JOURNAL OF
AMASYA UNIVERSITY
THE INSTITUTE OF SCIENCES
AND TECHNOLOGY
(JAUIST)

ON CONTACT PSEUDO-SLANT SUBMANIFOLDS IN (LCS)n-MANIFOLDS

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Research Type: Original Research Article

Received: 13/12/2022 Accepted: 27/12/2022

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Abstract

In this study, we investigate the differential geometry of contact pseudo-slant submanifolds of a (LCS)n -manifold. The necessary and sufficient conditions for contact pseudo-slant submanifolds of a (LCS)n-manifold are given.

Key Words: $(LCS)_n$-manifold, slant submanifold, contact pseudo-slant submanifold.

Özet

Bu çalışmada, bir (LCS)n -manifoldunun kontak pseudo-slant altmanifoldlarının diferansiyel geometrisini araştırıyoruz. Bir (LCS)n-manifoldunun kontak pseudo-slant altmanifoldları için gerekli ve yeterli koşullar verilmiştir.

Anahtar Kelimeler: $(LCS)_n$-manifold, slant altmanifold, kontak pseudo-slant altmanifold

1.Introduction

Chen [5],[6], first studied slant immersion in 1990 as a generalisation of both invariant and anti-invariant submanifolds in almost Hermitian manifolds. Later, Lotta[13], extended the concept of slant immersion into almost contact metric manifolds. After that such submanifolds of a Sasakian manifold were studied by Cabrerizo et al. [3], [4],

Papagiuc [16], introduced the concept of semi-slant submanifolds of an almost Hermitian manifold. Cabrerizo et al. investigated and characterised slant submanifolds of Sasakian manifolds and K-contact, providing several examples. Cabrerizo et al. [3], [4], defined and studied bi-slant submanifolds in an almost contact metric manifold and simultaneously gave the notion of pseudo-slant submanifolds. Khan et al. [12] have also investigated pseudo-slant submanifolds. Then, De et al. [7] studied and characterized pseudo-slant submanifolds of a trans-Sasakian manifold. Recently, in [2] ; Dirik, et al. [1], [8], [9], [10] studied slant and pseudo-slant submanifold in different manifolds.

Shaikh [17], [18], [19], recently introduced the concept of Lorentzian concircular structure manifolds (abbreviated (LCS)n-manifolds). giving an example which generalizes the notion of Lorentzian para Sasakian manifolds introduced by Matsumoto [14] and also by Mihai and Chen [15]. Then, Shaikh and Baishya [18] looked into how (LCS)n -manifolds could be used in general theory of relativity and cosmology. Also, the $(LCS)_n$ -manifolds are also studied by Shaikh, Kim and Hui [19].

The paper is structured as follows. In Section 2, Fundamental formulas and definitions for (LCS)n -manifolds and their submanifolds are reviewed.In Section 3we investigate the geometry of a (LCS)n - manifold's contact pseudo-slant submanifolds. In a (LCS)n -manifold, necessary and sufficient conditions are given for a submanifold to be a contact pseudo-slant submanifold.

2. Preliminaries

An n-dimensional Lorentzian manifold $\widetilde{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0, 2) such that for all point $x \in \widetilde{M}$, the tensor $g_x : T_x\widetilde{M} \times T_x\widetilde{M} \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +,\cdots, +)$, here $T_x\widetilde{M}$ denotes the tangenttial vector space of $\widetilde{M}$ at x and $\mathbb{R}$ is the real number space. A non-zero vector $p \in T_x\widetilde{M}$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_x(p, p) < 0$ (resp., ≤ 0, $= 0$, > 0).

