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On the Spectrum of the Non-Selfadjoint Differential Operator with an Integral Boundary Condition and Negative Weight Function

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Abstract

In this paper, we shall study the spectral properties of the non-selfadjoint operator in the space $L^2_\rho(\mathbb{R}_+)$ generated by the Sturm-Liouville differential equation

$$-y'' + q(x)y = \omega^2 \rho(x)y, \quad x \in \mathbb{R}_+$$

with the integral type boundary condition

$$\int_0^\infty G(x)y(x)dx + \gamma y'(0) - \theta y(0) = 0$$

and the non-standard weight function

$$\rho(x) = -1$$

where $|\gamma| + |\theta| \neq 0$. There are an enormous number of papers considering the positive values of $\rho(x)$ for both continuous and discontinuous cases. The structure of the weight function affects the analytical properties and representations of the solutions of the equation. Differently from the classical literature, we used the hyperbolic type representations of the fundamental solutions of the equation to obtain the spectrum of the operator. Moreover, the conditions for the finiteness of the eigenvalues and spectral singularities were presented. Hence, besides generalizing the recent results, Naimark's and Pavlov's conditions were adopted for the negative weight function case.

1. Introduction

Differential equations, particularly the ones with integral boundary conditions, have been inevitable tools in modeling natural phenomena such as thermodynamics, liquid flow, and demographics, see [1]. Modeling the vibration of a loaded string, equations of gas dynamics, and the theory of shock waves are a few quite interesting examples of a vast research area in mathematical physics that makes use of boundary value problems with a boundary condition involving spectral parameters in it [2]. Therefore in this paper we will focus on Sturm-Liouville operator generated by well-known one dimensional Schrödinger equation

$$-y'' + q(x)y = \omega^2 \rho(x)y, \quad x \in \mathbb{R}_+ \tag{1.1}$$

where ω is a spectral parameter and ρ is the weight function under the integral boundary condition.
The utility stemmed from the interconnection of studies on direct and inverse problems with the methods of solving many problems in mathematical analysis, keeps this research area vigorous [3–7]. This productive and efficient subject area, originated by the pioneer work of

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Naimark dealing with the singular non-self-adjoint problem for $\rho(x)=1$, finds itself specialized sub-areas governing different but connected techniques, for example, cases considering positive weight [8–13], non-continuous weight [14–17], sign-changing weight [18–20] as well as discrete cases [21–28]. Especially, the spectral singularities of the non-selfadjoint problem under the integral boundary condition has been investigated in [9, 10].

At first, Gasymov's approach in considering the sign-changing weight function for the inverse problem of the Sturm-Liouville type operator yielded different results from the previous literature [18]. Besides the fact that the appearance of these weight functions enriched the study area with applications in physics, the analytical difficulties arising from the negative sign made the problem even more attractive.

In the Sturm-Liouville problem, hyperbolic-type solutions obtained depending on the negative weight function cause some analytical difficulties, as well as the necessity of re-evaluation of conventional techniques. In this paper, the spectral properties of the non-selfadjoint singular Sturm-Liouville type operator, under the integral boundary condition and the non-standard weight function $\rho(x)=-1$ shall be analyzed. We engage with this problem owing to the deficiency in the studies investigating the requirements of the analytical solutions of Sturm-Liouville equation in distinct regions.

Let us also remark that, while the transformation chosen for the eigenparameter determines the analytical properties of the Jost solutions in discrete problems; the structure of the weight function affects the Jost solution in differential case. Hence, based on this idea, this paper may also lay the groundwork for new research topics in both inverse and direct problems. This paper has also a crucial importance since this is the first study which considers the negative value of a weight function for singular non-selfadjoint operators under the integral boundary condition. Therefore, we adopt the recent results to the negative weight function case and obtain new results which might give rise to the new research topics.

This article is structured as follows: Section 2 presents the general solution to (1.1) subject to the integral boundary condition in terms of the fundamental solutions to the boundary value problem (1.1) with negative weight function. Later in the same section, we obtain resolvent set in terms of these solutions. In Section 3, more general theorems for eigenvalues and spectral singularities concerning some additional and more strict conditions on the potential function are provided.

