Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 53 (5) (2024), 1218 – 1237
DOI : 10.15672/hujms.1219010

Research Article

Pointwise hemi-slant Riemannian maps ($\mathcal{PHSRM}$) from almost Hermitian manifolds

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Abstract

In 2022, the notion of pointwise slant Riemannian maps were introduced by Y. Gündüzalp and M. A. Akyol in [J. Geom. Phys. 179, 104589, 2022] as a natural generalization of slant Riemannian maps, slant Riemannian submersions, slant submanifolds. As a generalization of pointwise slant Riemannian maps and many subclasses notions, we introduce *p*ointwise hemi-slant Riemannian maps (briefly, $\mathcal{PHSRM}$) from almost Hermitian manifolds to Riemannian manifolds, giving a figure which shows the subclasses of the map and a non-trivial (proper) example and investigate some properties of the map, we deal with their properties: the J-pluriharmonicity, the J-invariant, and the totally geodesicness of the map. Finally, we study some curvature relations in complex space form, involving Chen inequalities and Casorati curvatures for $\mathcal{PHSRM}$, respectively.

Mathematics Subject Classification (2020). 53C15, 53B20

Keywords. Riemannian map, Hermitian manifold, slant Riemannian map, hemi-slant submersion, hemi-slant Riemannian map, pointwise hemi-slant Riemannian map

1. Introduction

In differential geometry, it is useful to define appropriate maps in order to compare differentiable manifolds. In this respect, there are some important maps between manifolds such as isometric immersions, Riemannian submersions and Riemannian maps which are natural generalizations of isometric immersions and Riemannian submersions.

The notion of isometric immersions included many subclasses of submanifolds including important submanifolds of Kaehler manifolds. More precisely, holomorphic and totally real submanifolds were submanifolds examples of Kaehler manifolds. As a generalization of holomorphic and totally real submanifolds, slant submanifolds were introduced by B. Y. Chen in [15]. We recall that a submanifold M is called slant submanifold if for all non-zero vector X tangent to M the angle $\theta(X)$ between JX and T_pM is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_pM$.

In the 1889's, Casorati introduced Casorati curvature which is a very natural concept for regular surfaces in the three-dimensional Euclidean space in [14]. In a Riemannian manifold, this curvature is defined as the normalized square of the length of the second

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Received: 14.12.2022; Accepted: 08.07.2023

fundamental form, and it is well known that this is an extrinsic invariant. Afterwards, many geometers studied some optimal inequalities involving Casorati curvatures in various ambient spaces, for example see ([7, 8, 30–32, 54, 57, 60, 61]).

In the 1960's, B. O'Neill [37] and A . Gray [21] independently introduced Riemannian submersions. More precisely, a differentiable map $\pi : (M_1, g_1) \longrightarrow (M_2, g_2)$ between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a Riemannian submersion if π_* is onto and it satisfies

$$g_2(\pi_* X_1, \pi_* X_2) = g_1(X_1, X_2) \tag{1.1}$$

for X_1, X_2 vector fields tangent to M_1, where π_* denotes the derivative map. The theory is also a very active research field not only in mathematics but also in mathematical physics. More precisely, some of them are the Yang-Mills theory ([11, 58]), the Kaluza-Klein theory ([12, 28]), supergravity and superstring theories ([29, 36]), etc.

In the 1990's, F. Etayo introduced the notion of pointwise slant submanifolds under the name of quasi-slant submanifolds in [19] and B. Y. Chen and O. Garay studied this kind of submanifolds and investigated the geometrical characterizations in [18].

In the 1990's, B. Y. Chen established some inequalities between the main extrinsic (the squared mean curvature) and main intrinsic invariants (the scalar curvature and the Ricci curvature) of a submanifold in a real space form [16]. The author also established a relation between the Ricci curvature and the squared mean curvature for a submanifold [17]. For the inequalities, see: ([9, 34, 35, 51, 55, 56]).

In the 1992's , A. E. Fischer [27] defined the notion of Riemannian maps as a generaliation of isometric immersions and Riemannian submersions. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For the geometry of Riemannian maps between various Riemannian manifolds and their applications in spacetime geometry, see: ([1–6, 20, 23–25, 38–40, 45–49, 52]).

In the 2010's, B. Şahin introduced the anti-invariant Riemannian submersions, semi-invariant Riemannian submersions and slant submersions from almost Hermitian manifolds to Riemannian manifolds. as an analogue of anti-invariant submanifolds, semi-invariant submanifolds and slant submanifolds, respectively in [49]. Afterwards, as a natural generalization of slant submersions, the notion of hemi-slant submersions has defined by H. M. Taştan et. al in [53].

In the 2014's, J. W. Lee and B. Şahin defined the notion of pointwise slant submersions, as a generalization of slant submersions which can be seen analogue of pointiwise slant submanifolds and obtained several basic results in this setting in [33]. More precisely, let σ be a Riemannian submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). If, at each given point $p \in M_1$, the Wirtinger angle $\theta(X)$ between $J_1 X$ and the space $(ker\sigma_*)_p$ is independent of the choice of the nonzero vector $X \in (\ker \sigma_*)$, then we say that σ is a pointwise slant submersion. In this case, the angle θ can be regarded as a function on M_1, which is called the slant function of the pointwise slant submersion. One can find many papers related to this notion see: ([41], [43], [42], [44]).

In [47], B. Şahin introduced slant Riemannian maps from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic Riemannian maps and anti-invariant Riemannian maps, anti-invariant submanifolds, anti-invariant Riemannian submersions, slant submanifolds, slant submersions, then he studied the geometry of such maps. As a generalization of these notions, he also defined the notion of hemi-slant Riemannian maps in [50] (see Figure 1).

In 2022, the present authors [24] introduced the notion of pointwise slant Riemannian maps as a generalization of many notions including slant submanifolds, slant Riemannian submersions, slant Riemannian maps, pointwise slant submanifolds, pointwise slant

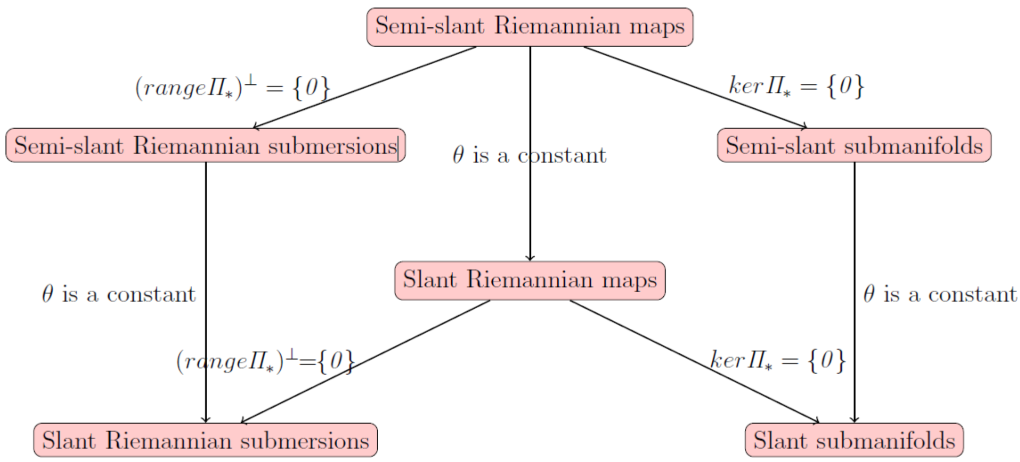

Figure 1. New class of Riemannian maps ($\mathcal{PHSRM}$)

submersions. The aim of the present paper is to introduce and study a new class of Riemannian maps called *p*ointwise hemi-slant Riemannian maps (briefly, $\mathcal{PHSRM}$) as a generalization of many concepts mentioned in Figure 2 below.

The paper is structured as follows. In Section 2 we recall some notions, which will be used in the following sections. In Section 3 we define the notion of $\mathcal{PHSRM}$ from almost Hermitian manifolds to Riemannian manifolds, giving a figure which shows the subclasses of the map and a non-trivial (proper) example and investigate some properties of the map, we deal with their properties: the J-pluriharmonicity of $\mathcal{PHSRM}$, the J-invariant of $\mathcal{PHSRM}$ and the totally geodesic maps of $\mathcal{PHSRM}$. In Section 5 we study some curvature relations in complex space form, involving Chen inequalities and Casorati curvatures for $\mathcal{PHSRM}$, respectively.

2. Preliminaries

In this section, recall some basic materials from [10, 27, 50, 59].

