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Constant Angle Ruled Surfaces in $\mathbb{E}_1^3$

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Abstract

In this study, for the first time, a method is given for a developable ruled surface to be a constant angle ruled surface. The general equations of constant angle surfaces have been shown in the studies done so far. In this study, a new method is given on how to obtain a constant angled surface when any constant direction is given in Minkowski $3-$space.

1. Introduction

A constant angle surface is a surface whose tangent planes make a constant angle with a fixed vector field of space. In other words, constant angle surfaces whose unit normal forms a constant angle with an assigned direction field in the Euclidean $3-$space. This surface is a generalization of a helical curve. An interesting motivation to study helix surfaces or constant angle surfaces arises from physics. The most basic known application areas of the constant angle surfaces are for light such as crystal, liquid and shape from shading problems. In recent years, many authors have studied these special surfaces to take advantage of their applications in mathematics and physics. Paolo and Scala discuss some properties of constant angel surfaces in terms of the Hamilton-Jacobi equation. They investigate the properties of a constant angle surface when the direction field is singular along a line or a point, [1]. Munteanu and Nistor obtain a classification for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to R in Euclidean $3-$space [2]. Many studies have been done on constant angle surfaces and developable surfaces [3, 4]. In [5], the author investigates the constant angle ruled surfaces generated by Frenet frame vectors. Recently the theory of constant angle surfaces is extended to other ambient spaces. For example; in [6, 7], they study these surfaces in $\mathbb{E}_1^3$. Also, in [8]- [11], the authors extend the concept of constant angle surfaces to a Lorentzian ambient space. Also, in product spaces $\mathbb{S}^2\times\mathbb{R}$ [12, 13], in $\mathbb{H}^2\times\mathbb{R}$ [14] and in Heisenberg group [15, 16].
On the other hand, an isophote curve is defined as the locus of the surface points whose normal vectors make a constant angle with a given constant vector as seen in Figure 1.1. Therefore, we can say that the curves on the constant angle surface are isophote curves. The isophote curve is a nice corollary to Lambert's law of cosines in the optics branch of physics. This law states that the illuminance intensity on a diffused surface is proportional to the cosine of the angle formed between the normal vector of the surface and the light vector. So, we can say the geometric description of isophote curves on surfaces which are the surface normal vectors in points of the curve make a constant angle with a fixed light direction [17]. In recent years, there have been many applications of these curves in different branches. In [18], the authors developed a novel technique to detect caries lesions using isophote concepts. Also, in [19], they present the implementation of a real-time eye detection method that uses the properties of isophotes, to achieve robustness against changes in illumination, eye rotation and pupil size.

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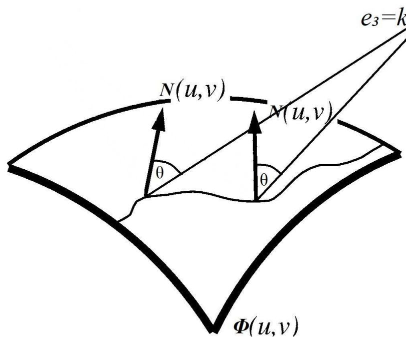

Figure 1.1: An isophote on a surface

In this present paper, we investigate the spherical circles and constant angle surfaces in $\mathbb{E}_1^3$. The difference of the present paper is a fixed angle surface is obtained with respect to any direction and some characterizations are given in three-dimensional Minkowski space. This constant angle surface is the developable ruled surface whose direction is the spherical circle in Minkowski space. Also, by the definition of isophote curves, the curves on this surface are isophote curves. These curves have applications in many fields. At the beginning of these is optics, which is its application in physics. There are many studies that bring together the optics branch of physics and the geometry branch of mathematics [20–24]. This study is one of them. Based on that, we can say that when we beam from a light source in a constant direction, the intensity of the light will be the same at every point on this constant angle surface. On the other hand, the singularity of the ruled surfaces has been studied by many authors. We also investigate the singularity types of this special surface. Finally, as an application, we give some illustrated examples which support the theory of the paper.