In a Lorentzian manifold $(\widetilde{M},g)$, a vector field K is said to be concircular[20], if the (1, 1)− tensor field A by defined by

$$g(Y,K) = A(Y), \tag{2.1}$$

for all $Y \in \Gamma(T\widetilde{M})$. It is satisfies

$$(\widetilde{\nabla}_Y A)X = \alpha\{g(X,Y) + \omega(X)A(Y)\}, \tag{2.2}$$

where $\alpha \neq 0$ and ω is a closed 1-form and $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let $\widetilde{M}$ be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we obtain

$$g(\xi,\xi) = -1. \tag{2.3}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1−form η such that

$$g(X,\xi) = \eta(X). \tag{2.4}$$

In an $(LCS)_n$-manifold, we obtain

$$(\widetilde{\nabla}_X \eta)Y = \alpha\{g(X,Y) + \eta(Y)\eta(X)\}, \ (\alpha \neq 0), \tag{2.5}$$

$$\widetilde{\nabla}_X \xi = \alpha\{X + \eta(X)\xi\} \ (\alpha \neq 0) \tag{2.6}$$

for all vector fields X, Y, where $\widetilde{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying

$$\widetilde{\nabla}_X \alpha = X(\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.7}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. Let us take

$$\varphi X = \frac{1}{\alpha}\widetilde{\nabla}_X \xi. \tag{2.8}$$

Then from (2.6) and (2.8) we have following equations

$$\varphi X = X + \eta(X)\xi \tag{2.9}$$

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{2.10}$$

from which it follows that φ is a symmetric (1, 1)-tensor and is called the structure tensor of the manifold. So, the Lorentzian manifold $\widetilde{M}$ together with the unit timelike concircular vector field ξ, its associated 1−form η and a (1, 1)−tensor field φ is said to be a Lorentzian concircular structure manifold (shortly, $(LCS)_n$-manifold) [17]. Particularly, if we take $\alpha = 1$, then we can obtain the Lorentzian para-Sasakian structure of Matsumoto (Matsumoto and Mihai, 1988). The following relationships hold in the (LCS)n-manifold (n > 2).

from which it follows that φ is a symmetric (1, 1)-tensor and is called the structure tensor of the manifold. So, the Lorentzian manifold $\widetilde{M}$ together with the unit timelike concircular vector field ξ, its associated 1−form η and a (1, 1)−tensor field φ is said to be a Lorentzian concircular structure manifold (shortly, $(LCS)_n$-manifold) (Shaikh, 2003). Particularly, if we take $\alpha = 1$, then we can obtain the Lorentzian para-Sasakian structure of Matsumoto [14]. The following relationships hold in the (LCS)n-manifold (n > 2).

$$\varphi\xi = 0, \eta(\xi) = -1, \eta(\varphi Y) = 0, g(\varphi Y, \varphi X) = g(Y, X) + \eta(Y)\eta(X), \tag{2.11}$$

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.12}$$

and

$$S(Y, \xi) = (n-1)(\alpha^2 - \rho)\eta(Y), \tag{2.13}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.14}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \tag{2.15}$$

$$(\widetilde{\nabla}_X \varphi)Y = \alpha\{g(X, Y)\xi + \eta(Y)X + 2\eta(Y)\eta(X)\xi\}, \tag{2.16}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.17}$$

$$R(X, Y)Z = \varphi R(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi, \tag{2.18}$$

for any vector field X, Y and Z on $\widetilde{M}$ and $\beta = -(\xi\rho)$ is a scalar function, where S and R are, respectively,the Ricci tensor and the curvature tensor of the manifold.

Let M be a submanifold of an $(LCS)_n$-manifold $\widetilde{M}$ with the induced metric g. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_Y X = \nabla_Y X + h(Y,X) \tag{2.19}$$

and

$$\widetilde{\nabla}_Y V = -A_V Y + \nabla_Y^{\perp} V, \tag{2.20}$$

respectively, where ∇ and $\nabla^{\perp}$ be the induced connections on the tangential bundle TM and the normal bundle $T^{\perp}M$ of M, where h and A_V are, respectively, the second fundamental form and the shape operator (corresponding to the normal vector field V) for the submanifold of M into $\widetilde{M}$. The second fundamental form h and shape operator A_V are related by

$$g(A_V Y, X) = g(h(Y,X), V), \tag{2.21}$$

for any X, $Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. If $h(Y,X)=0$, for any $Y,X \in \Gamma(TM)$, then M is said to be a totally geodesic submanifold.