Notation. *Let ω be a complex parameter. In this paper, for the complex left half-plane, we set the notation $\mathbb{C}_{left} := \{\omega \in \mathbb{C} : \operatorname{Re}\omega < 0\}$. As usual topological relatives, we use $\overline{\mathbb{C}}_{left}$ for its completion, and $\partial\mathbb{C}_{left}$ for its boundary set. We denote number of elements in a set A with $\#A$ and the linear Lebesgue measure of a Lebesgue measurable set A with $\mu(A)$.*

2. Solutions of the problem

In this part, we present some preliminary results for the negative weight function case which can be deduced similar to the theorems and techniques in [4–6, 8, 9].

Let $\mathscr{T}$ be the operator in $L^2_\rho(\mathbb{R}_+)$ generated by the differential equation

$$-y'' + q(x)y = \omega^2\rho(x)y, \quad x \in \mathbb{R}_+ \tag{2.1}$$

with the integral boundary condition

$$\int_0^\infty G(x)y(x)\,dx + \gamma y'(0) - \theta y(0) = 0 \tag{2.2}$$

and the non-standard weight function

$$\rho(x) = -1, \quad x \in \mathbb{R}_+ \tag{2.3}$$

where γ, θ are complex numbers with $|\gamma|+|\theta| \neq 0$, and ω is spectral parameter. Note that q and G are complex valued functions, such that $G \in L^1_\rho(\mathbb{R}_+) \cap L^2_\rho(\mathbb{R}_+)$, and q satisfies the following condition:

$$\int_0^\infty s|q(s)|\,ds < \infty. \tag{2.4}$$

Let us denote by $S(x,\omega)$ and $C(x,\omega)$, the solutions of (2.1) subject to the initial conditions

$$\begin{aligned} S(0,\omega) &= 0, \\ C(0,\omega) &= 1, \\ \frac{\partial}{\partial x}S(x,\omega)\Big|_{x=0} &= 1, \\ \frac{\partial}{\partial x}C(x,\omega)\Big|_{x=0} &= 0. \end{aligned}$$

Consider the case $q(x) \equiv 0$. Then, (2.1) takes the form

$$y'' = \omega^2 y, \quad x \in \mathbb{R}_+.$$

Thus, $S(x,\omega)$ and $C(x,\omega)$ can be represented by the hyperbolic type functions

$$\begin{aligned} S(x,\omega) &= \frac{\sinh \omega x}{\omega}, \\ C(x,\omega) &= \cosh \omega x. \end{aligned}$$

Using the results of [4] and constant coefficients method, one can easily show that the fundamental solutions $S(x,\omega)$ and $C(x,\omega)$ have the Volterra type integral representations as

$$S(x,\omega)=\frac{\sinh\omega x}{\omega}+\int_0^x\frac{\sinh\omega(x-t)}{\omega}q(t)S(t,\omega)\,dt,$$

and

$$C(x,\omega)=\cosh\omega x+\int_0^x\frac{\sinh\omega(x-t)}{\omega}q(t)S(t,\omega)\,dt.$$

Moreover, both functions $S(\cdot,\omega)$ and $C(\cdot,\omega)$ are entire in ω. They are also analytic on $\overline{\mathbb{C}}_{left}$. Existence and uniqueness results of the solutions $S(x,\omega)$ and $C(x,\omega)$ can also be proven analogous to [4]. Also, Wronskian of the solutions $S(x,\omega)$ and $C(x,\omega)$ can be written as

$$W[S(x,\omega),C(x,\omega)]=-1,\quad \omega\in\mathbb{C}.$$

Now, let us denote the Jost solution of the operator $\mathscr{T}$ by $e(x,\omega)$ which is the solution of (2.1) satisfying the asymptotic condition

$$\lim_{x\to\infty}e(x,\omega)e^{-\omega x}=1,\quad \omega\in\mathbb{C}_{left}. \tag{2.5}$$

Under the condition (2.4), this solution can be found as

$$e(x,\omega)=e^{\omega x}+\int_x^\infty K(x,s)e^{\omega s}ds, \tag{2.6}$$

where the kernel K is uniquely determined by the potential function q such that $K(x,.)\in L_1(0,\infty)$ and it is continuously differentiable with respect to its arguments.
On the same manner with [4], we deduce that the Jost solution $e(\cdot,\omega)$ is analytic in $\mathbb{C}_{left}$ and continuous on $\overline{\mathbb{C}}_{left}$ from the validity of the inequality

$$|K(x,s)|\le c\int_{\frac{x+s}{2}}^\infty|q(\tau)|\,d\tau,\quad x\le s<\infty, \tag{2.7}$$

for any constant $c>0$ independent of the variables x and s.
Denote by $g(x,\omega)$, another solution of (2.1) satisfying the asymptotic conditions

$$\begin{aligned}\lim_{x\to\infty}g(x,\omega)e^{\omega x}&=1,\quad \omega\in\overline{\mathbb{C}}_{left},\\ \lim_{x\to\infty}g_x(x,\omega)e^{\omega x}&=-\omega,\quad \omega\in\overline{\mathbb{C}}_{left}.\end{aligned} \tag{2.8}$$