2. Preliminaries

Let $\mathbb{E}_1^3 = (\mathbb{R}^3, \langle , \rangle_L)$ be Minkowski 3−space which is given with the Lorentzian metric as follows

$$\begin{array}{rcl} \langle , \rangle_L & = & \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \\ (u,v) & \to & \langle u,v \rangle_L = u_1v_1 + u_2v_2 - u_3v_3, \end{array}$$

where $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3)$ are general coordinates $\mathbb{E}_1^3$. In that case semi-Riemannian metric, an ordinary vector $u \in \mathbb{E}_1^3$ so-called spacelike if $\langle u,u \rangle_L > 0$ or $u = 0$, timelike if $\langle u,u \rangle_L < 0$ and null (lightlike) if $\langle u,u \rangle_L = 0$ but $u \neq 0$. The norm of a vector u is given by $\|u\|_L = \sqrt{|\langle u,u \rangle_L|}$ [25]. Considering the concept of the Lorentz cross product $\times : \mathbb{E}_1^3 \times \mathbb{E}_1^3 \to \mathbb{E}_1^3$. For $u,v \in \mathbb{E}_1^3$, the vector $u \times v$ is defined as

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_2v_1 - u_1v_2).$$

Definition 2.1 ([25]). *Let u and v be two vectors in Minkowski 3-space.*

a. Let u and v be two time-like vectors. If these vectors span a vector subspace, there is a unique real number $\theta \geq 0$ such that

$$\langle u,v \rangle_L = \|u\|_L \|v\|_L \cosh\theta.$$

b. Let u and v be vectors. If these two space-like vectors span a vector subspace, there is a unique real number $\theta \geq 0$ such that

$$\langle u,v \rangle_L = \|u\|_L \|v\|_L \cos\theta.$$

Definition 2.2 ([25]). *Let u and v be space-like and time-like vectors in $\mathbb{E}_1^3$, respectively. Then, there is a unique non-negative real number $\theta \geq 0$ satisfying*

$$\langle u,v \rangle_L = \|u\|_L \|v\|_L \sinh\theta.$$

Definition 2.3 ([25]). *Let u and v be in the same timecone of $\mathbb{E}_1^3$. In this case, there is a unique non-negative real number $\theta \geq 0$ as follows:*

$$\langle u,v \rangle_L = -\|u\|_L \|v\|_L \cosh\theta.$$

Let timelike and spacelike curves be vectors with spacelike or timelike normal vectors, respectively. Such curves are called Frenet curves. In this case, the Frenet equations are given by

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\delta\kappa(s) & 0 & \tau(s) \\ 0 & \varepsilon\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix},$$

where $\langle T,T\rangle_L = \varepsilon$ and $\langle N,N\rangle_L = \delta$ [27]. Let the position vector of the surface M in the standard form of Lorentz Minkowski space $\mathbb{E}_1^3$ is

$$\Phi(u,v) = (\Phi_1(u,v), \Phi_2(u,v), \Phi_3(u,v)).$$

Definition 2.4 ([28]). *Surfaces formed by the movement of a line along a curve in space are called ruled surfaces. The parameterization of the ruled surface for any two differentiable curves α and γ is*

$$\Phi(u,v) = \alpha(v) + u\gamma(v)$$

where $\alpha(v)$ is called base curve of the ruled surface and $\gamma(v)$ is a unit direction vector of an oriented line in $\mathbb{E}_1^3$.

Theorem 2.5 ([29,30]). *Let M be a regular ruled surface with the parameterization $\Phi(u,v) = \alpha(v) + u\gamma(v)$. If the Gaussian curvature of the surface M is zero, the surface M is called a developable surface. Also, another characterization for developable ruled surfaces is that* $\det(\alpha'(v), \gamma(v), \gamma'(v)) = 0$.

Definition 2.6 ([31]). *For the surface $\Phi(u,v) = \alpha(v) + u\gamma(v)$, line of striction is given by*

$$\bar{\alpha}(v) = \alpha(v) - \frac{\langle \gamma(v)\times\gamma'(v), \gamma(v)\times\alpha'(v)\rangle}{\|\gamma(v)\times\gamma'(v)\|^2}\gamma(v).$$

3. Main Results

This section is based on the definition of a constant angle ruled surface in 3-dimensional Minkowski space. In this section, constant angle ruled surfaces are studied with the help of any given direction and these surfaces are characterized. According to the casual characters of the orthonormal vectors, the direction vectors of the constant angle ruled surface can be obtained in different ways in the 3-dimensional Minkowski space.

