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ON VERTEX DECOMPOSABILITY AND REGULARITY OF GRAPHS

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Abstract. There are two motivating questions in [M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai and S. Yassemi, arXiv:1006.1087v1] and [M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai and S. Yassemi, J. Pure Appl. Algebra, 215(10) (2011), 2473-2480] about Castelnuovo-Mumford regularity and vertex decomposability of simple graphs. In this paper, we give negative answers to the questions by providing two counterexamples.

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1. Introduction

Throughout this paper, we assume that $R = K[x_1, \ldots, x_n]$ is the polynomial ring over a field K and suppose that G is a finite simple graph on the vertex set $V = \{x_1, \ldots, x_n\}$ and the edge set E. For a vertex v of G the set of all neighbors of v is denoted by $N(v)$ and we denote by $N[v]$ the set $N(v) \cup \{v\}$ and also we denote by $\deg(v)$ the number $|N(v)|$. An independent set of G is a subset A of $V(G)$ such that none of its elements are adjacent. The *edge ideal* of the graph G is the quadratic square-free monomial ideal $I(G) = \langle x_ix_j \mid \{x_i, x_j\} \in E\rangle$ and was first introduced by Villarreal [15]. Two edges $\{x, y\}$ and $\{z, u\}$ of G are called 3-*disjoint* if the induced subgraph of G on $\{x, y, z, u\}$ is disconnected or equivalently in the complement of G the induced graph on $\{x, y, z, u\}$ is a four-cycle. A subset A of edges of G is called a pairwise 3-disjoint set of edges in G if each pair of edges of A is 3-disjoint, see [10,12,17]. The maximum cardinality of all pairwise 3-disjoint sets of edges in G is denoted by $a(G)$, see [10,12,17]. Note that $a(G)$ is called *induced matching number*. The Castelnuovo-Mumford

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regularity of a graded R-module M is defined as $reg(M) = \max\{j-i| \beta_{i,j}(M) \neq 0\}$. Katzmann [8] proved that $reg(R/I(G)) \geq a(G)$ for every simple graph G. Stanley [13] defined a graded R-module M to be *sequentially Cohen-Macaulay* if there exists a finite filtration of graded R-modules $0 = M_0 \subset M_1 \subset \ldots \subset M_r = M$ such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing: $\dim(M_1/M_0) < \dim(M_2/M_1) < \ldots < \dim(M_r/M_{r-1})$. In particular, we call the graph G sequentially Cohen-Macaulay (resp., unmixed) if $R/I(G)$ is sequentially Cohen-Macaulay (resp., unmixed). Herzog and Hibi [5] defined the homogeneous ideal I to be *componentwise linear* if (I_d) has a linear resolution for all d, where (I_d) is the ideal generated by all degree d elements of I. They proved that if I is a square-free monomial ideal, then R/I is sequentially Cohen-Macaulay if and only if the square-free Alexander dual $I^\vee$ is componentwise linear. It is known that if I has a linear resolution, then I is componentwise linear. Note that for a square-free monomial ideal $I = \langle\{x_{i1} \ldots x_{in_i} \mid i = 1, \ldots, t\}\rangle$ of R the *Alexander dual* of I, denoted by $I^\vee$, is defined as $I^\vee = \cap_{i=1}^{t} \langle x_{i1}, \ldots, x_{in_i}\rangle$. For a monomial ideal I, we write (I_i) to denote the ideal generated by the degree i elements of I. The monomial ideal I is componentwise linear if (I_i) has a linear resolution for all i (see [5]). If I is generated by square-free monomials, then we denote by $I_{[i]}$ the ideal generated by the square-free monomials of degree i of I. Herzog and Hibi [5, Proposition 1.5] proved that the square-free monomial ideal I is componentwise linear if and only if $I_{[i]}$ has a linear resolution for all i.

Woodroofe [16] defined the graph G to be *vertex decomposable* if it is a totally disconnected graph (with no edges) or if the following recursive conditions hold:
(i) there is a vertex v in G such that $G\backslash v$ and $G\backslash N[v]$ are both vertex decomposable;
(ii) no independent set in $G \setminus N[v]$ is a maximal independent set in $G \setminus v$.