The mean curvature vector H of M is given by

$$H = \frac{tr(\mathrm{h})}{r}$$

where r is the dimension of M . A submanifold is said to be totally umbilical if it is completely umbilical.

$$\mathrm{h}(X,Y) = \mathrm{g}(X,Y)H$$

- If $\mathrm{h}(X,Y) = 0$, a submanifold is said to be totally geodesic, where for all X,Y∈$\Gamma(TM)$.
- If $\mathrm{H} = 0$, a submanifold is said to be minimal.

Now, let M be a submanifold of an $(LCS)_n$-manifold $\widetilde{M}$, then for any $X \in \Gamma(TM)$, we may write

$$\varphi X = TX + NX, \tag{2.22}$$

where TX is the tangent component and NX is the normal component of φX. Also, for any $V \in \Gamma(T^{\perp}M)$, we have

$$\varphi V = tV + nV, \tag{2.23}$$

where tV and nV are called tangent and normal parts of φV. Thus, by using (2.12), (2.22) and (2.23), we obtain

$$T^2 + tN = I + \eta \otimes \xi, \quad NT + nN = 0 \tag{2.24}$$

and

$$n^2 = I - Nt, \; Tt + tn = 0. \tag{2.25}$$

Moreover, the covariant derivatives of the tensor fields *T, N, t* and *n* are, respectively, defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.26}$$

$$(\nabla_X N)Y = \nabla_X^{\perp} NY - N\nabla_X Y, \tag{2.27}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla_X^{\perp} V \tag{2.28}$$

and

$$(\nabla_X n)V = \nabla_X^{\perp} nV - n\nabla_X^{\perp} V. \tag{2.29}$$

The covariant derivative of φ, $\nabla\varphi$ can be defined by

$$(\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi\tilde{\nabla}_X Y \tag{2.30}$$

for any $X, Y \in \Gamma(TM)$ and $\tilde{\nabla}$ is the Riemannian connection on $\tilde{M}$.

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(TX, Y) = g(X, TY)$ and for $V, W \in \Gamma(T^{\perp}M)$, we get $g(U, nW) = g(nU, W)$. These show that T and n are also symmetric tensor fields. Moreover, for any $Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we can write

$$g(NY, V) = g(Y, tV), \tag{2.31}$$

which is the relation between N and t.

A submanifold M is said to be invariant if N is identically zero, that is, $\varphi Y \in \Gamma(TM)$ for all $Y \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if T is identically zero, that is, $\varphi W \in \Gamma(T^{\perp}M)$ for all $W \in \Gamma(TM)$.

The Gauss and Weingarten formulas together with (2.16), (2.22), (2.23) and (2.30) yield

$$(\nabla_X T)Y = A_{NY}X + th(X,Y) + \alpha\{g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\} \tag{2.32}$$

and

$$(\nabla_X N)Y = nh(X,Y) - h(X,TY). \tag{2.33}$$

for any $X, Y \in \Gamma(TM)$. Similarly, we obtain

$$(\nabla_X t)V = A_{nV}X - TA_V X \tag{2.34}$$

and

$$(\nabla_X n)V = -h(tV, X) - NA_V X. \tag{2.35}$$

for any $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

The canonical structures *T, N, t* and *n* on a submanifold M are said to be parallel if $\nabla T = 0$, $\nabla N = 0$, $\nabla t = 0$ and $\nabla n = 0$, respectively.
Since *M* is tangent to ξ, making use of $\varphi X = \frac{1}{\alpha} \tilde{\nabla}_X \xi$, (2.19), (2.20), (2.21) and (2.22), we obtain

$$\nabla_X \xi = \alpha TX, \quad h(X, \xi) = \alpha NX, \quad A_V \xi = \alpha tV, \tag{2.36}$$

for all $V \in \Gamma(T^{\perp}M)$ and $X \in \Gamma(TM)$.

From (2,24), (2.32) and (2.33), we obtain

$$(\nabla_X T)\xi = -\alpha \{ -tNX + X + \eta(X) \xi \} \tag{2.37}$$

and

$$(\nabla_X N)\xi = -\alpha NTX \tag{2.38}$$

for any $X, \xi \in \Gamma(TM)$.

Similarly, we get

$$(\nabla_\xi t)V = 2\alpha tnV \tag{2.39}$$

and

$$(\nabla_\xi n)V = -2\alpha NtV. \tag{2.40}$$

for any $V \in \Gamma(T^{\perp}M)$ and $\xi \in \Gamma(TM)$.