Case 1. Let $\vec{e}_3$ be a timelike vector. So, $\{\vec{e}_1, \vec{e}_2\}$ are spacelike vectors. The Lorentz circle with the help of these orthonormal vectors in this space is as follows

$$\alpha(v) = \cosh\theta(\cos v\vec{e}_1 + \sin v\vec{e}_2) + \sinh\theta(\vec{e}_1\times\vec{e}_2). \tag{3.1}$$

Examining the casual character of the defined above curve α, provides

$$\langle \alpha, \alpha\rangle = 1.$$

So, we can easily say that α is spacelike and $\alpha \in S_1^2$. We take the derivative of the equation (3.1) with respect to v

$$\alpha'(v) = \cosh\theta(-\sin v\vec{e}_1 + \cos v\vec{e}_2). \tag{3.2}$$

The norm of the equation (3.2) is

$$\left\|\alpha'(v)\right\| = \cosh\theta.$$

Hence, the unit tangent vector of $\alpha(v)$ is obtained as follows

$$T(v) = \frac{\alpha'(v)}{\|\alpha'(v)\|} = -\sin v\vec{e}_1 + \cos v\vec{e}_2.$$

Examining the casual character of the tangent vector, we can see that it is a spacelike vector as

$$\langle T, T\rangle = 1.$$

If we cross product the spacelike curve $\alpha(v)$ and the spacelike tangent vector $T(v)$, we get

$$S(v) = \alpha(v)\times T(v) = -\sinh\theta(\cos v\vec{e}_1 + \sin v\vec{e}_2) - \cosh\theta\vec{e}_3$$

and we obtain the casual character of S is a timelike vector as

$$\langle S, S\rangle = -1.$$

Thus, the Sabban frame $\{\alpha(v), T(v), S(v)\}$ is obtained on S_1^2. If the necessary calculations are made, the derivative change of the frame is found

$$\frac{d}{dv}\begin{bmatrix}\alpha(v)\\ T(v)\\ S(v)\end{bmatrix}=\begin{bmatrix}0 & \cosh\theta & 0\\ -\cosh\theta & 0 & -\sinh\theta\\ 0 & -\sinh\theta & 0\end{bmatrix}\begin{bmatrix}\alpha(v)\\ T(v)\\ S(v)\end{bmatrix}.$$

In addition, the Darboux vector of the Lorenz circle $\alpha(v)$ is a vector that determines the constant direction as

$$\omega=\sinh\theta\,\alpha(v)+\cosh\theta S(v).$$

In fact, if the necessary calculations are made here, it is easily seen that

$$\omega=(\vec{e}_1\times\vec{e}_2)=-\vec{e}_3.$$

Theorem 3.1. *Let $\vec{e}_3$ be timelike and $\{\vec{e}_1,\vec{e}_2\}$ be spacelike vectors in $3-$dimensional Minkowski space. The spacelike Lorentz circle on S_1^2 with the help of the orthonormal vectors in this space is*

$$\alpha(v)=\cosh\theta(\cos v\vec{e}_1+\sin v\vec{e}_2)+\sinh\theta\vec{e}_3,\quad \theta\neq 0.$$

The surface defined below is a spacelike ruled surface

$$\Phi(u,v)\quad\rightarrow\quad \Phi(u,v)=\int_0^v[f(v)\alpha(v)+g(v)\alpha'(v)]\,dv+u\alpha(v) \tag{3.3}$$

and $S(v)=\alpha(v)\times T(v)$ is the timelike unit normal to ruled surface where f and g are the differentiable functions.

Proof. Considering the definition of ruled surfaces,

$$\int_0^v\left[f(v)\alpha(v)+g(v)\alpha'(v)\right]dv$$

is defined as the ruled surface directrix (also called the base curve) and the vector $\alpha(v)$ is defined as the direction vector of the surface. So, we can easily see that the surface $\Phi(u,v)$ is a ruled surface in $3-$dimensional Minkowski space. To find the normal of the surface, we calculate the parameter curves of the surface

$$N=\frac{\Phi_u\times\Phi_v}{\|\Phi_u\times\Phi_v\|}.$$

If the derivatives of equation (3.3) are taken with respect to u and v, respectively, one immediately has

$$\Phi_u=(\cosh\theta\cos v,\cosh\theta\sin v,\sinh\theta)$$

and

$$\Phi_v=(f(v)\cosh\theta\cos v-(g(v)+u)\cosh\theta\sin v, f(v)\cosh\theta\sin v+(g(v)+u)\cosh\theta\cos v, f(v)\sinh\theta).$$

If the following calculations are made to find the normal of the surface, we obtain

$$\Phi_u\times\Phi_v=(-(g(v)+u)\cosh\theta\sinh\theta\cos v,-(g(v)+u)\cosh\theta\sinh\theta\sin v,-(g(v)+u)\cosh^2\theta)$$

and

$$\|\Phi_u\times\Phi_v\|=(g(v)+u)\cosh\theta.$$

Therefore, we can easily find the normal of the surface as follows:

$$N=(-\sinh\theta\cos v,-\sinh\theta\sin v,-\cosh\theta). \tag{3.4}$$

If necessary arrangements are made in equation (3.4), it can be seen that

$$\begin{aligned} N &= -\sinh\theta\,(\cos v\vec{e}_1+\sin v\vec{e}_2)-\cosh\theta\vec{e}_3,\\ N &= S. \end{aligned}$$

Thus, we can say that $S(v)$ is the unit normal vector to the ruled surface $\Phi(u,v)$. If the casual character of the normal vector is computed here, one has

$$\langle N,N\rangle=-1.$$

Hence, the ruled surface $\Phi(u,v)$ is a spacelike surface.