By the help of the asymptotic identities (2.5) and (2.8), the Wronskian of e and g can be found as

$$W[e(x,\omega),g(x,\omega)]=-2\omega,\quad \omega\in\overline{\mathbb{C}}_{left}, \tag{2.9}$$

which concludes that e and g form the fundamental system of solutions for (2.1) on $\partial\mathbb{C}_{left}$.
For the complex parameter ω, define the functions

$$N(\omega)=\int_0^\infty G(s)e(s,\omega)\,ds+\gamma e_x(0,\omega)-\theta e(0,\omega),$$

$$M(\omega)=\int_0^\infty G(s)g(s,\omega)\,ds+\gamma g_x(0,\omega)-\theta g(0,\omega),$$

and for $t\in\mathbb{R}_+$,

$$u(t,\omega)=\frac{-1}{2\omega}\left\{g(t,\omega)\int_t^\infty G(s)e(s,\omega)\,ds-e(t,\omega)\int_t^\infty G(s)g(s,\omega)\,ds+M(\omega)e(t,\omega)\right\}.$$

Clearly, resolvent operator of $\mathscr{T}$ can be obtained as

$$R_\omega(\mathscr{T})\phi=\int_0^\infty\mathscr{G}(x,t;\omega)\phi(t)\,dt,\quad \phi\in L_\rho^2(\mathbb{R}_+).$$

Here we set the notation $\mathscr{G}(x,t;\omega)$ for the Green's function of $\mathscr{T}$ defined as

$$\mathscr{G}(x,t;\omega)=\mathscr{G}^{(1)}(x,t;\omega)+\mathscr{G}^{(2)}(x,t;\omega),$$

with the functions

$$\mathscr{G}^{(1)}(x,t;\omega) := \frac{-e(x,\omega)\,u(t,\omega)}{N(\omega)},$$

$$\mathscr{G}^{(2)}(x,t;\omega) := -\begin{cases} \frac{e(x,\omega)u(t,\omega)}{2\omega}, & 0 \le t < x, \\ \frac{e(t,\omega)u(x,\omega)}{2\omega}, & x \le t < \infty. \end{cases}$$

Hence, the resolvent set $R_\omega(\mathscr{T})$ is given by

$$R_\omega(\mathscr{T}) = \left\{\eta : \eta = \omega^2, \operatorname{Re}\omega < 0, N(\omega) \neq 0\right\}.$$

3. Spectrum of $\mathscr{T}$

In this section in order to deal with the quantitative structure of the spectrum of $\mathscr{T}$, we will investigate the sets of zeros of the function N on left half plane and on its boundary, respectively. Let us denote these sets by

$$Z_1 := \left\{\omega : \omega \in \mathbb{C}_{left}, N(\omega) = 0\right\},$$
$$Z_2 := \left\{\omega : \omega \in \partial\mathbb{C}_{left}, N(\omega) = 0\right\}.$$

Recall that the multiplicity of a zero in the region $\overline{\mathbb{C}}_{left}$ is called the multiplicity of the corresponding eigenvalue and spectral singularity of the operator [7,8]. According to this definition, Z_3 denotes the set of all the accumulation points of Z_1 and Z_4 denotes the set of all zeros of N in $\overline{\mathbb{C}}_{left}$ with infinite multiplicity.
Notice that the set of eigenvalues of $\mathscr{T}$ is related to the set of zeros Z_1

$$\sigma_d(\mathscr{T}) = \left\{\mathfrak{z} : \mathfrak{z} = \omega^2, \omega \in Z_1\right\}, \tag{3.1}$$

and the set of spectral singularities $\mathscr{T}$ is related to the set of zeros Z_2

$$\sigma_{ss}(\mathscr{T}) = \left\{\mathfrak{z} : \mathfrak{z} = \omega^2, \omega \in Z_2\right\} \setminus \{0\}. \tag{3.2}$$

Within the same circle of ideas in the proofs of the theorems from [4,8,9], by the classical definition of spectrum of a differential operator we obtain that the set $\sigma_c(\mathscr{T})$ defined as

$$\sigma_c(\mathscr{T}) = \{\mathfrak{z} : \mathfrak{z} = i\tau, \tau \ge 0\}$$

is the continuous spectrum of $\mathscr{T}$.