Now, we put $Q = T^2$, Then the covariant derivative of Q, ∇Q can be defined by

$$(\nabla_X Q)Y = \nabla_X QY - Q\nabla_X Y \tag{2.41}$$

for any $X, Y \in \Gamma(TM)$.

A. Lotta introduced slant submanifolds in contact geometry as follows: [13]

Definition 2.1. Let *M* be a submanifold of an almost contact metric manifold $\tilde{M}(\varphi, \xi, \eta, g)$. Then *M* is said to be a contact slant submanifold if the angle $\theta(X)$ between φX and $T_x(M)$ is constant at any point $x \in M$ for any *X* linearly independent with ξ. Thus the totally real and totally real submanifolds are special classes of slant submanifolds with slant angles $\theta = 0$ and θ

$=\frac{\pi}{2}$, respectively. If the slant angle θ is neither zero nor $\frac{\pi}{2}$, then the slant submanifold is said to be a proper contact slant submanifold.

The following theorem is well known for the slant submanifolds of an almost contact metric manifold [13].

Theorem 2.1. Let M be a submanifold of an $(LCS)_n$-manifold $\widetilde{M}$, such that ξ is tangent to M. Then M is a slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$T^2 = \lambda(I + \eta \otimes \xi). \tag{2.42}$$

Moreover, if θ is the slant angle of M, then it satisfies $\lambda = \cos^2\theta$ [13] .

Corollary 2.2. Let M be a slant submanifold of an $(LCS)_n$-manifold $\widetilde{M}$ with slant angle θ. Then for any $X, Y \in \Gamma(TM)$, we have

$$g(TX,TY) = \cos^2\theta\,\{g(X,Y) + \eta(X)\,\eta(Y)\} \tag{2.43}$$

and

$$g(NX,NY) = \sin^2\theta\,\{g(X,Y) + \eta(X)\,\eta(Y)\}. \tag{2.44}$$

3 Contact pseudo-slant submanifolds in an (LCS)n-manifold

In this section, In a (LCS)n-manifold, necessary and sufficient conditions are given for a submanifold to be a contact pseudo-slant submanifold.

Definition 3.1. [12]Let M be a submanifold of an $(LCS)_n$-manifold $\widetilde{M}(\varphi, \xi, \eta, g)$. We say that M is a contact pseudo-slant submanifold if there exists a pair of orthogonal distributions $D^{\perp}$ and D_θ on M such that

(i) The distribution $D^{\perp}$ is a anti-invariant, i.e., $\varphi(D^{\perp}) \subseteq T^{\perp}M$,

(ii) The distribution D_θ is a slant with slant angle θ,

(iii) The tangent space TM admits the orthogonal direct decomposition

$$TM = D^{\perp} \oplus D_\theta, \xi \in \Gamma(D_\theta).$$

Let d_1 and d_2 be the dimensions of $D^{\perp}$ and D_θ, respectively, Thus if

(i) $d_2 = 0$, then M is a anti-invariant submanifold.

(ii) $d_1 = 0$ and $\theta = 0$, then M is a invariant submanifold.

(iii) $d_1 = 0$ and $0 < \theta < \frac{\pi}{2}$, then M is a proper contact slant submanifold.

(iv) $\theta = \frac{\pi}{2}$, then M is a anti-invariant submanifold.

(v) $d_2\, d_1 \neq 0$ ve $0 < \theta < \frac{\pi}{2}$, then M is a proper contact pseudo-slant submanifold.

If we denote the projections on $D^{\perp}$ and D_θ by ϖ_1 and ϖ_2, respectively, then for any $X \in \Gamma(TM)$, we have

$$X = \varpi_1 X + \varpi_2 X + \eta(X)\xi .$$

If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of a contact pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as follows

$$T^{\perp}M = \varphi\,(D^{\perp}) \oplus N(D_\theta) \oplus \mu,\ \varphi\,(D^{\perp}) \perp N(D_\theta).$$

Theorem 3.1. Let M proper contact pseudo-slant submanifold of a $(LCS)_n$-manifold $\widetilde{M}$. If t is parallel, then M is either mixed geodesic or anti-invariant submanifold.