□

Corollary 3.2. *Let $\vec{e}_3$ be timelike and $\{\vec{e}_1, \vec{e}_2\}$ be spacelike vectors in $3-$dimensional Minkowski space. Suppose that the normal of the spacelike surface $\Phi(u,v)$ is N and $\omega = (\vec{e}_1 \times \vec{e}_2) = -\vec{e}_3$ is the axis of the constant direction. Then, the surface $\Phi(u,v)$ is a spacelike constant angle ruled surface.*

Proof. Let the normal of the spacelike surface $\Phi(u,v)$ be N and $\omega = (\vec{e}_1 \times \vec{e}_2) = -\vec{e}_3$ be the axis of the constant direction. Considering equation (3.4) and ω axis of the constant direction, we can write that

$$\langle N, \omega \rangle = -\cosh\theta = \text{constant}.$$

So, we can say that the surface $\Phi(u,v)$ is a spacelike constant angle ruled surface. □

Corollary 3.3. *The surface $\Phi(u,v)$ is a developable spacelike ruled surface.*

Proof. If we restate the base curve of the surface $\Phi(u,v)$ as

$$\varphi = \int_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv,$$

and use the developable ruled surface condition, we obtain that

$$\det\left(\varphi'(v), \alpha(v), \alpha'(v)\right) = \det\left(f(v)\alpha(v) + g(v)\alpha'(v), \alpha(v), \alpha'(v)\right).$$

If necessary calculations are made, it can be easily seen that this determinant value is zero. So, we can say that $\Phi(u,v)$ is a spacelike developable ruled surface. □

Corollary 3.4. *The line of striction of the spacelike surface $\Phi(u,v)$ is*

$$\bar{\varphi} = \varphi + g(v)\alpha(v)$$

where $\varphi = \int_0^v [f(v)\alpha(v) + g(v)\alpha'(v)]\, dv$.

Proof. The line of striction of the surface is computed as follows

$$\bar{\varphi} = \varphi - \frac{\langle \alpha(v) \times \alpha'(v), \alpha(v) \times \varphi'(v) \rangle}{\|\alpha(v) \times \alpha'(v)\|^2} \alpha(v). \tag{3.5}$$

If the necessary calculations are made in equation (3.5), we find

$$\begin{aligned} \alpha(v) \times \alpha'(v) &= \left(-\sinh\theta\cosh\theta\cos v, -\sinh\theta\cosh\theta\sin v, \cosh^2\theta\right), \\ \alpha(v) \times \varphi'(v) &= \left(-g(v)\cosh\theta\sinh\theta\cos v, -g(v)\cosh\theta\sinh\theta\sin v, g(v)\cosh^2\theta\right). \end{aligned}$$

If the above equations are substituted in equation (3.5), the line of striction is obtained as

$$\begin{aligned} \bar{\varphi} &= \varphi + \frac{g(v)\cosh^2\theta}{\cosh^2\theta}\alpha(v), \\ \bar{\varphi} &= \varphi + g(v)\alpha(v). \end{aligned}$$

□

Corollary 3.5. *Considering the theory in the study, we can say that when we are given any axis, we can create a constant angle surface with the help of this axis. For example, we examine the problem of creating a constant angle ruled surface with axis $k = -\vec{e}_3$. To find the Lorentz circle $\alpha(v)$, the circle whose normal is $k = -\vec{e}_3$ must be written. This is found by writing the intersection curve of the light cone and the plane with $-\vec{e}_3$ normal. Let the $\{\vec{e}_1, \vec{e}_2\}$ be an orthonormal frame obtained in the plane whose normal is $-\vec{e}_3$. In this case, the intersection curve of the light cone and the plane is as follows*

$$\cos v\vec{e}_1 + \sin v\vec{e}_2.$$

This curve is the Lorentz circle with radius $r = \cosh\theta$ given by

$$\alpha(v) = \cosh\theta\cos v\vec{e}_1 + \cosh\theta\sin v\vec{e}_2 + \sinh\theta\vec{e}_3.$$

The surface

$$\Phi(u,v) = \int_0^v \left[f(v)\alpha(v) + g(v)\alpha'(v)\right] dv + u\alpha(v)$$

obtained by this circle $\alpha(v)$ is a constant hyperbolic angle ruled surface with the axis $k=-\vec{e}_3$. The normal to this surface is

$$N=(-\sinh\theta\cos v,-\sinh\theta\sin v,-\cosh\theta)$$

and $\langle N,\vec{e}_3\rangle=-\cosh\theta$. According to the state of the θ hyperbolic angle, the angle that the surface makes with the axis is determined. Also, when the functions f and g are changed, they change on the constant angle surfaces.