A $2n-$dimensional Riemannian manifold (M_1, g_1, J) is called an almost Hermitian manifold if there exists a tensor field J of type $(1,1)$ on M such that $J^2 = -I$ and

$$g_1(X,Y) = g_1(JX, JY), \quad \forall X, Y \in \Gamma(TM_1), \tag{2.1}$$

where I denotes the identity transformation of T_pM_1. Consider an almost Hermitian manifold (M_1, g_1, J) and denote by ∇ the Levi-Civita connection on M_1 with respect to g_1. Then M_1 is called a Kaehler manifold [59] if J is parallel with respect to ∇, i.e.

$$(\nabla_X J)Y = 0, \tag{2.2}$$

$\forall X, Y \in \Gamma(TM_1)$.

As a generalization of isometric immersions and Riemannian submersions, the notion of Riemannin maps was defined by Fischer in [27] as follows;

Let σ be a C^∞-map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). The second fundamental form of σ is given by

$$(\nabla\sigma_*)(X,Y) = \nabla^\sigma_X \sigma_* Y - \sigma_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM_1), \tag{2.3}$$

where ∇^σ is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_1 and g_2 [10].

We call the map σ a totally geodesic map if $(\nabla\sigma_*)(X,Y) = 0$ for $X, Y \in \Gamma(TM_1)$. [10]

Denote the range of σ_* by $range\sigma_*$ as a subset of the pullback bundle $\sigma^{-1}TM_2$. With its orthogonal complement $(range\sigma_*)^{\perp}$ we obtain the following decomposition

$$\sigma^{-1}TM_2 = \mathrm{range}\,\sigma_* \oplus (\mathrm{range}\sigma_*)^{\perp}.$$

Moreover, we have

$$TM_1 = \ker\sigma_* \oplus (\ker\sigma_*)^{\perp}.$$

Finally, B. Şahin proved the following lemma in [45].

Theorem 2.1 ([45]). *Let σ be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). Then*

$$(\nabla\sigma_*)(X, Y) \in \Gamma((\mathrm{range}\,\sigma_*)^{\perp}) \quad \textit{for } X, Y \in \Gamma((\ker\sigma_*)^{\perp}). \tag{2.4}$$

Let σ be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). Then, we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{T}_{\xi_1}\xi_2 = h\nabla_{v\xi_1}v\xi_2 + v\nabla_{v\xi_1}h\xi_2 \tag{2.5}$$

and

$$\mathcal{A}_{\xi_1}\xi_2 = v\nabla_{h\xi_1}h\xi_2 + h\nabla_{h\xi_1}v\xi_2 \tag{2.6}$$

for every $\xi_1, \xi_2 \in \Gamma(TM_1)$, where ∇ is the Levi-Civita connection of g_1. In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined for Riemannian submersions. For any $\xi_1 \in \Gamma(TM_1)$, $\mathcal{T}_{\xi_1}$ and $\mathcal{A}_{\xi_1}$ are skew-symmetric operators on $(\Gamma(TM_1), g_1)$ reversing the horizontal and the vertical distributions. We note that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy

$$\mathcal{T}_{\eta_1}\eta_2 = \mathcal{T}_{\eta_2}\eta_1,\ \ \mathcal{A}_{\xi_1}\xi_2 = -\mathcal{A}_{\xi_2}\xi_1,\ \ \forall\eta_1, \eta_2 \in \Gamma(ker\sigma_*),\ \forall\xi_1, \xi_2 \in \Gamma((ker\sigma_*)^{\perp}). \tag{2.7}$$

Using (2.5) and (2.6), we obtain

$$\nabla_{\eta_1}\eta_2 = \mathcal{T}_{\eta_1}\eta_2 + \hat{\nabla}_{\eta_1}\eta_2; \tag{2.8}$$

$$\nabla_{\eta_1}\xi_1 = \mathcal{T}_{\eta_1}\xi_1 + h\nabla_{\eta_1}\xi_1; \tag{2.9}$$

$$\nabla_{\xi_1}\eta_1 = \mathcal{A}_{\xi_1}\eta_1 + v\nabla_{\xi_1}\eta_1; \tag{2.10}$$

$$\nabla_{\xi_1}\xi_2 = \mathcal{A}_{\xi_1}\xi_2 + h\nabla_{\xi_1}\xi_2, \tag{2.11}$$

for any $\xi_1, \xi_2 \in \Gamma((ker\sigma_*)^{\perp})$, $\eta_1, \eta_2 \in \Gamma(ker\sigma_*)$, here $\hat{\nabla}_{\eta_1}\eta_2 = v\nabla_{\eta_1}\eta_2$.

3. $\mathcal{PHSRM}$ from Kaehler manifolds

In this section, we are going to introduce pointwise hemi-slant Riemannian maps (briefly, $\mathcal{PHSRM}$) from almost Hermitian manifolds to Riemannian manifolds, provide some examples and investigate the geometry of foliations and their geometric properties. We first deal with the J-pluriharmonicity, the J-invariant of the map and obtain necessary and sufficient conditions for the image of σ_* to be a local product Riemannian manifold and give necessary and sufficient conditions for σ to be totally geodesic. Finally, we give some theorems on the harmonicity of the $\mathcal{PHSRM}$ maps.

Definition 3.1. Let (M_1, g_1, J) be an almost Hermitian manifold and (M_2, g_2) be a Riemannian manifold. Then we say that a Riemannian map $\sigma : M_1 \to M_2$ is a pointwise hemi-slant Riemannian map ($\mathcal{PHSRM}$) if there exists a pair of orthogonal distributions $\mathcal{D}^{\theta}$ and $\mathcal{D}^{\perp}$ on $ker\sigma_*$ such that

(1) The space $ker\sigma_*$ admits the orthogonal direct decomposition $\mathcal{D}^{\theta} \oplus \mathcal{D}^{\perp}$.
(2) The distribution $\mathcal{D}^{\perp}$ is totally real (anti-invariant).
(3) The distribution $\mathcal{D}^{\theta}$ is pointwise slant with slant function θ.

In this case, the angle θ can be regarded as a function on M_1, which is called the hemi-slant function of the $\mathcal{PHSRM}$.

Figure 2 shows some examples for $\mathcal{PHSRM}$.

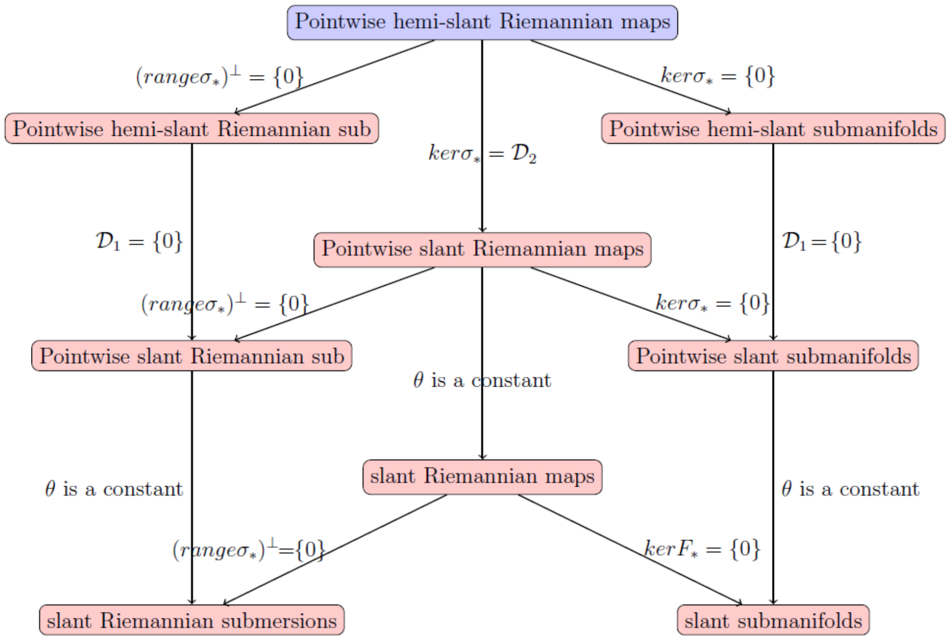

Figure 2. Examples of $\mathcal{PHSRM}$

We now give two non-trivial examples for $\mathcal{PHSRM}$.

