11.3%)

H2 must be answered: 9 students (20.4%)

H3 must be answered: 8 students (18.1%)

It was observed that some of the students did not raise their hands or declare ideas when answering *H1, H2 and H3*. When the percentages are taken into consideration, it is observed that the percentage of distress that is determined at least 8% in the midterm exam is at least 20.4%. This phase of the study lasted 1 class hour.

Findings on stage 4:

In the course, the students should be informed about the correct breakdown. When students were asked their thoughts about the correct answer, it was observed that some students could not perceive the situation, that is, they did not understand the wrongness of their answers. Although some students used expressions of approval, the researcher had the impression that they did not actually understand the event. Some students have stated that they have never done this conversion before, that they have not encountered a similar situation in the book, and that no similar teaching application has been made by the researcher (despite the examples in Stage 2). A few students with high self-confidence, albeit in a slightly low voice "*teacher, we couldn't do it, but you didn't teach it*" they accused. This phase of the study lasted 1 class hour.

4. DISCUSSION and CONCLUSION

Some "*if and only if*" propositions include acceptance or hypothesis (s) alongside the necessary condition or sufficient condition. This type of propositions should be given a new name and taught to students in Abstract Mathematics course or in Introduction to Algebra-Algebra I course. These propositions can be called "one-sided hypothesized *if and only if* proposition".

Some "*if and only if*" propositions include acceptance or hypothesis (s) alongside the necessary condition and sufficient condition. These propositions should be taught to students in Abstract Mathematics course or in Introduction to Algebra-Algebra I course. This type of proposition can also be called "hypothesized if and only if proposition".

"What is an assumption?" should be explained, defined and exemplified in Abstract Mathematics course. It should also be exemplified and discussed with students how the proposition turns into acceptance when "*if and only if*" propositions are broken down into "if" propositions. It is known that examining and discussing the correctness or falsity of proof by students is effective in students' understanding of proof (Doruk, 2019). Moore (1994) stated that undergraduate mathematics students have difficulties such as being unable to understand and use mathematical language and symbols. In this sense, the idea and the literature agree that theorem1 type propositions proposed in this study should be explained and discussed in more detail.

It is understood that is not sufficient for students to follow only the proofs of the teachers to learn to prove high-level theorems of type theorem1 (especially from Stage 3). For this reason, especially during the course process, students should be provided with the opportunity to make proof with the help of the teacher. The theorem-proving ways of pre-service mathematics teachers in this study show that their perspectives on the nature of proof do immature. This result is compatible with the literature (Güner, 2012).

The situation of these students can also be examined thus the difficulties in learning theorems of type theorem 1 can be determined in more detail. In-class discussions increase students' understanding of the theorem-proof. During the proof of the theorems, students should be asked questions that will improve their understanding by focusing on critical points and discussing differences of opinion by taking the students' opinions.

This study shows us that, although there are some ontological problems with the definition of theorems, solving the theorem is a high-level cognitive competence. Therefore, for the student to reach a level of competence in solving the theorem, it is imperative that the student reveals a consciousness, work and spend labor. It is clear that defining and exemplifying these new types' theorems will not serve as an elixir either. The type of theorem already discussed here is usually theorems of the type encountered in undergraduate courses in the third year. It is also known that students at lower levels have problems with proving the "*if and only if theorems*".

The instructors who carry out the course can repeat the study in the Faculty of Science while the students are studying. It is known that gaining the proficiency of proving is important for graduates of both faculties (Anapa & Şamkar, 2010)

Anapa and Şamkar (2010) suggest that activities that provide students with proving skills should be done more in high school years. They also predict that asking questions about proof in central selection exams will affect the adequacy of proof. In this sense, the effect of the education of students on proof at the high school level on students' ability to prove at the university level can be examined by considering school types.

It is seen that some of the students who realized their mistakes after the 4th stage held the instructors responsibly and made various accusations against them. Actually, the accusations indicate that the instructors who conduct the courses should know pedagogical content knowledge.

It is seen that the steps followed by the students while proving propositions are procedural. It is seen that they do not question the stages of the proof and the reasons for the actions taken. It is known that pre-service teachers choose to memorize while proving (Anapa & Şamkar, 2010).

The level of failure in proving the proposition discussed in this study is high. The failure of undergraduate students to prove is defined as a disappointment by Jones (2000). In other words, it is known that there are studies at the undergraduate level where similar results were obtained with the results obtained in this study conducted at the university level.

Doruk, Özdemir and Kaplan (2015) state that pre-service mathematics teachers lack self-confidence in proving. In this sense, the relationship between pre-service teachers' difficulties in proving their "*if and only if propositions*" and self-confidence can be investigated.

Students' problems regarding the solution of theorems should be resolved as soon as possible by discussing and giving feedback on the exam papers. Although students have the right to object to the exam papers after the exam, they do not have the right to look at the exam paper. However, the researcher thinks that students should have a legal right to look at how their exam papers are scored and where they have mistaken. The instructors should give the student detailed information and feedback about the exam paper. The material and legal infrastructure of this should be established.

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