Theorem 3.6. *Let $\Phi: I\times J\to\mathbb{E}^3$, $\Phi(u,v)=\int_0^v[f(v)\alpha(v)+g(v)\alpha'(v)]\,dv+u\alpha(v)$ be a spacelike constant angle ruled surface and $f,g: I\to\mathbb{R}$ be smooth functions with*

$$\frac{d}{dv}\left(\int_0^v\left[f(v)\alpha(v)+g(v)\alpha'(v)\right]dv\right)=f(v)\alpha(v)+g(v)\alpha'(v).$$

Also, let $(u_0,v_0)\in I\times J$ be a singular point of $\Phi(u,v)$ and put

$$x_0=\int_0^v\left[f(v_0)\alpha(v_0)+g(v_0)\alpha'(v_0)\right]dv+u_0\alpha(v_0)=\Phi(u_0,v_0).$$

The germ of $\Phi(u,v)$ at x_0 is locally diffeomorphic to $C\times\mathbb{R}$ and SW. Also, the germ of $\Phi(u,v)$ at x_0 isn't locally diffeomorphic to CCR.

Proof. Let $\Phi: I\times J\to\mathbb{E}^3$ be a spacelike constant angle ruled surface and $f,g: I\to\mathbb{R}$ be smooth functions. Considering the theory in [32, 33], we calculated that

$$\det\left(\alpha(v),\alpha'(v),\alpha''(v)\right)=\sinh\theta\cosh^2\theta.$$

1. For $\theta\neq 0$ $(\theta\neq\frac{\pi}{2},\pi,\ldots)$, $\det(\alpha(v),\alpha'(v),\alpha''(v))\neq 0$. Then,
   a. Since $u_0=g(v_0)$ and $f(v_0)\neq g'(v_0)$, the germ of $\Phi(u,v)$ at x_0 is locally diffeomorphic to $C\times\mathbb{R}$.
   b. Since $u_0=g(v_0)$, $f(v_0)=g'(v_0)$ and $f'(v_0)\neq g''(v_0)$ the germ of $\Phi(u,v)$ at x_0 is locally diffeomorphic to SW.
2. For $\theta=0$ $\left(\theta=\frac{\pi}{2},\pi,\ldots\right)$, $\det(\alpha(v),\alpha'(v),\alpha''(v))=0$. Then, although $u_0=g(v_0)$, $f(v_0)\neq g'(v_0)$, $\det\left(\alpha(v),\alpha'(v),\alpha^{(3)}(v)\right)=0$. Hence, the germ of $\Phi(u,v)$ at x_0 isn't locally diffeomorphic to CCR.

□

Case 2. Let $\vec{e}_3$ be a timelike vector and $\{\vec{e}_1,\vec{e}_2\}$ be spacelike vectors. The Lorentz circle with the help of these orthonormal vectors in this space is

$$\alpha(v)=\sinh\theta(\cos v\vec{e}_1+\sin v\vec{e}_2)+\cosh\theta(\vec{e}_1\times\vec{e}_2). \tag{3.6}$$

If we examine the casual character of the defined above curve α, we get

$$\langle\alpha,\alpha\rangle=-1.$$

So, we can easily say that α is a timelike vector and $\alpha\in H_0^2$. If we take the derivative of the equation (3.6) with respect to v, we have

$$\alpha'(v)=\sinh\theta(-\sin v\vec{e}_1+\cos v\vec{e}_2). \tag{3.7}$$

The norm of the equation (3.7) is

$$\left\|\alpha'(v)\right\|=\sinh\theta.$$

Thus, the unit tangent vector of $\alpha(v)$ is obtained as follows

$$T(v)=\frac{\alpha'(v)}{\|\alpha'(v)\|}=-\sin v\vec{e}_1+\cos v\vec{e}_2.$$

Examining the casual character of the tangent vector, yields

$$\langle T,T\rangle=1.$$

Taking cross product the timelike curve $\alpha(v)$ and the spacelike tangent vector $T(v)$, provides