The equality $reg(R/I(G)) = a(G)$ was proved in the following cases: (i) G is a tree graph; (ii) G is a chordal graph, where the graph G is called *chordal* if every cycle of length > 3 has a chord; (iii) G is a bipartite graph and unmixed; (iv) G is a bipartite graph and sequentially Cohen-Macaulay; (v) G is a very well-covered graph, where the graph G is called *very well-covered* if it is unmixed without an isolated vertices and $2ht(I(G)) = |V|$; (vi) G is a C_5-free vertex decomposable graph; (vii) G is an almost complete multipartite graph such that it is sequentially Cohen-Macaulay or unmixed. For details see [4,7,8,9,12,14,17].

Mahmoudi et al. in [11, Question 4.11] and in [12, Question 4.13] raised the following question:

Question 1.1. *Let G be a sequentially Cohen-Macaulay graph with $2n$ vertices which are not isolated and with $ht(I(G)) = n$. Then do we have the following statements?*

(1) *G has a vertex v such that* $\deg(v) = 1$.
(2) *G is vertex decomposable.*
(3) $reg(R/I(G)) = a(G)$.

In this paper we give a negative answer to this question by providing two counterexamples. For every unexplained notion or terminology, we refer the reader to [6].

2. Counterexamples

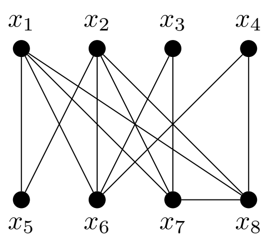
We start this section by recalling the following definition:

Definition 2.1. Let I be a monomial ideal of R all of whose generators have degree d. Then I has a linear resolution if for all $i \geq 0$ and for all $j \neq i + d$, $\beta_{i,j}(I) = 0$. In particular, I has a linear resolution if and only if $reg(I) = d$.

Lemma 2.2. ([1, Lemma 2.3]) *Let $I = \langle u_1, \ldots, u_m \rangle$ be a monomial ideal with* $\deg(u_i) = d_i$ *and $d_i \leq d_{i+1}$ for $1 \leq i \leq m-1$. If (I_i) has a linear resolution for all $i < d_m$ and $reg(I) = d_m$, then I is componentwise linear.*

By the following example we show that the Question 1.1(1) and (3) have negative answers:

Example 2.3. Let G be the following graph:



Then we may consider the edge ideal

$$I = (x_1x_5, x_1x_6, x_1x_7, x_1x_8, x_2x_5, x_2x_6, x_2x_7, x_2x_8, x_3x_6, x_3x_7, x_4x_6, x_4x_8, x_7x_8)$$

of $R = K[x_1, \ldots, x_8]$. This ideal has the following primary decomposition

$$\begin{aligned} I =& (x_5, x_6, x_7, x_8) \cap (x_1, x_2, x_3, x_4, x_7) \cap (x_1, x_2, x_3, x_4, x_8) \cap (x_1, x_2, x_3, x_6, x_8) \\ & \cap (x_1, x_2, x_4, x_6, x_7) \cap (x_1, x_2, x_6, x_7, x_8). \end{aligned}$$

So $ht(I) = 4$ and

$$I^\vee = (x_5x_6x_7x_8, x_1x_2x_3x_4x_7, x_1x_2x_3x_4x_8, x_1x_2x_3x_6x_8, x_1x_2x_4x_6x_7, x_1x_2x_6x_7x_8).$$

Hence by using Macaulay2 [3], we have $reg(R/I) = 2$ and $reg(I^\vee) = 5$. Therefore by Lemma 2.2 it readily follows that G is sequentially Cohen-Macaulay. One can easily check that for any two edges $\{x_i, x_j\}$ and $\{x_k, x_l\}$ of G such that i, j, l, k are different positive integers, the induced subgraph of G on the vertices $\{x_i, x_j, x_k, x_l\}$ is connected. Therefore, $a(G) = 1 \neq reg(R/I)$ giving a negative answer to Question 1.1.(1) and, in addition, G does not have a vertex of degree 1 contradicting Question 1.1.(3).