Lemma 3.1. *Suppose $G \in L^1_\rho(\mathbb{R}_+) \cap L^2_\rho(\mathbb{R}_+)$ and (2.4) holds, then*

(i) Z_1 is bounded, $\#Z_1$ is at most countable, and Z_3 is a subset of a bounded interval of $\partial\mathbb{C}_{left}$,
(ii) Z_2 is a compact set with $\mu(Z_2) = 0$.

Proof. Using the inequality (2.6) and the expression of $N(\omega)$, it can be easily seen that $N(\omega)$ is analytic with respect to ω in $\mathbb{C}_{left}$ and continuous on the imaginary axis. Also, it yields the asymptotic

$$N(\omega) = \gamma\omega + \theta + o(1), \quad \omega \in \overline{\mathbb{C}}_{left}, |\omega| \to \infty, \tag{3.3}$$

for $|\gamma| + |\theta| \neq 0$. The boundedness of the sets Z_1 and Z_2 follows from (3.3). Hence, the proof of part (a) results from analicity of $N(\omega)$ in $\mathbb{C}_{left}$ and continuity on the imaginary axis. For the part (b), we shall consider the boundary uniqueness theorems of analytic functions [29]. Using these theorems, we get that Z_2 is a closed set and $\mu(Z_2) = 0$. □

The following theorem can be stated easily using (3.1), (3.2) and Lemma 3.1 :

Theorem 3.2. *Suppose $G \in L^1_\rho(\mathbb{R}_+) \cap L^2_\rho(\mathbb{R}_+)$ and (2.4) holds. Then,*

(i) $\sigma_d(\mathscr{T})$ is bounded, $\#\sigma_d(\mathscr{T})$ is at most countable, and the set of its limit points is contained in a bounded interval of $\partial\mathbb{C}_{left}$.
(ii) $\sigma_{ss}(\mathscr{T})$ is a bounded set with zero measure.

From now on, we will consider the spectral properties of $\mathscr{T}$ under more strict conditions on the potential. Firstly we consider the Naimark's condition

$$\int_0^\infty e^{\varepsilon\tau}\left(|q(\tau)| + |G(\tau)|\right)d\tau < \infty, \tag{3.4}$$

for any $\varepsilon > 0$, which enables us to use analytic continuation properties of the Jost function for the proof.

Theorem 3.3. *Suppose the condition (3.4) holds true. Then $\mathscr{T}$ possesses finitely many eigenvalues and spectral singularities and each one has finite multiplicity.*

Proof. (2.7) and (3.4) make it clear that

$$|K(x,s)| \le Ae^{-\frac{\varepsilon(x+s)}{2}}, \tag{3.5}$$

for arbitrary positive constant A. Considering the expression of $N(\omega)$ and (3.5), it is clear that $N(\omega)$ continues analytically from $\mathbb{C}_{left}$ to the right half-plane $\{\omega : \mathrm{Re}\omega < \frac{\varepsilon}{4}\}$. As a consequence, the limit points of the zeros of $N(\omega)$ in $\mathbb{C}_{left}$ cannot lie in the imaginary axis. From the results of Lemma 3.1, we can see that the sets Z_1 and Z_2 are bounded and both have a finite number of elements. Also taking into account the analicity of $N(\omega)$ for $\{\omega : \mathrm{Re}\omega < \frac{\varepsilon}{4}\}$, we deduce that the zeros of $N(\omega)$ in $\mathbb{C}_{left}$ are of finite number and they are of finite multiplicity, which concludes the assertion of theorem. □