$$S(v) = \alpha(v) \times T(v) = \cosh\theta(\cos v\vec{e}_1 + \sin v\vec{e}_2) - \sinh\theta\vec{e}_3$$

and we obtain the casual character of S is spacelike as

$$\langle S, S\rangle = 1.$$

Hence, the Sabban frame $\{\alpha(v), T(v), S(v)\}$ is obtained on H_0^2. Moreover, the Darboux vector of the $\alpha(v)$ is the vector that determines the constant direction as

$$\omega = \cosh\theta\alpha(v) + \sinh\theta S(v).$$

If the necessary calculations are made here, we can easily see the timelike Darboux vector as

$$\langle\omega, \omega\rangle = -1.$$

With the same method as above, theorems and corollaries given in Case 1 can also be given for this case and other cases.

Case 3. Let $\vec{e}_3$ and $\vec{e}_1$ be a spacelike vectors and $\vec{e}_2$ be timelike vector. With the help of these orthonormal vectors, we get the Lorentz circle as

$$\alpha(v) = \cos\theta(\cosh v\vec{e}_1 + \sinh v\vec{e}_2) + \sin\theta(\vec{e}_1 \times \vec{e}_2). \tag{3.8}$$

Examining the casual character of the curve α, we get it as spacelike

$$\langle\alpha, \alpha\rangle = 1.$$

Taking the derivative of the equation (3.8) with respect to v, gives

$$\alpha'(v) = \cos\theta(\sinh v\vec{e}_1 + \cosh v\vec{e}_2).$$

The unit tangent vector of $\alpha(v)$ is obtained

$$T(v) = \sinh v\vec{e}_1 + \cosh v\vec{e}_2.$$

If we examine the casual character of the tangent vector, we have that it is a timelike vector. If we cross product the spacelike curve $\alpha(v)$ and the timelike tangent vector $T(v)$, one has

$$S(v) = \sin\theta(-\cosh v\vec{e}_1 + \sinh v\vec{e}_2) - \cos\theta\vec{e}_3$$

and we obtain the casual character of S vector is spacelike. Thus, the Sabban frame $\{\alpha(v), T(v), S(v)\}$ is obtained on S_1^2. So, from here we can say that if S and $\vec{e}_3$ span the spacelike sub-vector space, there is a single non-negative real number $\theta \geq 0$ such that $\langle S, \vec{e}_3\rangle = \cos\theta$.

Case 4. Let $\vec{e}_3$ and $\vec{e}_2$ be a spacelike vectors and $\vec{e}_1$ be timelike vector. Then the Lorentz circle is

$$\alpha(v) = \cos\theta(\sinh v\vec{e}_1 + \cosh v\vec{e}_2) + \sin\theta(\vec{e}_1 \times \vec{e}_2).$$

If we examine the casual character of the defined above α curve, we get it spacelike. The unit tangent vector of $\alpha(v)$ is obtained as follows

$$T(v) = \cosh v\vec{e}_1 + \sinh v\vec{e}_2.$$

If we cross product the spacelike curve $\alpha(v)$ and the timelike tangent vector $T(v)$, we get

$$S(v) = \sin\theta(-\sinh v\vec{e}_1 + \cosh v\vec{e}_2) - \cos\theta\vec{e}_3$$

and we obtain the casual character of S vector is spacelike. Thus, the Sabban frame $\{\alpha(v), T(v), S(v)\}$ is obtained on S_1^2. So, from here we can say that if S and $\vec{e}_3$ span the spacelike sub-vector space, there is a single non-negative real number $\theta \geq 0$ such that $\langle S, \vec{e}_3\rangle = \cos\theta$.
Given any direction, the equations and figures of the associated constant angle surfaces are discussed in the examples below.

Example 3.7. *We consider the equation of the constant angle surface with the axis*

$$k=\vec{e}_3=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{2}{\sqrt{2}}\right)$$

and draw its graph.

$$\vec{e}_1=(1,1,1), \quad \vec{e}_2=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right).$$

Note that $\vec{e}_3$ is a timelike vector, $\vec{e}_1$ is a spacelike vector, and $\vec{e}_2$ is a spacelike vector and $\vec{e}_1,\vec{e}_2,\vec{e}_3$ are the orthonormal vectors. Then, for $\theta=\ln 2$ the spherical circle $\alpha(v)$ is obtained as follows

$$\alpha(v)=\left(\frac{5}{4}\cos v+\frac{5}{4\sqrt{2}}\sin v-\frac{3}{4\sqrt{2}},\frac{5}{4}\cos v-\frac{5}{4\sqrt{2}}\sin v-\frac{3}{4\sqrt{2}},\frac{5}{4}\cos v-\frac{6}{4\sqrt{2}}\right).$$