Recall that a *circulant graph* is defined as follows: let $n \geq 1$ be an integer and let $S \subseteq \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The circulant graph $C_n(S)$ is the graph on n vertices $V = \{x_1, \ldots, x_n\}$ such that $\{x_i, x_j\}$ is an edge of $C_n(S)$ if and only if $\min\{|i-j|, n-|i-j|\} \in S$. For ease of notation, we write $C_n(a_1, \ldots, a_t)$ instead of $C_n(\{a_1, \ldots, a_t\})$, for more details see [2]. Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$. Members of Δ are called faces of Δ and a facet of Δ is a maximal face of Δ with respect to inclusion. The simplicial complex Δ is pure if every facet has the same cardinality. Also, the simplicial complex Δ with the facets $F_1, \ldots, F_r$ is denoted by $\Delta = \langle F_1, \ldots, F_r \rangle$. The simplicial complex Δ is called a *simplex* when it has a unique facet. For the simplicial complex Δ and the face $F \in \Delta$, one can introduce two new simplicial complexes. The *deletion* of F from Δ is $del_\Delta(F) = \{A \in \Delta | F \cap A = \emptyset\}$. The *link* of F in Δ is $lk_\Delta(F) = \{A \in \Delta | F \cap A = \emptyset, A \cup F \in \Delta\}$. If $F = \{v\}$, we write $del_\Delta v$ (resp. $lk_\Delta v$) instead of $del_\Delta(\{v\})$ (resp. $lk_\Delta(\{v\})$); see [6] for details information. The *Stanley-Reisner* ideal of Δ over K is the ideal I_Δ of R which is generated by those square-free monomials x_F with $F \notin \Delta$, where $x_F = \prod_{x_i \in F} x_i$. Let I be an arbitrary square-free monomial ideal. Then there is a unique simplicial complex Δ such that $I = I_\Delta$. Following [16] a simplicial complex Δ is recursively defined to be *vertex decomposable* if it is either a simplex or else has some vertex v so that (i) both $del_\Delta v$ and $lk_\Delta v$ are vertex decomposable, and (ii) no face of $lk_\Delta v$ is a facet of $del_\Delta v$.

A simplicial complex Δ is *shellable* if the facets of Δ can be ordered, say $F_1, \ldots, F_s$, such that for all $1 \leq i < j \leq s$, there exists some $x \in F_j \setminus F_i$ and some $k \in \{1, 2, \ldots, j-1\}$ with $F_j \setminus F_k = \{x\}$. Hence if Δ is shellable with shelling order $F_1, \ldots, F_s$, then for each $2 \leq j \leq s$, the subcomplex $\langle F_1, \ldots, F_{j-1} \rangle \cap \langle F_j \rangle$ is pure of dimension $\dim F_j - 1$, for detials see [6, Section 8.2]. The following implications hold:

vertex decomposable $\Longrightarrow$ shellable $\Longrightarrow$ sequentially Cohen-Macaulay.
Also, both implications are known to be strict.

The independence complex of the graph G is defined by $Ind(G) = \{F \subseteq V \mid F$ is an independence set in $G\}$. It is clear $I(G) = I_{Ind(G)}$. Let v be a vertex of G. By [7] we have the following relations:
$del_{Ind(G)}v = Ind(G \setminus v)$ and $lk_{Ind(G)}v = Ind(G \setminus N[v])$. Therefore one can deduce that the graph G is vertex decomposable if and only if the independence complex $Ind(G)$ is vertex decomposable.

Theorem 2.4. ([2, Theorem 6.1 (iii)]) *The graph* $C_{16}(1,4,8)$ *is the smallest well-covered circulant that is shellable but not vertex decomposable.*

By the following example we show that Question 1.1(2) has a negative answer:

Example 2.5. Let I be an ideal of $R = K[x_1, \ldots, x_{26}]$ generated by the following monomials