Proof. For any $X \in \Gamma(D_\theta)$, $Y \in \Gamma(D^{\perp})$, from (2,33) and (2,34) we have t parallel if and only if N parallel, so $\nabla F=0$.
This implies

$$Ch(X,Y) - h\,(X,TY) = 0. \tag{3.1}$$

When we replace X in the above equation with TX, we get

$$nh(TX,Y) - h\,(TX,TY) = 0 \tag{3.2}$$

for $Y \in \Gamma(D^{\perp})$, $TY=0$, so

$$nh(TX,Y) = 0. \tag{3.3}$$

We get by replacing X in the above equation with TX.

$$nh(T^2X,Y) = -n\,cos^2\,\theta\,h(X,Y) = 0. \tag{3.4}$$

As a result, we have $\theta=\frac{\pi}{2}$ (M is anti-invariant) or $h=0$ (M is mixed geodesic).

Theorem 3.2. Let M be a contact pseudo-slant submanifold of a $(LCS)_n$-manifold $\widetilde{M}$. Then the covariant derivative of T is symetric.

Proof. For any $X,Y,Z \in \Gamma(TM)$, we have (2,32)

$$g((\nabla_X T)Y,Z) = g(A_{NY}X + th(X,Y) + \alpha\,\{g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi\},Z). \tag{3,35}$$

If equation (3,35) is used, we obtain

$$\begin{aligned} g((\nabla_X T)Y,Z) &= g(h(X,Z),NY) + g(t\,h(X,Y),Z) \\ &\quad + \alpha\,\{g(X,Y)\,\eta(Z) + \eta(Y)g(X,Z) + 2\,\eta(X)\,\eta(Y)\,\eta(Z)\} \\ &= g(t\,h(X,Z),Y) + g(h(X,Y),NZ) \\ &\quad + \alpha\,\{g(\eta(Z)X + g(X,Z)\xi + 2\,\eta(X)\eta(Z)\xi, Y)\} \\ &= g(A_{NZ}X + th(X,Z) + \alpha\,\{g(X,Z)\xi + \eta(Z)X \\ &\quad + 2\,\eta(X)\,\eta(Z)\xi, Y) \\ &= g((\nabla_X T)Z,Y). \end{aligned}$$

Which supports our claim.

Theorem 3.3. Let M be a proper pseudo-slant submanifold of a $(LCS)_n$-manifold $\widetilde{M}$. The tensor field N is parallel if and only if shape operatory A_V satisfies

$$A_{nV}TY = \cos^2\theta(A_V Y + \eta(Y)\xi) \tag{3.5}$$

for any $Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Proof. If N is parallel, we get from (2,33)

$$nh(X,Y) - h(X,TY) = 0 \tag{3.6}$$

for any X, Y$\in \Gamma(TM)$, This implies

$$nh(X,TY) - h(X,T^2Y) = 0 \tag{3.7}$$

so,

$$nh(X,TY) = \cos^2\theta\, h(X,Y + \eta(Y)\xi). \tag{3.8}$$

As a result, we have

$$g(nh(X,TY),V) = \cos^2\theta\, g(h(X,Y + \eta(Y)\,\xi),V)$$
$$g(h(X,TY),nV) = \cos^2\theta\, g(A_V X, Y + \eta(Y)\,\xi)$$
$$g(A_{NV}TY,X) = \cos^2\theta\, g(A_V Y + \eta(Y)\xi, X)$$

for any $V \in \Gamma(TM^{\perp})$. This equivalent to

$$A_{nV}TY = \cos^2\theta(A_V Y + A_V \eta(Y)\xi). \tag{3.9}$$

The proof is now complete.

Theorem 3.4. Let M be a proper contact pseudo-slant submanifold of a $(LCS)_n$-manifold $\widetilde{M}$. Then n is parallel if and only if the shape operatör A_V of M satisfies the condition $A_U\,tV = -A_V\,tU$ for all $U,V \in \Gamma(T^{\perp}M)$.