For the functions $f(v)=\sin v$ and $g(v)=\cos v$, if the necessary calculations are made in equation (3.3), the equation of the spacelike constant angle ruled surface can be easily written as

$$\Phi(u,v)=(\Phi_1,\Phi_2,\Phi_3)$$

where

$$\begin{aligned}
\Phi_1(u,v) &= \frac{5v+3\cos v}{4\sqrt{2}}+\frac{5}{4}u\cos v+\frac{5}{4\sqrt{2}}u\sin v-\frac{3}{4\sqrt{2}}u,\\
\Phi_2(u,v) &= -\frac{5v-3\cos v}{4\sqrt{2}}+\frac{5}{4}u\cos v-\frac{5}{4\sqrt{2}}u\sin v-\frac{3}{4\sqrt{2}}u,\\
\Phi_3(u,v) &= \frac{3\cos v}{2\sqrt{2}}+\frac{5}{4}u\cos v-\frac{6}{4\sqrt{2}}u.
\end{aligned}$$

If we calculate the singular points for this surface according to Theorem 3.6, we can write that

$$\det\left(\alpha(v),\alpha'(v),\alpha''(v)\right)\neq 0 \quad \text{for} \quad \theta\neq 0\ \left(\theta\neq\frac{\pi}{2},\pi,\ldots\right).$$

a. *For $f(v_0)=\sin v_0$, $g'(v_0)=-\sin v_0$,*

$$\sin v_0\neq-\sin v_0$$

and

$$v_0\neq 0,\pi,2\pi,3\pi,\ldots,k\pi \quad \text{for} \quad k\in\mathbb{Z}.$$

Since $u_0=g(v_0)$, we can say that all points as (u_0,v_0) of $\Phi(u,v)$ satisfying the following condition are locally diffeomorphic to $C\times\mathbb{R}$

$$u_0=\cos v_0 \quad \text{for} \quad v_0\neq 0,\pi,2\pi,3\pi,\ldots,k\pi \quad \text{for} \quad k\in\mathbb{Z}.$$

b. *For $f'(v)=\cos v$, $g'(v)=-\sin v$, $g''(v)=-\cos v$,*

$$\begin{aligned}
f(v_0) &= \sin v,\\
g'(v_0) &= -\sin v.
\end{aligned}$$

From the equality of the above equations, we obtain that

$$v_0=0,\pi,2\pi,3\pi,\ldots,k\pi \quad \text{for} \quad k\in\mathbb{Z}.$$

Considering the following equations,

$$\begin{aligned}
f'(v_0) &= \cos v,\\
g''(v_0) &= -\cos v,
\end{aligned}$$

we have

$$f'(v_0)\neq g''(v_0) \quad \text{for} \quad v_0\neq\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots,\frac{k\pi}{2} \quad \text{for} \quad k=2n+1,\quad n\in\mathbb{Z}.$$

We obtain the other singular point as follows

$$u_0=g(v_0)=g\left(\frac{k\pi}{2}\right)=0 \quad \text{for} \quad k=2n+1,\quad n\in\mathbb{Z}.$$

So, the point $\Phi(u_0,v_0)=\Phi\left(0,\frac{k\pi}{2}\right)$ is locally diffeomorphic to SW. Also, according to Theorem 3.6, we know that the germ of $\Phi(u,v)$ isn't locally diffeomorphic to CCR. So, we give the figure of the spacelike constant angle ruled surface in Figure 3.1 as,

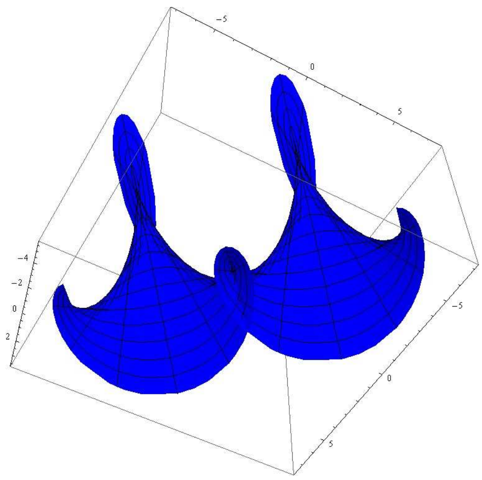

Figure 3.1: Spacelike constant angle ruled surface for $\ln 2$

Example 3.8. *We consider the equation of the constant angle surface with the axis*

$$k=\vec{e}_3=\left(\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{6}\right)$$

and

$$\vec{e}_1=\left(\frac{2}{\sqrt{131}},\frac{3}{\sqrt{131}},\frac{12}{\sqrt{131}}\right),\quad \vec{e}_2=\left(-\frac{7\sqrt{3}}{2\sqrt{131}},\frac{17\sqrt{3}}{3\sqrt{131}},\frac{5\sqrt{3}}{6\sqrt{131}}\right).$$