$x_{16}x_{26}$	$x_{15}x_{26}$	$x_{13}x_{26}$	$x_{12}x_{26}$	$x_{10}x_{26}$	x_8x_{26}	x_7x_{26}	x_6x_{26}	x_5x_{26}	x_4x_{26}	x_3x_{26}	x_2x_{26}	x_1x_{26}
$x_{16}x_{25}$	$x_{15}x_{25}$	$x_{13}x_{25}$	$x_{12}x_{25}$	$x_{10}x_{25}$	x_8x_{25}	x_7x_{25}	x_6x_{25}	x_5x_{25}	x_4x_{25}	x_3x_{25}	x_2x_{25}	x_1x_{25}
$x_{16}x_{24}$	$x_{15}x_{24}$	$x_{13}x_{24}$	$x_{12}x_{24}$	$x_{10}x_{24}$	x_8x_{24}	x_7x_{24}	x_6x_{24}	x_5x_{24}	x_4x_{24}	x_3x_{24}	x_2x_{24}	x_1x_{24}
$x_{16}x_{23}$	$x_{15}x_{23}$	$x_{13}x_{23}$	$x_{12}x_{23}$	$x_{10}x_{23}$	x_8x_{23}	x_7x_{23}	x_6x_{23}	x_5x_{23}	x_4x_{23}	x_3x_{23}	x_2x_{23}	x_1x_{23}
$x_{16}x_{22}$	$x_{15}x_{22}$	$x_{13}x_{22}$	$x_{12}x_{22}$	$x_{10}x_{22}$	x_8x_{22}	x_7x_{22}	x_6x_{22}	x_5x_{22}	x_4x_{22}	x_3x_{22}	x_2x_{22}	x_1x_{22}
$x_{16}x_{21}$	$x_{15}x_{21}$	$x_{13}x_{21}$	$x_{12}x_{21}$	$x_{10}x_{21}$	x_8x_{21}	x_7x_{21}	x_6x_{21}	x_5x_{21}	x_4x_{21}	x_3x_{21}	x_2x_{21}	x_1x_{21}
$x_{16}x_{20}$	$x_{15}x_{20}$	$x_{13}x_{20}$	$x_{12}x_{20}$	$x_{10}x_{20}$	x_8x_{20}	x_7x_{20}	x_6x_{20}	x_5x_{20}	x_4x_{20}	x_3x_{20}	x_2x_{20}	x_1x_{20}
$x_{16}x_{19}$	$x_{15}x_{19}$	$x_{13}x_{19}$	$x_{12}x_{19}$	$x_{10}x_{19}$	x_8x_{19}	x_7x_{19}	x_6x_{19}	x_5x_{19}	x_4x_{19}	x_3x_{19}	x_2x_{19}	x_1x_{19}
$x_{16}x_{18}$	$x_{15}x_{18}$	$x_{13}x_{18}$	$x_{12}x_{18}$	$x_{10}x_{18}$	x_8x_{18}	x_7x_{18}	x_6x_{18}	x_5x_{18}	x_4x_{18}	x_3x_{18}	x_2x_{18}	x_1x_{18}
$x_{16}x_{17}$	$x_{15}x_{17}$	$x_{13}x_{17}$	$x_{12}x_{17}$	$x_{10}x_{17}$	x_8x_{17}	x_7x_{17}	x_6x_{17}	x_5x_{17}	x_4x_{17}	x_3x_{17}	x_2x_{17}	x_1x_{17}
$x_{15}x_{16}$	$x_{12}x_{16}$	x_8x_{16}	x_4x_{16}	x_1x_{16}	$x_{14}x_{15}$	$x_{11}x_{15}$	x_7x_{15}	x_3x_{15}	$x_{13}x_{14}$	$x_{10}x_{14}$	x_6x_{14}	x_2x_{14}
$x_{12}x_{13}$	x_9x_{13}	x_5x_{13}	x_1x_{13}	$x_{11}x_{12}$	x_8x_{12}	x_4x_{12}	$x_{10}x_{11}$	x_7x_{11}	x_3x_{11}	x_9x_{10}	x_6x_{10}	x_2x_{10}
x_8x_9	x_5x_9	x_1x_9	x_7x_8	x_4x_8	x_6x_7	x_3x_7	x_5x_6	x_2x_6	x_4x_5	x_1x_5	x_3x_4	x_2x_3
x_1x_2												

The ideal I is an edge ideal of a graph, say G. This ideal has the form

$$I = (J, x_{17}, x_{18}, \cdots, x_{26}) \cap (x_1, \cdots, x_8, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}),$$

where J is the edge ideal of circulant graph $C_{16}(1,4,8)$. This ideal has the following primary decomposition

$$I = \bigcap_{i=1}^{80} (\mathfrak{p}_i, x_{17}, x_{18}, \cdots, x_{26}) \cap (x_1, \cdots, x_8, x_{10}, x_{12}, x_{13}, x_{15}, x_{16});$$

where $\mathfrak{p}_i$ for $1 \leq i \leq 80$ is an associated prime of circulant graph $C_{16}(1,4,8)$. Therefore $ht(I) = 13$ and the simplicial complex $Ind(G)$ has 81 facets as follows:

$F_0 = \{x_9, x_{11}, x_{14}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26}\}$,

$F_1 = \{x_9, x_{11}, x_{14}, x_{16}\}$, $F_2 = \{x_5, x_{11}, x_{14}, x_{16}\}$, $F_3 = \{x_7, x_9, x_{14}, x_{16}\}$, $F_4 = \{x_3, x_9, x_{14}, x_{16}\}$, $F_5 = \{x_5, x_7, x_{14}, x_{16}\}$,
$F_6 = \{x_3, x_5, x_{14}, x_{16}\}$, $F_7 = \{x_6, x_9, x_{11}, x_{16}\}$, $F_8 = \{x_5, x_7, x_{10}, x_{16}\}$, $F_9 = \{x_2, x_5, x_{11}, x_{16}\}$, $F_{10} = \{x_2, x_9, x_{11}, x_{16}\}$,
$F_{11} = \{x_2, x_7, x_{13}, x_{16}\}$, $F_{12} = \{x_7, x_{10}, x_{13}, x_{16}\}$, $F_{13} = \{x_2, x_{11}, x_{13}, x_{16}\}$, $F_{14} = \{x_6, x_{11}, x_{13}, x_{16}\}$, $F_{15} = \{x_3, x_5, x_{10}, x_{16}\}$,
$F_{16} = \{x_3, x_{10}, x_{13}, x_{16}\}$, $F_{17} = \{x_3, x_6, x_{13}, x_{16}\}$, $F_{18} = \{x_2, x_7, x_9, x_{16}\}$, $F_{19} = \{x_3, x_6, x_9, x_{16}\}$, $F_{20} = \{x_2, x_5, x_7, x_{16}\}$,
$F_{21} = \{x_7, x_9, x_{12}, x_{14}\}$, $F_{22} = \{x_1, x_4, x_{10}, x_{15}\}$, $F_{23} = \{x_1, x_8, x_{10}, x_{15}\}$, $F_{24} = \{x_5, x_8, x_{10}, x_{15}\}$, $F_{25} = \{x_1, x_{10}, x_{12}, x_{15}\}$,
$F_{26} = \{x_4, x_{10}, x_{13}, x_{15}\}$, $F_{27} = \{x_8, x_{10}, x_{13}, x_{15}\}$, $F_{28} = \{x_5, x_{10}, x_{12}, x_{15}\}$, $F_{29} = \{x_3, x_9, x_{12}, x_{14}\}$, $F_{30} = \{x_3, x_8, x_{10}, x_{13}\}$,
$F_{31} = \{x_3, x_5, x_8, x_{14}\}$, $F_{32} = \{x_5, x_8, x_{11}, x_{14}\}$, $F_{33} = \{x_6, x_8, x_{11}, x_{13}\}$, $F_{34} = \{x_6, x_8, x_{13}, x_{15}\}$, $F_{35} = \{x_4, x_6, x_{13}, x_{15}\}$,
$F_{36} = \{x_2, x_8, x_{13}, x_{15}\}$, $F_{37} = \{x_2, x_8, x_{11}, x_{13}\}$, $F_{38} = \{x_1, x_4, x_6, x_{15}\}$, $F_{39} = \{x_4, x_6, x_9, x_{15}\}$, $F_{40} = \{x_6, x_9, x_{12}, x_{15}\}$,
$F_{41} = \{x_1, x_6, x_{12}, x_{15}\}$, $F_{42} = \{x_1, x_6, x_8, x_{15}\}$, $F_{43} = \{x_2, x_4, x_{13}, x_{15}\}$, $F_{44} = \{x_2, x_9, x_{12}, x_{15}\}$, $F_{45} = \{x_2, x_4, x_9, x_{15}\}$,
$F_{46} = \{x_4, x_6, x_{11}, x_{13}\}$, $F_{47} = \{x_4, x_9, x_{11}, x_{14}\}$, $F_{48} = \{x_4, x_7, x_9, x_{14}\}$, $F_{49} = \{x_2, x_4, x_{11}, x_{13}\}$, $F_{50} = \{x_5, x_7, x_{10}, x_{12}\}$,
$F_{51} = \{x_1, x_3, x_8, x_{14}\}$, $F_{52} = \{x_1, x_8, x_{11}, x_{14}\}$, $F_{53} = \{x_1, x_3, x_{12}, x_{14}\}$, $F_{54} = \{x_1, x_7, x_{12}, x_{14}\}$, $F_{55} = \{x_1, x_7, x_{10}, x_{12}\}$,
$F_{56} = \{x_3, x_6, x_8, x_{13}\}$, $F_{57} = \{x_5, x_7, x_{12}, x_{14}\}$, $F_{58} = \{x_3, x_5, x_{12}, x_{14}\}$, $F_{59} = \{x_3, x_5, x_{10}, x_{12}\}$, $F_{60} = \{x_1, x_3, x_{10}, x_{12}\}$,
$F_{61} = \{x_2, x_7, x_9, x_{12}\}$, $F_{62} = \{x_3, x_6, x_9, x_{12}\}$, $F_{63} = \{x_2, x_5, x_7, x_{12}\}$, $F_{64} = \{x_2, x_5, x_8, x_{11}\}$, $F_{65} = \{x_1, x_6, x_8, x_{11}\}$,
$F_{66} = \{x_2, x_4, x_9, x_{11}\}$, $F_{67} = \{x_4, x_6, x_9, x_{11}\}$, $F_{68} = \{x_1, x_3, x_6, x_{12}\}$, $F_{69} = \{x_2, x_5, x_8, x_{15}\}$, $F_{70} = \{x_2, x_5, x_{12}, x_{15}\}$,
$F_{71} = \{x_1, x_4, x_6, x_{11}\}$, $F_{72} = \{x_1, x_4, x_{11}, x_{14}\}$, $F_{73} = \{x_1, x_4, x_7, x_{14}\}$, $F_{74} = \{x_3, x_5, x_8, x_{10}\}$, $F_{75} = \{x_1, x_3, x_8, x_{10}\}$,
$F_{76} = \{x_1, x_4, x_7, x_{10}\}$, $F_{77} = \{x_4, x_7, x_{10}, x_{13}\}$, $F_{78} = \{x_1, x_3, x_6, x_8\}$, $F_{79} = \{x_2, x_4, x_7, x_9\}$, $F_{80} = \{x_2, x_4, x_7, x_{13}\}$