Example 3.2. Let $(\mathbb{R}^8, g_{\mathbb{R}^8})$ be the Euclid space. Consider $\{J_1, J_2\}$ a pair of almost complex structures on $\mathbb{R}^8$ satisfying $J_1 J_2 = -J_2 J_1$, here

$$J_1(a_1, ..., a_8) = (-a_3, -a_4, a_1, a_2, -a_7, -a_8, a_5, a_6)$$

and

$$J_2(a_1, ..., a_8) = (-a_2, a_1, a_4, -a_3, -a_6, a_5, a_8, -a_7).$$

For any real-valued function $\lambda : \mathbb{R}^8 \to \mathbb{R}$, we define new almost complex structure J_λ on $\mathbb{R}^8$ by $J_\lambda = (\cos\lambda)J_1 + (\sin\lambda)J_2$. Then, $\mathbb{R}^8_\lambda = (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8})$ is an almost Hermitian manifold. Consider a Riemannian map $\sigma : \mathbb{R}^8_\lambda \to \mathbb{R}^8$ by

$$\sigma(x_1, ..., x_8) = (x_2, x_3, x_6, x_8, 1992, 2014, 2018, 2022).$$

Then, by direct calculations, we obtain the Jacobian matrix of σ as:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{8\times 8}$$

Then the map σ is a $\mathcal{PHSRM}$ such that

$$\mathcal{D}^\theta = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} \right\rangle \ and \ \mathcal{D}^\perp = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4} \right\rangle.$$

Also, we obtain

$$(ker\sigma_*)^\perp = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8} \right\rangle,$$

with the slant function $\theta = f$.

Let σ be a $\mathcal{PHSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2). Then for any $V \in \Gamma(\ker\sigma_*)$, we put

$$JV = \phi V + \omega V, \tag{3.1}$$

where $\phi V \in \Gamma(\ker\sigma_*)$ and $\omega V \in \Gamma(ker\sigma_*)^\perp$. Also for any $\xi \in \Gamma(ker\sigma_*)^\perp$, we have

$$J\xi = \mathcal{B}\xi + \mathcal{C}\xi, \tag{3.2}$$

where $\mathcal{B}\xi \in \Gamma(\ker\sigma_*)$ and $\mathcal{C}\xi \in \Gamma(ker\sigma_*)^\perp$.

The proof of the following result is exactly the same as that for slant immersions (see [15] or [13] for Sasakian case), so we omit its proof.

Theorem 3.3. *Let σ be a $\mathcal{PHSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2). Then σ is a $\mathcal{PHSRM}$ if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$\phi^2 U = \lambda U \tag{3.3}$$

for $U \in \Gamma(\mathcal{D}^\theta)$. If σ is a $\mathcal{PHSRM}$, then $\lambda = -\cos^2\theta$.

By using the above theorem, it is easy to see that

$$g_2(\phi\sigma_*(U), \phi\sigma_*(V)) = \cos^2\theta g_1(U, V),$$
$$g_2(\omega\sigma_*(U), \omega\sigma_*(V)) = \sin^2\theta g_1(U, V),$$

for any $U, V \in \Gamma(\mathcal{D}^\theta)$.

Now, we are going to investigate the $J-$pluriharmonicity of the $\mathcal{PHSRM}$ with respect to the distibutions on the total space. First, we have the following definition.

Definition 3.4. Let σ be a $\mathcal{PHSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. A $\mathcal{PHSRM}$ is called $J-$pluriharmonic, $(ker\sigma_*)^\perp$-J-pluriharmonic, $ker\sigma_*$-J-pluriharmonic, $\mathcal{D}^\perp$-J-pluriharmonic, $\mathcal{D}^\theta$-J-pluriharmonic, $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-pluriharmonic and $((ker\sigma_*)^\perp - ker\sigma_*)$-$J$-pluriharmonic if

$$(\nabla\sigma_*)(X, Y) + (\nabla\sigma_*)(JX, JY) = 0 \tag{3.4}$$

for any $X, Y \in \Gamma(TM_1)$, for any $X, Y \in \Gamma((ker\sigma_*)^\perp)$, for any $X, Y \in \Gamma(ker\sigma_*)$, for any $X, Y \in \Gamma(\mathcal{D}^\perp)$, for any $X, Y \in \Gamma(\mathcal{D}^\theta)$, for any $X \in \Gamma((ker\sigma_*)^\perp)$, $Y \in \Gamma(ker\sigma_*)$,

We first have the following theorem.

Theorem 3.5. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. Suppose that the map σ is a $\mathcal{D}^\perp$-J-pluriharmonic. Then the map σ is a $ker\sigma_*$-geodesic map if and only if $\mathcal{T} = \{0\}$ which gives that the fibres are totally geodesic submanifolds.*

Proof. For any $U, V \in \Gamma(\mathcal{D}^\perp)$, since $\mathcal{D}^\perp$-J-pluriharmonic, by virtue of (2.3) we have

$$0 = (\nabla\sigma_*)(U, V) + (\nabla\sigma_*)(JU, JV)$$
$$= -\sigma_*(\mathcal{T}_U V) + (\nabla\sigma_*)(JU, JV)$$

which gives the proof. □

For the slant distribution $\mathcal{D}^\theta$, we have

Theorem 3.6. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function ϕ. Suppose that the map σ is a $\mathcal{D}^\theta$-J-pluriharmonic. Then the map σ is a $\omega\mathcal{D}^\theta$-geodesic map if and only if $\mathcal{T}_U V + \mathcal{T}_{\phi U}\phi V + \mathcal{H}\nabla_{\phi V}\omega W + \mathcal{A}_{\omega V}\phi W$.*

Proof. Given $U, V \in \Gamma(\mathcal{D}^\theta)$, since $\mathcal{D}^\theta$-J-pluriharmonic, by virtue of (2.3) we obtain

$$
\begin{aligned}
0 &= (\nabla\sigma_*)(V, W) + (\nabla\sigma_*)(JV, JW)\\
&= -\sigma_*(\mathcal{T}_V W) + (\nabla\sigma_*)(\omega V, \omega W) - \sigma_*(\mathcal{T}_{\phi V}\phi W + \mathcal{H}\nabla_{\phi V}\omega W + \mathcal{A}_{\omega V}\phi W)\\
(\nabla\sigma_*)(\omega V, \omega W) &= -\sigma_*(\mathcal{T}_V W + \mathcal{T}_{\phi V}\phi W + \mathcal{H}\nabla_{\phi V}\omega W + \mathcal{A}_{\omega V}\phi W)
\end{aligned}
$$

which completes the proof. □

For $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-pluriharmonicity, we have the following theorem.

Theorem 3.7. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. Suppose that the map σ is a $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-pluriharmonic. Then the following assertions are equivalent.*

(i) *The anti-invariant distribution $\mathcal{D}^\perp$ defines a totally geodesic foliations on M_1.*

(ii) $\nabla^{M_2}_{\sigma_* JV}\sigma_*\omega W = \sigma_*(\mathcal{C}\mathcal{A}_{JV}W + \omega\mathcal{V}\nabla_{JV}W)$

Proof. For $V \in \Gamma(\mathcal{D}^\perp)$ and $W \in \Gamma(\mathcal{D}^\theta)$, since the map σ is a $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-pluriharmonic, by using (2.3), we get

$$
\begin{aligned}
0 &= (\nabla\sigma_*)(V, W) + (\nabla\sigma_*)(JV, JW)\\
&= -\sigma_*(\nabla_V W) + \nabla^{M_2}_{\sigma_*(JV)}\sigma_*(\omega W) - \sigma_*(\nabla_{JV} JW)\\
&= -\sigma_*(\nabla_V W) + \nabla^{M_2}_{\sigma_*(JV)}\sigma_*(\omega W) - \sigma_*(J\nabla_{JV} W)\\
&= -\sigma_*(\nabla_V W) + \nabla^{M_2}_{\sigma_*(JV)}\sigma_*(\omega W) - \sigma_*(\mathcal{C}\mathcal{A}_{JV}W + \omega\mathcal{V}\nabla_{JV}W)\\
\sigma_*(\nabla_V W) &= \nabla^{M_2}_{\sigma_*(JV)}\sigma_*(\omega W) - \sigma_*(\mathcal{C}\mathcal{A}_{JV}W + \omega\mathcal{V}\nabla_{JV}W)
\end{aligned}
$$

which gives the proof. □

Finally, for $((ker\sigma_*)^\perp - ker\sigma_*)$-$J$-pluriharmonicity, we have the following theorem.