Note that $\vec{e}_3$ is a spacelike vector, $\vec{e}_1$ is a timelike vector, $\vec{e}_2$ is a spacelike vector and $\vec{e}_1,\vec{e}_2,\vec{e}_3$ are the orthonormal vectors. In this case, for $\theta=\frac{\Pi}{4}$ the spherical circle $\alpha(v)$ is obtained as follows

$$\alpha(v)=\left(\frac{2\sinh v}{\sqrt{262}}-\frac{7\sqrt{3}\cosh v}{2\sqrt{262}}+\frac{\sqrt{3}}{2\sqrt{2}},\frac{3\sinh v}{\sqrt{262}}+\frac{17\sqrt{3}\cosh v}{3\sqrt{262}}+\frac{\sqrt{3}}{3\sqrt{2}},\frac{12\sinh v}{\sqrt{262}}+\frac{5\sqrt{3}\cosh v}{6\sqrt{262}}+\frac{\sqrt{3}}{6\sqrt{2}}\right).$$

For the functions $f(v)=\cosh v$ and $g(v)=\sinh v$, if the necessary calculations are made, the equation of the timelike constant angle ruled surface can be written as

$$\Phi(u,v)=(\Phi_1,\Phi_2,\Phi_3)$$

where

$$\begin{aligned}
\Phi_1(u,v) &= \frac{1}{524}\left(2\sqrt{262}\cosh 2v+131\sqrt{6}\sinh v-7\sqrt{\frac{393}{2}}\sinh 2v\right)+\frac{2u\sinh v}{\sqrt{262}}-\frac{7\sqrt{3}u\cosh v}{2\sqrt{262}}+\frac{\sqrt{3}u}{2\sqrt{2}},\\
\Phi_2(u,v) &= \frac{3\cosh 2v}{2\sqrt{262}}+\frac{\sinh v}{\sqrt{6}}+\frac{17\sinh 2v}{2\sqrt{786}}+\frac{3u\sinh v}{\sqrt{262}}+\frac{17\sqrt{3}u\cosh v}{3\sqrt{262}}+\frac{\sqrt{3}u}{3\sqrt{2}},\\
\Phi_3(u,v) &= 3\sqrt{\frac{2}{131}}\cosh 2v+\frac{\sinh v}{2\sqrt{6}}+\frac{5\sinh 2v}{4\sqrt{786}}+\frac{12u\sinh v}{\sqrt{262}}+\frac{5\sqrt{3}u\cosh v}{6\sqrt{262}}+\frac{\sqrt{3}u}{6\sqrt{2}}.
\end{aligned}$$

So, we give the figure of the timelike constant angle ruled surface for $\theta=\frac{\Pi}{4}$ in Figure 3.2 as,

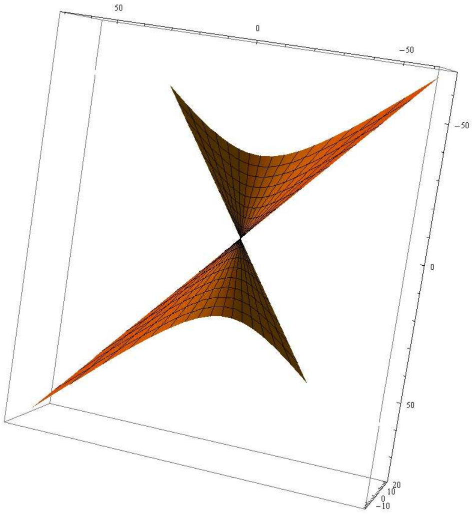

Figure 3.2: Timelike constant angle ruled surface for $\theta = \frac{\Pi}{4}$

4. Conclusion

In this paper, constant angle surfaces with respect to any direction are obtained and their characterizations are given in $\mathbb{E}_1^3$. The constant angle surface mentioned here is the developable ruled surface whose direction is the spherical circle in Minkowski space. It is also clear that the curves on this surface are isophote curves. This curve has many applications in physics, especially optics. Finally, the topics discussed in this article are expressed with some illustrated examples to support the theory of the article.

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