By the proof of Theorem 2.4, we have $F_1, \dots, F_{80}$ is a shelling order of $Ind(C_{16}(1,4,8))$ and the graph $C_{16}(1,4,8)$ is the smallest well-covered circulant that is shellable but not vertex decomposable. We claim that $F_0, F_1, \dots, F_{80}$ is a shelling order of $Ind(G)$. Since $F_1, \dots, F_{80}$ is a shelling order, it is enough to show that for each i, there exists some $v \in F_i \setminus F_0$ and some $k < i$ such that $F_i \setminus F_k = \{v\}$. If $i = 1$, then it is clear $F_1 \setminus F_0 = \{x_{16}\}$. Now we assume that $1 \neq i \leq 80$. Since $F_i \setminus F_1 \subseteq F_i \setminus F_0$, we may choose $v \in F_i \setminus F_1$ and so there exists some $1 \leq k < i$ such that $F_i \setminus F_k = \{v\}$. Therefore $Ind(G)$ is shellable and so G is sequentially Cohen-Macaulay.

Now, we claim that for each element x_t with $1 \leq t \leq 26$, $del_{Ind(G)}(x_t)$ is not vertex decomposable. If $x_t \in \{x_9, x_{11}, x_{14}, x_{17}, \dots, x_{26}\}$, then by using the definition on the above facets it is obvious that $del_{Ind(G)}(x_t)$ has a facet, say F', such that $F' \neq F_i$ for $0 \leq i \leq 80$, and in this case $del_{Ind(G)}(x_t)$ is not vertex decomposable. For the remaining claim, we assume that $x_t \in \{x_1, \dots, x_8, x_{10}, x_{12}, x_{13}, x_{15}, x_{16}\}$ and we will show that $del_{Ind(G)}(x_t)$ is not shellable and so it is not vertex decomposable. By contrary, let $del_{Ind(G)}(x_t)$ be shellable and so we may consider the shelling order $F_0 = F_{s_0}, F_{s_1}, \dots, F_{s_r}$. By this shelling order we have $F_0 = (F_{s_1} \setminus \{x_m\}) \cup \{x_{17}, \dots, x_{26}\}$ for some $x_m \in F_{s_1}$ and for all i and $j < i$ there exists $x_l \in F_{s_i} \setminus F_{s_j}$ and $k < i$ such that $F_{s_i} \setminus F_{s_k} = \{x_l\}$. By this assumption we claim that $F_{s_1}, \dots, F_{s_r}$ is shellable and for this it is enough for such k to assume $F_{s_k} = F_0$. In this case $F_{s_i} = (F_0 \setminus \{x_{17}, \dots, x_{26}\}) \cup \{x_l\} = \{x_9, x_{11}, x_{14}, x_l\}$. We may assume $F_{s_i} \neq F_{s_1}$. Since $F_{s_i} = \{x_9, x_{11}, x_{14}, x_l\}$ and $F_{s_1} = \{x_9, x_{11}, x_{14}, x_m\}$, we have $F_{s_i} \setminus F_{s_1} = \{x_l\}$. It therefore follows that $F_{s_1}, \dots, F_{s_r}$ is a shelling order. Hence $del_{Ind(C_{16}(1,4,8))}(x_t) = \langle F_{s_1}, \dots, F_{s_r} \rangle$ and this means that $del_{Ind(C_{16}(1,4,8))}(x_t)$ is pure shellable and Cohen-Macaulay. This is a contradiction by the proof of Theorem 2.4. Thus $del_{Ind(G)}(x_t)$ is not shellable and so G is not vertex decomposable.