Theorem 3.8. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function ϕ. Suppose that the map σ is a $(ker\sigma_*)^\perp$-$ker\sigma_*$-J-pluriharmonic. Then the following assertions are equivalent.*

(i) *The horizontal distribution $(ker\sigma_*)^\perp$ defines a totally geodesic foliations on M_1.*

(ii) $(\nabla\sigma_*)(\mathcal{C}X, \omega U) = -\sigma_*(\mathcal{T}_{\mathcal{B}X}\phi U + \mathcal{H}\nabla_{\mathcal{B}X}\omega U + \mathcal{A}_{\mathcal{C}X}\phi U)$

for any $X \in \Gamma(ker\sigma_*)^\perp$ and $U \in \Gamma(ker\sigma_*)$.

Proof. For $X \in \Gamma(ker\sigma_*)^\perp$ and $U \in \Gamma(ker\sigma_*)$, since the map σ is a $((ker\sigma_*)^\perp - ker\sigma_*)$-$J$-pluriharmonic, by using (2.3), we get

$$
\begin{aligned}
0 &= (\nabla\sigma_*)(X, U) + (\nabla\sigma_*)(JX, JU)\\
&= -\sigma_*(\nabla_X U) + (\nabla\sigma_*)(\mathcal{B}X, \phi U) + (\nabla\sigma_*)(\mathcal{B}X, \omega U)\\
&\quad + (\nabla\sigma_*)(\mathcal{C}X, \phi U) + (\nabla\sigma_*)(\mathcal{C}X, \omega U)\\
&= -\sigma_*(\nabla_X U) - \sigma_*(\mathcal{T}\mathcal{B}X, \phi U) - \sigma_*(\mathcal{H}\nabla_{\mathcal{B}X}\omega U)\\
&\quad - \sigma_*(\mathcal{A}_{\mathcal{C}X}\phi U) + (\nabla\sigma_*)(\mathcal{C}X, \omega U)\\
(\nabla\sigma_*)(\mathcal{C}X, \omega U) &= -\sigma_*(\nabla_X U) - \sigma_*(\mathcal{T}\mathcal{B}X, \phi U + \mathcal{H}\nabla_{\mathcal{B}X}\omega U) + \mathcal{A}_{\mathcal{C}X}\phi U)
\end{aligned}
$$

which completes the proof. □

Finally, we will find necessary and sufficient conditions for the $\mathcal{PHSRM}$ to be the $J-$invariant of the distibutions on the total space. First, we have the following definition.

Definition 3.9. Let σ be a $\mathcal{PHSRM}$ from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. A $\mathcal{PHSRM}$ is called $J-$invariant, $(ker\sigma_*)^\perp$-J-invariant, $ker\sigma_*$-J-invariant, $\mathcal{D}^\perp$-J-invariant, $\mathcal{D}^\theta$-J-invariant, $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-invariant and $((ker\sigma_*)^\perp - ker\sigma_*)$-$J$-invariant if

$$(\nabla\sigma_*)(Z, W) = (\nabla\sigma_*)(JZ, JW) \tag{3.5}$$

for any $Z, W \in \Gamma(TM_1)$, for any $Z, W \in \Gamma((ker\sigma_*)^\perp)$, for any $Z, W \in \Gamma(ker\sigma_*)$, for any $Z, W \in \Gamma(\mathcal{D}^\perp)$, for any $Z, W \in \Gamma(\mathcal{D}^\theta)$, for any $Z \in \Gamma((ker\sigma_*)^\perp)$, $W \in \Gamma(ker\sigma_*)$,

We first have the following theorem.

Theorem 3.10. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. Suppose map σ is a $\mathcal{D}^\perp$-J-invariant. The following assertiones are equivalent.*

(i) *The anti-invariant distribution $\mathcal{D}^\perp$ defines a totally geodesic foliations on M_1.*

(ii) $\nabla^{M_2}_{\sigma_* JX}\sigma_* JZ = \sigma_*(\mathcal{C}\mathcal{A}_{JX}Z + \omega\mathcal{V}\nabla_{JX}Z)$

for any $X, Z \in \Gamma(\mathcal{D}^\perp)$.

Proof. Given $X, Z \in \Gamma(\mathcal{D}^\perp)$, since the map is $\mathcal{D}^\perp$-J-invariant, by using (2.3), we get the proof. □

For the slant distribution $\mathcal{D}^\theta$, we have

Theorem 3.11. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. Suppose map σ is a $\mathcal{D}^\theta$-J-invariant. The following assertiones are equivalent.*

(i) *The fibres are totally geodesic submanifolds in M_1.*

(ii) $\nabla\sigma_*(\omega U, \omega V) = \sigma_*(\mathcal{T}_{\phi U}\phi U + \mathcal{H}\nabla_{\phi U}\omega V - \mathcal{A}_{\omega U}\phi U)$

for any $U, V \in \Gamma(\mathcal{D}^\theta)$.

Proof. Given $U, V \in \Gamma(\mathcal{D}^\theta)$, since $\mathcal{D}^\theta$-J-invariant, by virtue of (2.3), we obtain

$$\begin{aligned}
&(\nabla\sigma_*)(U, V) = (\nabla\sigma_*)(JU, JV)\\
&-\sigma_*(\nabla_U V) = (\nabla\sigma_*)(\phi U, \phi V) + (\nabla\sigma_*)(\phi U, \omega V) + (\nabla\sigma_*)(\omega U, \phi V) + (\nabla\sigma_*)(\omega U, \omega V)\\
&-\sigma_*(\nabla_U V) = -\sigma_*(\nabla_{\phi U}\phi V) - \sigma_*(\nabla_{\phi U}\omega V) - \sigma_*(\nabla_{\omega U}\phi V) - \sigma_*(\nabla_{\omega U}\omega V)\\
&-\sigma_*(\nabla_U V) = -\sigma_*(\mathcal{T}_{\phi U}\phi V + \mathcal{H}\nabla_{\phi U}\omega V - \mathcal{A}_{\omega U}\phi V) - \sigma_*(\nabla_{\omega U}\omega V).
\end{aligned}$$

which completes the proof. □

For $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-invariant, we have the following theorem.

Theorem 3.12. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. The map σ is a $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-invariant if and only if $\nabla^{M_2}\sigma_*(JX)\sigma_*(\omega U) = \sigma_*(\mathcal{A}_{JX}\phi U + \mathcal{H}\nabla_{JX}\omega U - \mathcal{A}_X U)$ for any $X \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$.*

Proof. Given $X \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$. since $(\mathcal{D}^\perp - \mathcal{D}^\theta)$-$J$-invariant, by virtue of (2.3) we obtain

$$\begin{aligned}
&(\nabla\sigma_*)(X, U) = (\nabla\sigma_*)(JX, JU)\\
&-\sigma_*(\nabla_X U) = (\nabla\sigma_*)(JX, \phi U) + (\nabla\sigma_*)(JX, \omega U)\\
&-\sigma_*(\nabla_X U) = -\sigma_*(\nabla_{JX}\phi U) - \nabla^{M_2}\sigma_*(JX)\sigma_*(\omega U) - \sigma_*(\nabla_{JX}\omega U)\\
&-\sigma_*(\nabla_U V) = -\sigma_*(\mathcal{A}_{JX}\phi U + \mathcal{H}\nabla_{JX}\omega U - \sigma_*(\mathcal{H}\nabla_{JX}\omega U).
\end{aligned}$$

which gives the proof. □

Finally, for $((ker\sigma_*)^\perp - ker\sigma_*)$-$J$-invariant, we have the following theorem.