However, there is more strict condition for the potential called Pavlov's condition which pushes us to use new methods to prove the finiteness of the sets $\sigma_d(\mathscr{T})$ and $\sigma_{ss}(\mathscr{T})$. Let the following integral condition holds true:

$$\int_0^\infty \exp(\varepsilon\tau^\delta)(|q(\tau)|+|G(\tau)|)\,d\tau < \infty, \quad \frac{1}{2} \le \delta < 1 \tag{3.6}$$

for any $\varepsilon > 0$. Clearly, $N(\omega)$ is analytic in the complex left-half plane $\mathbb{C}_{left}$ and continuous on the imaginary axis. Nevertheless, analytic continuation property does not hold from the left-half plane to the right-half plane. We will also benefit from the subsequent relations between the sets Z_1, Z_2, Z_3 and Z_4 in order to verify the following theorem which can be inferred directly from the boundary uniqueness theorems of analytic functions [29]:

$$Z_1 \cap Z_4 = \emptyset, \quad Z_3 \subset Z_4 \subset Z_2, \tag{3.7}$$

and

$$\mu(Z_3) = \mu(Z_4) = 0.$$

Theorem 3.4. *If the condition for the potential (3.6) holds to be true, then $Z_4 = \emptyset$.*

Proof. Using Lemma 3.1., we obtain that

$$\left|\int_{-\infty}^{-T} \frac{\ln|N(\omega)|}{1+\omega^2}d\omega\right| < \infty, \quad \left|\int_T^\infty \frac{\ln|N(\omega)|}{1+\omega^2}d\omega\right| < \infty, \tag{3.8}$$

for sufficiently large values of $T > 0$. Moreover, $N(\omega)$ is analytic in $\mathbb{C}_{left}$, all its derivatives are continuous up to the imaginary axis and

$$\left|N^{(r)}(\omega)\right| \le C_r, \quad \omega \in \overline{\mathbb{C}}_{left}, \quad r = 1,2,..., \quad |\omega| < 2T, \tag{3.9}$$

where

$$C_r := c\int_0^\infty s^r |K(0,s)|\,ds. \tag{3.10}$$

If we make use of (3.8), (3.9) and Pavlov's theorem, we get

$$\int_0^\omega \ln t(s)\,d\mu(Z_{4,s}) > -\infty, \tag{3.11}$$

where $t(s) := \inf_r \frac{C_r s^r}{r!}$ for $s \ge 0$, and $\mu(Z_{4,s})$ is the linear Lebesgue measure of the $s-$neighborhood of Z_4 [8,9]. We can also estimate C_r from above

$$C_r = c\int_0^\infty s^r |K(0,s)|\,ds \le c\int_0^\infty s^r e^{-\frac{\varepsilon}{4}s}\,ds \le Bb^r r^r r!, \tag{3.12}$$

for constants B and b depending on c and δ. When estimate (3.12) is substituted in the definition of t implies that

$$t(s) = \inf_r \frac{C_r s^r}{r!} \le B\inf_r\{b^r s^r r^r\} \le Be^{-s^{-1}e^{-1}b^{-1}}. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\int_0^\omega s^{-\frac{\delta}{1-\delta}}\,d\mu(Z_{4,s}) < \infty. \tag{3.14}$$

Then the inequality $\frac{\delta}{1-\delta} \ge 1$, together with (3.14) ensures that, for arbitrary s, $\mu(Z_{4,s}) = 0$ or $Z_4 = \emptyset$. □

Theorem 3.5. *Suppose that the condition (3.6) holds true. Then $\mathscr{T}$ possesses finitely many eigenvalues and spectral singularities and each one is of finite multiplicity..*

Proof. It would clearly have been necessary to show that $N(\omega)$ acquires finitely many zeros with finite multiplicities in $\overline{\mathbb{C}}_{left}$. When we applied the previous theorem, the relation (3.7) just amounts to saying that $Z_3 = \emptyset$. That is to say, the bounded sets Z_1 and Z_2 cannot possess accumulation points. Therefore, the zeros of $N(\omega)$ in $\overline{\mathbb{C}}_{left}$ are finitely many. The fact $Z_4 = \emptyset$ concludes that these zeros are of finite multiplicity. □

4. Conclusion

In this paper, we investigated the spectrum of the operator constructed by the help of differential Sturm-Liouville type operator and negative valued weight function. The specific feature of this study is that we obtain the spectrum using the hyperbolic type fundamental solutions. We also impose an integral boundary condition and this also effects the structure of the Naimark's and Pavlov's conditions. There are so many papers considering the trigonometric type fundamental solutions. Also, this paper is the differential analog of the hyperbolic type problems in discrete operators. Therefore, we bring a novel viewpoint to the recent papers and this paper may lay the goundwork for future studies.

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