Hence we construct a sequentially Cohen-Macaulay graph with 26 vertices such that $ht(I) = 13$ but it is not vertex decomposable.

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References

[1] H. N. Aziz, A. Mafi and F. Seyfpour, *Bi-sequentially Cohen-Macaulay bipartite graphs*, to appear in J. Algebra Appl., (2023).

[2] J. Earl, K. N. Vander Meulen and A. Van Tuyl, *Independence complexes of well-covered circulant Graphs*, Exp. Math., 25(4) (2016), 441-451.

[3] D. R. Grayson and M. E. Stillman, Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/.

[4] H. T. Ha and A. Van Tuyl, *Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers*, J. Algebraic Combin., 27(2) (2008), 215-245.

[5] J. Herzog and T. Hibi, *Componentwise linear ideals*, Nagoya Math. J., 153 (1999), 141-153.

[6] J. Herzog and T. Hibi, Monomial Ideals, Graduate Texts in Mathematics, 260, Springer-Verlag London, Ltd., London, 2011.

[7] H. Higashidaira, *On the sequentially Cohen-Macaulay properties of almost complete multipartite graphs*, Comm. Algebra, 45(6) (2017), 2478-2493.

[8] M. Katzmann, *Characteristic-independence of Betti numbers of graph ideals*, J. Combin. Theory Ser. A, 113(3) (2006), 435-454.

[9] F. Khosh-Ahang and S. Moradi, *Regularity and projective dimension of the edge ideal of C_5-free vertex decomposable graphs*, Proc. Amer. Math. Soc., 142(5) (2014), 1567-1576.

[10] M. Kummini, *Regularity, depth and arithmetic rank of bipartite edge ideals*, J. Algebraic Combin., 30(4) (2009), 429-445.

[11] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai and S. Yassemi, *Vertex decomposability and regularity of very well-covered graphs*, arXiv:1006.1087v1.

[12] M. Mahmoudi, A. Mousivand, M. Crupi, G. Rinaldo, N. Terai and S. Yassemi, *Vertex decomposability and regularity of very well-covered graphs*, J. Pure Appl. Algebra, 215(10) (2011), 2473-2480.

[13] R. P. Stanley, Combinatorics and Commutative Algebra, Second edition, Progress in Mathematics, 41. Birkhäuser Boston, Inc., Boston, MA, 1996.

[14] A. Van Tuyl, *Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity*, Arch. Math. (Basel) 93(5) (2009), 451-459.

[15] R. H. Villarreal, *Cohen-Macaulay graphs*, Manuscripta Math., 66(3) (1990), 277-293.

[16] R. Woodroofe, *Vertex decomposable graphs and obstructions to shellability*, Proc. Amer. Math. Soc., 137(10) (2009), 3235-3246.

[17] X. Zheng, *Resolutions of facet ideals*, Comm. Algebra, 32(6) (2004), 2301-2324.

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