Theorem 3.13. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ. If the map σ is a $((ker\sigma_*)^\perp - ker\sigma_*)$-$J$-invariant if and only if $\mathcal{C}(\mathcal{T}_{\mathcal{B}X}U + \mathcal{A}_{\mathcal{C}X}U) + \omega(\hat{\nabla}_{\mathcal{B}X}U + \mathcal{V}\nabla_{\mathcal{C}X}U) + \mathcal{A}_X U = 0$ for any $X \in \Gamma(ker\sigma_*)^\perp$ and $U \in \Gamma(ker\sigma_*)$.*

Proof. Given $X \in \Gamma(ker\sigma_*)^\perp$ and $U \in \Gamma(ker\sigma_*)$. We assume that the map is invariant. In this case, by virtue of (2.3) we have

$$\begin{aligned}(\nabla\sigma_*)(X,U) &= (\nabla\sigma_*)(JX, JU)\\ -\sigma_*(\nabla_X U) &= (\nabla\sigma_*)(\mathcal{B}X, JU) + (\nabla\sigma_*)(\mathcal{C}X, JU)\\ -\sigma_*(\nabla_X U) &= -\sigma_*(\nabla_{\mathcal{B}X} JU) - -\sigma_*(\nabla_{\mathcal{C}X} JU)\\ -\sigma_*(\nabla_U V) &= -\sigma_*(J(\mathcal{T}_{\mathcal{B}X}U + \hat{\nabla_{\mathcal{B}X}}U) + J(\mathcal{A}_{\mathcal{C}X}U + \mathcal{V}\nabla_{\mathcal{C}X}U)\\ 0 &= \sigma_*(\mathcal{C}(\mathcal{T}_{\mathcal{B}X}U + \mathcal{A}_{\mathcal{C}X}U) + \omega(\hat{\nabla_{\mathcal{B}X}}U + \mathcal{V}\nabla_{\mathcal{C}X}U + \mathcal{A}_X U)\end{aligned}$$

which completes the proof. □

Recall that a map σ is called totally geodesic if $(\nabla\sigma_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM_1)$. Geometrically the notion implies that for each geodesic β in M_1 the image $\sigma(\beta)$ is a geodesic in M_2.

Theorem 3.14. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2). Then σ is totally geodesic if and only if*

$$\omega\mathcal{T}_U JV + \mathcal{C}\mathcal{H}\nabla_U JV = 0$$
$$\sin 2\theta U(\theta)Z + \mathcal{H}\nabla_U \omega\phi Z + \mathcal{C}\mathcal{H}\nabla_U \omega Z + \omega\mathcal{T}_U \omega Z = 0$$
$$\sin 2\theta X(\theta)Z + \mathcal{H}\nabla_X \omega\phi Z + \mathcal{C}\mathcal{H}\nabla_X \omega Z + \omega\mathcal{A}_X \omega Z = 0$$

and

$$\nabla^\sigma_X \sigma_*(Y) = -\sigma_*\left(\mathcal{A}_X \phi\mathcal{B}Y + \mathcal{H}\nabla_X \omega\mathcal{B}Y\right) + \mathcal{C}\mathcal{H}\nabla_X \mathcal{C}Y + \omega\mathcal{A}_X \mathcal{C}Y)$$

for $\xi \in \Gamma(ker\sigma_*), U, V \in \Gamma\left(\mathcal{D}^\perp\right), Z \in \Gamma\left(\mathcal{D}^\theta\right)$ and $X, Y \in \Gamma\left((ker\sigma_*)^\perp\right)$

Proof. For $U, V \in \Gamma\left(\mathcal{D}^\perp\right)$, from (2.2), we have

$$(\nabla\sigma_*)(U, V) = \sigma_*(J\nabla_U JV).$$

By virtue of (2.9), (3.1) and and (3.2), we get

$$(\nabla\sigma_*)(U, V) = \sigma_*(\omega\mathcal{T}_U JV + \mathcal{C}\mathcal{H}\nabla_U JV). \tag{3.6}$$

For $U \in \Gamma(\ker\sigma_*)$ and $Z \in \Gamma\left(\mathcal{D}^\theta\right)$, (2.3), (2.2) and (3.1) imply

$$(\nabla\sigma_*)(U, Z) = \sigma_*\left(\nabla_U \phi^2 Z + \nabla_U \omega\phi Z + \omega\mathcal{T}_U \omega Z + \mathcal{C}\mathcal{H}\nabla_U \omega Z\right).$$

Then by using (3.3), we derive

$$\sin^2\theta\,(\nabla\sigma_*)(U, Z) = \sigma_*(\sin 2\theta U(\theta)Z + \mathcal{H}\nabla_U \omega\phi Z + \mathcal{C}\mathcal{H}\nabla_U \omega Z + \omega\mathcal{T}_U \omega Z). \tag{3.7}$$

In a similar way, for $X \in \Gamma\left((ker\sigma_*)^\perp\right)$ and $Z \in \Gamma\left(\mathcal{D}^\theta\right)$, we obtain

$$\sin^2\theta\,(\nabla\sigma_*)(X, Z) = \sigma_*(\sin 2\theta X(\theta)Z + \mathcal{H}\nabla_X \omega\phi Z + \mathcal{C}\mathcal{H}\nabla_X \omega Z + \omega\mathcal{A}_X \omega Z). \tag{3.8}$$

For $X, Y \in \Gamma\left((\ker\sigma_*)^\perp\right)$, from (2.3), (2.2) and (2.10), we have

$$\begin{aligned}(\nabla\sigma_*)(X, Y) &= \nabla^\sigma_X \sigma_*(Y) + \sigma_*(\nabla_X J\mathcal{B}Y) + \sigma_*(J\nabla_X \mathcal{C}Y)\\ &= \nabla^\sigma_X \sigma_*(Y) + \sigma_*(\mathcal{A}_X \phi\mathcal{B}Y + \mathcal{H}\nabla_X \omega\mathcal{B}Y + \mathcal{C}\mathcal{H}\nabla_X \mathcal{C}Y + \omega\mathcal{A}_X \mathcal{C}Y).\end{aligned} \tag{3.9}$$

Thus proof is complete due to (3.6)-(3.9). □

4. Chen-Ricci inequality and Casorati curvatures of $\mathcal{PHSRM}$

In the present section, we aim to obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for $\mathcal{PHSRM}$ from a Kaehler manifold to a Riemannian manifold. We also consider the equality cases of these inequalities. Finally, we study Casorati curvatures in comlex space form for $\mathcal{PHSRM}$.

Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$ and $\dim(\ker\sigma_*) = r = k_1 + 2k_2$. For every $q \in M_1$, we consider $\{X_1, X_2, \ldots, X_{k_1}, X_{k_1+1}, X_{k_1+2}, \ldots, X_{k_1+k_2}, \sec\theta\phi X_{k_1+1}, \ldots, \sec\theta\phi X_{k_1+k_2}\}$ and $\{X_{r+1}, \ldots, X_m\}$ two orthonormal bases of $(\ker\sigma_*)$ and $(\ker\sigma_*)^\perp$. From [26] and [49], we have

$$\begin{aligned}\hat{R}(U,V,F,W) = \frac{v}{4}\{&g_1(V,F)g_1(U,W) - g_1(U,F)g_1(V,W)\\ &+ g_1(U,JF)g_1(JV,W) - g_1(V,JF)g_1(JU,W)\\ &+ 2g_1(U,JV)g_1(JF,W)\} - g_1(\mathcal{T}_U W, \mathcal{T}_V F) + g_1(\mathcal{T}_V W, \mathcal{T}_U F),\end{aligned} \tag{4.1}$$

for all vector fields $U, V, F, W \in \Gamma(\ker\sigma_*)$ and

$$\begin{aligned}R^*(X,Y,Z,H) = \frac{v}{4}\{&g_1(Y,Z)g_1(X,H) - g_1(X,Z)g_1(Y,H)\\ &+ g_1(JY,Z)g_1(JX,H) - g_1(JX,Z)g_1(JY,H)\\ &+ 2g_1(X,JY)g_1(JZ,H)\} + g_1(\mathcal{A}_X Y, \mathcal{A}_Z H) - g_1(\mathcal{A}_Y Z, \mathcal{A}_X H)\\ &+ g_1(\mathcal{A}_X Z, \mathcal{A}_Y H)\end{aligned} \tag{4.2}$$

for all vector fields $X, Y, Z, H \in \Gamma(\ker\sigma_*)^\perp$.

Theorem 4.1. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_{M_2}) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$. Then, we have*

$$\widehat{Ric}(U) \geq \frac{v}{4}(r - 1 + 3\cos^2\theta) - rg_1(\mathcal{T}_U U, \mathcal{H}). \tag{4.3}$$

for a unit vector field $U \in D^\theta$. The equality case of (4.3) holds for a unit vertical vector U if and only if each fiber is totally geodesic.

Proof. From (4.4), we obtain

$$\widehat{Ric}(U) = \frac{v}{4}\{(r-1)g_1(U,U) + 3\sum_{i=1}^{r} g_1^2(U, JU_i)\} - rg_1(\mathcal{T}_U U, H) + \|\mathcal{T}_U U_i\|^2 \tag{4.4}$$

where

$$\widehat{Ric}(U) = \sum_{i=1}^{r} g_1(U, U_i, U_i, U). \tag{4.5}$$

Obviously, One can get easily,

$$g_1^2(JX_k, X_s) = \begin{cases} 0, & \text{for } i \in \{1, ..., k_1 - 1\}, \\ cos^2\theta, & \text{for } i \in \{k_1 + 1, ..., k_1 + 2k_2 - 1\}, \end{cases}$$

Since

$$\sum_{k,s=1}^{r} g_1^2(JX_k, X_s) = 2k_2 cos^2\theta. \tag{4.6}$$

using last equation (4.4), we drive (4.3). □

In a similar way, we have the following theorem.