Proof. Let n be parallel. Then from (2,35), we have

$$g((\nabla_X n)V,U) = g(-h(X,tV) - N A_V X, U) = 0$$
$$= -g(A_U tV, X) - g(A_V X, tU) = 0$$
$$= -g(A_U tV + A_V tU, X) = 0.$$

Hence we get $A_U tV = -A_V tU$ for $X \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.
which proves our assertion.

Theorem3.5: Let be M be a contact pseudo-slant submanifold of a $(LCS)_n$ – manifold $\widetilde{M}$. Then, we get

$$\cos^2\theta g([X,Y],Z) = g(TA_{NZ}X - A_{NZ}TX, Y)$$

for any $X,Y \in \Gamma(D_\theta)$ and $Z \in (D^{\perp})$.

Proof: for any $X,Y \in \Gamma(D_\theta)$ and $Z \in (D^{\perp})$, by direct calculation using (2,32) and (2,33) we obtain

$$+A_{NZ}X + th(X,Z) = (\nabla_X T)Z = -T\nabla_X Z$$

and

$$(\nabla_X N)Z = nh(X,Z).$$

Also by using (2,27) and (3,40), we conclude that

$$g([X,Y],Z) = g(A_{NZ}X, TY) - g(A_{NZ}Y, TX) + g(\nabla_Y^{\perp} NZ, NX) - g(\nabla_X^{\perp} NZ, NY)$$

$$= g(TA_{NZ}X, Y) - g(A_{NZ}TX, Y) + g((\nabla_Y N)Z + N\nabla_Y Z, NX)$$
$$- g((\nabla_X N)Z + N\nabla_X Z, NY)$$

$$= g(TA_{NZ}X - A_{NZ}TX, Y) + g(\nabla_Y Z, NX) - g(N\nabla_X Z, NY)$$

$$= g(TA_{NZ}X - A_{NZ}TX, Y) + \sin^2\theta\{g(\nabla_X Y, Z) - g(\nabla_Y X, Z)\}$$

$$= g(TA_{NZ}X - A_{NZ}TX, Y) + \sin^2\theta\{g([X,Y],Z)\},$$

thus, we concdude

$$cos^2\theta g([X,Y],Z) = g(TA_{NZ}X - A_{NZ}TX,Y).$$

Theorem 3.6. Let M be a totally umbilical submanifold of an $(LCS)_n$-manifold $\widetilde{M}$. Then at least one of the following satements is true.

(i). M is proper (LCS)n.

(ii). $H \in \Gamma(\nu)$.

(iii).Dim $(D^{\perp}) = 1$.

Proof: Let $X \in \Gamma(D^{\perp})$ and using (2.6), we obtain

$$(\tilde{\nabla}_X\varphi)X = \alpha\, g(X,X)\xi.$$

On applying (2.19), (2.20), (2.22) and (2.23), we get

$$\tilde{\nabla}_X NX - \varphi(\nabla_X X + h(X,X)) - \alpha\, g(X,X)\xi = 0.$$

$$-A_{NX}X + \nabla_X^{\perp}NX - N\nabla_X X - th(X,X) - nh(X,X) - \alpha\, g(X,X)\xi = 0.$$

The tangential components are compared

$$A_{NX}X + th(X,X) + \alpha\, g(X,X)\xi = 0.$$

Taking the product by $W \in \Gamma(D^{\perp})$,we obtain

$$g(A_{NX}X,W) + g(th(X,X),W) = 0.$$

Since M is totally umbilical submanifold, we obtain

$$g(A_{NX}W,X) + (th(X,X),W) = 0$$

$$g(h(W,X),NX) - g(h(X,X),NW) = 0$$

$$g(W,X)g(H,NX) - g(X,X)g(H,NW) = 0$$

$$-g(X,X)g(tH,W) + g(X,X)g(tH,X) = 0$$

that is

$$g(tH,X)W - g(tH,W)X = 0$$

Here tH is either zero or X and W are linearly dependent vector fields. If tH ≠ 0, than dim $(D^{\perp}) = 1$. Othervise H ∈ Γ(μ). Since $D_{\theta} \neq 0$ M is (lcs). Since $\theta \neq 0$ and $d_1 d_2 \neq 0$ proper (LCS)n.

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