Theorem 4.2. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$. Then, we have*

$$\widehat{Ric}(U) \geq \frac{v}{4}(r-1) - rg_1(\mathcal{T}_U U, \mathcal{H}). \tag{4.7}$$

for a unit vector field $U \in \Gamma\left(D^\perp\right)$. The equality case of (4.7) holds for a unit vertical vector $U \in \Gamma\left(D^\perp\right)$ if and only if each fiber is totally geodesic.

Theorem 4.3. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$. Then, the Ricci tensor $S^{\ker\sigma_*}$ on the vertical distribution satisfies,*

$$S^{\ker\sigma_*}(U,V) \geq \frac{v}{4}(r-1+3\cos^2\theta)g_1(U,V) - rg_1(\mathcal{T}_U V, \mathcal{H}) \tag{4.8}$$

for $U, V \in \Gamma(\ker\sigma_*)$, the equality status of the inequality satisfies if and only if every fibre is totally geodesic.

Proof. By virtue of (4.4), for $U, V \in \Gamma(\ker\sigma_*)$, we have

$$S^{\ker\sigma_*}(U,V) = \frac{v}{4}(r-1+3\cos^2\theta)g_1(U,V) - rg_1(\mathcal{T}_U V, \mathcal{H}) + \sum_{i=1}^{r} g_1(\mathcal{T}_{U_i}V, \mathcal{T}_U U_i). \tag{4.9}$$

Hence, the equality status of the inequality satisfies if and only if every fibre is totally geodesic. □

Similarly, the following theorem can be given.

Theorem 4.4. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$.*

$$2\rho^{\ker\sigma_*} = \frac{v}{4}\{r^2 - r + 6k_2\cos^2\theta\} - r^2\left\|H\right\|^2 + \|\mathcal{T}_{U_i}U_i\|^2 \tag{4.10}$$

for $U, V \in \Gamma(\ker\sigma_*)$.

Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$ and $\dim(\ker\sigma_*) = r = k_1 + 2k_2$. For every $q \in M_1$, we consider $\{X_1, X_2, \ldots, X_{k_1}, X_{k_1+1}, X_{k_1+2}, \ldots, X_{k_1+k_2}, \sec\theta\phi X_{k_1+1}, \ldots, \sec\theta\phi X_{k_1+k_2}\}$ and $\{X_{r+1}..., X_m\}$ two orthonormal bases of $(\ker\sigma_*)$ and $(\ker\sigma_*)^\perp$.

Now we denote $\mathcal{T}_{ij}^s$ by

$$\mathcal{T}_{ij}^s = g_1(\mathcal{T}_{U_i}U_j, X_s), \tag{4.11}$$

where $1 \leq i, j \leq r$ and $1 \leq s \leq n$.

Similarly, we denote $\mathcal{A}_{ij}^\alpha$ by

$$\mathcal{A}_{ij}^\alpha = g_1(\mathcal{A}_{X_i}X_j, U_\alpha), \tag{4.12}$$

where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq r$. From [22], we use

$$\delta(N) = \sum_{i=1}^{n}\sum_{k=1}^{r} g_1((\nabla_{X_i}\mathcal{T})_{U_k}U_k, X_i)). \tag{4.13}$$

From the Binomial theorem there is such as the following equation between the tensor fields T:

$$\begin{aligned}\sum_{s=1}^{n}\sum_{i,j=1}^{r}(\mathcal{T}_{ij}^s)^2 &= \frac{1}{2}r^2\left\|H\right\|^2 + \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - ... - \mathcal{T}_{rr}^s)^2 \\ &+ 2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2 - 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s - (\mathcal{T}_{ij}^s)^2).\end{aligned} \tag{4.14}$$

Theorem 4.5. *Let $\sigma : M_1 \to M_2$ be a $\mathcal{PHSRM}$ with $(range\sigma_*)^\perp = \{0\}$. Then*

$$2\rho^{\ker\sigma_*} \geq \frac{v}{4}\{r^2 - r + 6k_2\cos^2\theta\} - r^2\|\mathcal{H}\|^2 \tag{4.15}$$

The equality case of (4.15) holds if and only if each fiber is totally geodesic.

Proof. Using (4.11) in (4.15), we can write

$$2\rho^{\ker\sigma_*} = \frac{v}{4}\{r^2 - r + 6k_2\cos^2\theta\} - r^2\|\mathcal{H}\|^2 + \sum_{\alpha=p+1}^{b_1}\sum_{k,s=1}^{r}(\mathcal{T}^\alpha_{ks})^2 \tag{4.16}$$

If (4.14) is used in (4.16), then (4.16) can be written as

$$\begin{aligned}2\rho^{\ker\sigma_*} = \;& \frac{v}{4}\{r^2 - r + 6k_2\cos^2\theta\} - \frac{1}{2}r^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{\alpha=p+1}^{b_1}(\mathcal{T}^s_{11} - \mathcal{T}^s_{22} - ... - \mathcal{T}^s_{rr})^2\\ &+2\sum_{\alpha=p+1}^{b_1}\sum_{s=2}^{r}(\mathcal{T}^\alpha_{1s})^2 - 2\sum_{\alpha=p+1}^{b_1}\sum_{2\leq k<s\leq r}(\mathcal{T}^\alpha_{kk}\mathcal{T}^\alpha_{ss} - (\mathcal{T}^\alpha_{ks})^2).\end{aligned} \tag{4.17}$$

Thus from (4.37) we derive

$$\begin{aligned}2\rho^{\ker\sigma_*} \geq\;& \frac{v}{4}r(r - 1 + 3\cos^2\theta) - \frac{1}{2}r^2\|H\|^2\\ &+ \frac{1}{2}(T^s_{11} - T^s_{22} - ... - T^s_{rr})^2 - 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(T^s_{ii}T^s_{jj} - (T^s_{ij})^2).\end{aligned} \tag{4.18}$$

Furthermore, taking $U = W = U_i$, $V = F = U_j$, we obtain

$$\begin{aligned}2\sum_{2\leq i<j\leq r}R(U_i, U_j, U_j, U_i) =\;& 2\sum_{2\leq i<j\leq r}\hat{R}(U_i, U_j, U_j, U_i)\\ &+ 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(T^s_{ii}T^s_{jj} - (T^s_{ij})^2).\end{aligned} \tag{4.19}$$

Using (4.19) in (4.38), we derive

$$\begin{aligned}2\rho^{\ker\sigma_*} \geq\;& \frac{v}{4}\{r^2 - r + 6k_2\cos^2\theta\} - \frac{1}{2}r^2\|H\|^2\\ &+ 2\sum_{2\leq k<s\leq r}R^{\ker\sigma_*}(U_k, U_s, U_s, U_k) - 2\sum_{2\leq k<s\leq r}R(U_k, U_s, U_s, U_k).\end{aligned} \tag{4.20}$$

Besides, we have

$$2\rho^{\ker\sigma_*} = 2\sum_{2\leq i<j\leq r}\hat{R}(U_i, U_j, U_j, U_i) + 2\sum_{j=1}^{r}\hat{R}(U_1, U_j, U_j, U_1). \tag{4.21}$$

Considering (4.21) in (4.19), we derive

$$2\widehat{Ric}(U_1) \geq \frac{v}{4}\{r^2 - r + 6k_2\cos^2\theta\} - \frac{1}{2}r^2\|\mathcal{H}\|^2 - 2\sum_{2\leq k<s\leq r}R(U_k, U_s, U_s, U_k). \tag{4.22}$$

Since $M(c)$ is a complex space form, its curvature tensor R satisfies the we get

$$\widehat{Ric}(U_1) \geq \frac{v}{4}\{r - 1\} - \frac{1}{4}r^2\|\mathcal{H}\|^2. \tag{4.23}$$

From (4.22) and (4.23) we obtain (4.15). □

Hence, we have the following theorem.

Theorem 4.6. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J_1) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$. Then, for any unit vector field $U_1 \in \Gamma\left(D^\perp\right)$, it follows that*

$$\widehat{Ric}(U_1) \geq \frac{v}{4}\{r^2 - r + 6k_2 \cos^2\theta\} - \frac{1}{2}r^2 \|\mathcal{H}\|^2 \tag{4.24}$$

The equality case of the inequality satisfies if and only if

$$\mathcal{T}_{11}^\alpha = \mathcal{T}_{22}^\alpha + ... + \mathcal{T}_{rr}^\alpha,$$
$$\mathcal{T}_{1s}^\alpha = 0, \qquad s = 2, ..., r.$$

Theorem 4.7. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J_1) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$. Then, we have (4.2),*

$$Ric^*(X) = \frac{v}{4}\{(n-1)g_1(X,X) + 3\|\mathcal{C}\|^2\} - 2\|\mathcal{A}_X X_i\|^2 \tag{4.25}$$

where

$$Ric^*(X) = \sum_{i=1}^{n} R^*(X, X_i, X_i, X).$$

The equality case of (4.25) holds if and only if

$$A_{1j}^\alpha = 0, \qquad j = 2, ..., n.$$

Proof. By using (4.2), we have

$$2\tau^* = \frac{v}{4}(n(n-1) + 3\|C\|^2) - 3\sum_{\alpha=1}^{r}\sum_{i,j=1}^{n}(A_{ij}^\alpha)^2. \tag{4.26}$$

Thus (4.26) can be written as

$$2\tau^* = \frac{v}{4}(n(n-1) + 3\|C\|^2) - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(A_{1j}^\alpha)^2 - 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}(A_{ij}^\alpha)^2. \tag{4.27}$$

Moreover, taking $X = H = X_i$, $Y = Z = X_j$ in (4.2), we obtain

$$2\sum_{2\leq i<j\leq n} R(X_i, X_j, X_j, X_i) = 2\sum_{2\leq i<j\leq n} R^*(X_i, X_j, X_j, X_i) + 6\sum_{\alpha=1}^{r}\sum_{2\leq i<j\leq n}(A_{ij}^\alpha)^2. \tag{4.28}$$

Using (4.28) in (4.27), we derive

$$\begin{aligned} 2\tau^* = \frac{(v+3)}{4}n(n-1) + \frac{3(v-1)}{4}\|C\|^2 - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(A_{1j}^\alpha)^2 \\ + 2\sum_{2\leq i<j\leq n} R^*(X_i, X_j, X_j, X_i) - 2\sum_{2\leq i<j\leq n} R(X_i, X_j, X_j, X_i). \end{aligned} \tag{4.29}$$

Since $M(v)$ is a complex space form, its curvature tensor R satisfies the equality (4.2), we get

$$\sum_{2\leq i<j\leq n} R(X_i, X_j, X_j, X_i) = \frac{v}{8}((n-2)(n-1) + 3\sum_{2\leq i<j\leq n} g_1^2(CX_i, X_j)). \tag{4.30}$$

Then from (4.29) and (4.30) we get

$$2Ric^*(X_1) = \frac{(v+3)}{2}((n-1) + 3\|CX_1\|^2) - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(A_{1j}^\alpha)^2, \tag{4.31}$$

which gives (4.25). This completes the proof. □

From the above theorem, we have the following.

Theorem 4.8. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$.*

$$Ric^*(X) \leq \frac{v}{4}\{(n-1)g_1(X,X) + 3\|\mathcal{C}\|^2\}. \tag{4.32}$$

The equality case of the inequality holds if and only if the horizontal distribution is integrable.

Theorem 4.9. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$. If X is a unit vector, then we have*

$$Ric^*(X) \leq \frac{v}{4}\{(n-1) + 3\|\mathcal{C}\|^2\}. \tag{4.33}$$

The equality case of the inequality holds if and only if the horizontal distribution is integrable.

Theorem 4.10. *Let σ be a $\mathcal{PHSRM}$ from a Kaehler manifold (M_1^m, g_1, J) to a Riemannian manifold (M_2, g_2) with the slant function θ and $(range\sigma_*)^\perp = \{0\}$. Then we have*

$$2\tau^* = \sum_{1\leq i\leq j\leq n} R^*(X_i, X_j, X_j, X_i) = \frac{v}{4}\{n(n-1) + 3\|\mathcal{C}\|^2\} - 3\|\mathcal{A}_X X_i\|^2. \tag{4.34}$$

for any $X \in \Gamma\left((ker\sigma_*)^\perp\right)$.

Proof. Using the anti-symmetry of $\mathcal{A}$ and (4.2), we obtain

$$2\tau^* = \frac{v}{4}(n(n-1) + 3\sum_{i,j=1}^{n} g_1(\mathcal{C}X_i, X_j)g_1(\mathcal{C}X_i, X_j)) - 3\sum_{i,j=1}^{n} g_1(\mathcal{A}_{X_i}X_j, \mathcal{A}_{X_i}X_j), \tag{4.35}$$

where

$$\tau^* = \sum_{1\leq i<j\leq n} \hat{R}(X_i, X_j, X_j, X_i). \tag{4.36}$$

Let define

$$\|\mathcal{C}\|^2 = \sum_{i=1}^{n} g_1^2(\mathcal{C}X_i, X_j), \tag{4.37}$$

then from (4.35) and (4.37) we obtain

$$2\tau^* = \frac{v}{4}(n(n-1) + 3\,\|C\|^2) - 3\|\mathcal{A}_X X_i\|^2 \tag{4.38}$$

which completes the proof. □

Now, we are going to obtain Casorati curvatures of $\mathcal{PHSRM}$. The following lemma plays a key role in the proof of our theorem:

Lemma 4.11. *Let $W = \{(y_1, y_2, ..., y_m) \in R^m : y_1 + y_2 + ... + y_m = z\}$ be a hyperplane of R^m, and $g : R^m \to R$ a quadratic form given by*

$$g(y_1, y_2, ..., y_m) = c\Sigma_{k=1}^{m-1}(y_k)^2 + d(y_m)^2 - 2\Sigma_{1\leq k<s\leq m} y_k y_s,\ c > 0,\ d > 0.$$

Then the constrained extremum problem $min_{(y_1,y_2,...,y_m)\in W}\, g$ has the following solution:

$$y_1 = y_2 = ... = y_{m-1} = \frac{z}{c+1},\ y_m = \frac{z}{d+1} = \frac{z(m-1)}{(c+1)d} = (c-m+2)\frac{z}{c+1},$$

provided that $d = \frac{m-1}{c-m+2}$, [54].

Let σ be a $\mathcal{PHSRM}$ from a complex space form $(M_1^{b_1}(\nu), g_1, J)$ to a Riemannian manifold $(M_2^{b_2}, g_2)$ with $(range\sigma_*)^{\perp} = \{0\}$. Suppose $\{X_1, ..., X_p\}$ is an orthonormal basis of the vertical space $ker\sigma_{*q}$, for $q \in M_1$, and $\{X_{p+1}, ..., X_{b_1}\}$ be an orthonormal basis of the horizontal space $(ker\sigma_{*q})^{\perp}$.
We defined the scalar curvature $\tau^{ker\sigma_*}$ on the vertical space $ker\sigma_{*q}$ by

$$\tau^{ker\sigma_*} = \Sigma_{k,s=1}^{p} g_1(R^{ker\sigma_*}(X_k, X_s)X_s, X_k)$$

and the normalized scalar curvature $\kappa^{ker\sigma_*}$ of $ker\sigma_{*q}$ as

$$\kappa^{ker\sigma_*} = \frac{2\tau^{ker\sigma_*}}{p(p-1)}.$$

Then, we can write

$$\begin{aligned} \mathcal{T}_{ks}^{\beta} &= g_1(\mathcal{T}(X_k, X_s), X_\beta),\ k, s = 1, ..., p,\ \beta = p+1, ..., b_2, \\ \|\mathcal{T}\|^2 &= \Sigma_{k,s=1}^{p} g_1(\mathcal{T}(X_k, X_s), \mathcal{T}(X_k, X_s)), \\ trace\mathcal{T} &= \Sigma_{k=1}^{p} \mathcal{T}(X_k, X_k),\ \|trace\mathcal{T}\|^2 = g_1(trace\mathcal{T}, trace\mathcal{T}) \end{aligned}$$

and the squared norm of $\mathcal{T}$ over the manifold M_1, denoted by $\mathcal{C}^{ker\sigma_*}$, is called the vertical Casorati curvatures of the vertical space $(ker\sigma_*)_q$. Thus, we get

$$\mathcal{C}^{ker\sigma_*} = \frac{1}{p}\|\mathcal{T}\|^2 = \frac{1}{p}\Sigma_{\beta=p+1}^{b_1}\Sigma_{k,s=1}^{p}(\mathcal{T}_{ks}^{\beta})^2.$$

Now, assume that $\mathrm{L}^{ker\sigma_*}$ is a $t-$dimensional subspace $(ker\sigma_*)_q$, $2 \leq t$ and let $\{X_1, X_2, ..., X_t\}$ be an orthonormal basis of $\mathrm{L}^{ker\sigma_*}$. Then the Casorati curvature $\mathcal{C}^{ker\sigma_*}(\mathrm{L}^{ker\sigma_*})$ of $\mathrm{L}^{ker\sigma_*}$ defined as

$$\mathcal{C}^{ker\sigma_*}(\mathrm{L}^{ker\sigma_*}) = \frac{1}{t}\|\mathcal{T}\|^2 = \frac{1}{t}\Sigma_{\beta=p+1}^{b_1}\Sigma_{k,s=1}^{t}(\mathcal{T}_{ks}^{\beta})^2.$$

The normalized $\varphi^{ker\sigma_*}-$ Casorati curvatures $\varphi_{\mathcal{C}}^{ker\sigma_*}(p-1)$ and $\bar{\varphi}_{\mathcal{C}}^{ker\sigma_*}(p-1)$ of $ker\sigma_*)_q$ are given by
$[\varphi_{\mathcal{C}}^{ker\sigma_*}(p-1)]_q = \frac{1}{2}\mathcal{C}_q^{ker\sigma_*} + \frac{p+1}{2p} inf\{\mathcal{C}^{ker\sigma_*}(\mathrm{L}^{ker\sigma_*}) : \mathrm{L}^{ker\sigma_*}$ a hyperplane of $(ker\sigma_*)_q\}$, and
$[\bar{\varphi}_{\mathcal{C}}^{ker\sigma_*}(p-1)]_q = 2\mathcal{C}_q^{ker\sigma_*} - \frac{2p-1}{2p} inf\{\mathcal{C}^{ker\sigma_*}(\mathrm{L}^{ker\sigma_*}) : \mathrm{L}^{ker\sigma_*}$ a hyperplane of $(ker\sigma_*)_q\}$.

Theorem 4.12. *Let σ be a $\mathcal{PHSRM}$ from a complex space form $(M_1^{b_1}(\nu), g_1, J)$ to a Riemannian manifold $(M_2^{b_2}, g_2)$ with $(range\sigma_*)^{\perp} = \{0\}$ and $3 \leq p$. Then the normalized $\varphi-$ Casorati curvatures $\varphi_{\mathcal{C}}^{ker\sigma_*}$ and $\bar{\varphi}_{\mathcal{C}}^{ker\sigma_*}$ on $(ker\sigma_*)_q$ satisfy*

$$(i)\ \kappa^{ker\sigma_*} \leq \varphi_{\mathcal{C}}^{ker\sigma_*}(p-1) + \frac{\nu}{4} + \frac{3\nu}{2p(p-1)}(k_2\cos^2\theta), \tag{4.39}$$

$$(ii)\ \kappa^{ker\sigma_*} \leq \bar{\varphi}_{\mathcal{C}}^{ker\sigma_*}(p-1) + \frac{\nu}{4} + \frac{3\nu}{2p(p-1)}(k_2\cos^2\theta). \tag{4.40}$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ if and only if with respect to suitable orthonormal basis $\{X_1, ..., X_p\}$ on $(ker\sigma_*)_q$ and $\{X_{p+1}, ..., X_{b_1}\}$ on $((ker\sigma_*)_q)^{\perp}$, the components of $\mathcal{T}$ satisfy

$$\mathcal{T}_{11}^{\beta} = \mathcal{T}_{22}^{\beta} = ... = \mathcal{T}_{p-1p-1}^{\beta} = \frac{1}{2}\mathcal{T}_{pp}^{\beta},\ \ \beta \in \{p+1, p+2, ..., b_1\},$$

$$\mathcal{T}_{ks}^{\beta} = 0,\ \ k, s \in \{1, , ..., p\}(k \neq s),\ \ \beta \in \{p+1, p+2, ..., b_1\}.$$

Proof. Using (1.27) of [26] and (4.4) we have

$$\begin{aligned} 2\tau^{ker\sigma_*} &= \frac{\nu}{4}(p^2-p)+\frac{3\nu}{2}(k_2\cos^2\theta) \\ &- p\mathcal{C}^{ker\sigma_*}+\|trace\mathcal{T}\|^2. \end{aligned} \quad (4.41)$$

Now we define a function $\mathcal{Q}^{ker\sigma_*}$ associated with the following quadratic polynomial with respect to the components of $\mathcal{T}$:

$$\begin{aligned} \mathcal{Q}^{ker\sigma_*} &= \frac{1}{2}[(p^2-p)\mathcal{C}^{ker\sigma_*}+(p^2-1)\mathcal{C}^{ker\sigma_*}(\mathrm{L}^{ker\sigma_*})] \\ &- 2\tau^{ker\varphi_*}+\frac{\nu}{4}(p^2-p)+\frac{3\nu}{2}(k_2\cos^2\theta). \end{aligned}$$

Without loos of generality, by supposing that the hyperplane $\mathrm{L}^{ker\sigma_*}$ is spanned by $\{X_1,...,X_{p-1}\}$, one can produce

$$\begin{aligned} \mathcal{Q}^{ker\sigma_*} &= \Sigma_{\beta=p+1}^{b_1}\Sigma_{k=1}^{p-1}[p(\mathcal{T}_{kk}^{\beta})^2+(p+1)(\mathcal{T}_{kp}^{\beta})^2] \\ &+ \Sigma_{\beta=p+1}^{b_1}[2(p+1)\Sigma_{1=k<s}^{p-1}(\mathcal{T}_{ks}^{\beta})^2 \\ &- 2\Sigma_{1=k<s}^{p}\mathcal{T}_{kk}^{\beta}T_{ss}^{\beta}+\frac{p-1}{2}(\mathcal{T}_{pp}^{\beta})^2]. \end{aligned} \quad (4.42)$$

Using (4.42), we obtain the critical points

$$\mathcal{T}^c=(\mathcal{T}_{11}^{p+1},\mathcal{T}_{12}^{p+1},...,\mathcal{T}_{pp}^{p+1},...,\mathcal{T}_{11}^{b_1},...,\mathcal{T}_{pp}^{b_1})$$

of $\mathcal{Q}^{ker\sigma_*}$ are solutions of the next system of equations:

$$\begin{cases} \frac{\partial\mathcal{Q}^{ker\sigma_*}}{\partial\mathcal{T}_{kk}^{\beta}}=2(r+1)\mathcal{T}_{kk}^{\beta}-2\Sigma_{t=1}^{p}\mathcal{T}_{tt}^{\beta}= & 0 \\ \frac{\partial\mathcal{Q}^{ker\sigma_*}}{\partial\mathcal{T}_{pp}^{\beta}}=(r-1)\mathcal{T}_{pp}^{\beta}-2\Sigma_{t=1}^{p-1}\mathcal{T}_{tt}^{\beta}= & 0 \\ \frac{\partial\mathcal{Q}^{ker\sigma_*}}{\partial\mathcal{T}_{ks}^{\beta}}=4(r+1)\mathcal{T}_{ks}^{\beta}= & 0 \\ \frac{\partial\mathcal{Q}^{ker\sigma_*}}{\partial\mathcal{T}_{kp}^{\beta}}=2(r+1)\mathcal{T}_{kp}^{\beta}= & 0, \end{cases} \quad (4.43)$$

here $k,s\in\{1,2,...,p-1\}$, $k\neq s$ and $\beta\in\{p+1,...,b_1\}$. Frankly (4.43) is a system consisting only in linear homogeneous equations and it is easy to checky that every solution $\mathcal{T}^c$ has $\mathcal{T}_{ks}^{\beta}=0$ for $k\neq s$, and the determinant corresponding to the first two series of linear homogeneous equations in (4.43) has vanishes. Furthermore, the Hessian matrix of $\mathcal{Q}^{ker\sigma_*}$ is defined as

$$\mathcal{H}(\mathcal{Q}^{ker\sigma_*})=\begin{pmatrix} \mathcal{H}_1 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 \\ 0 & 0 & \mathcal{H}_3 \end{pmatrix},$$

here

$$\mathcal{H}_1=\begin{pmatrix} 2p & -2 & ... & -2 & -2 \\ -2 & 2p & ... & -2 & -2 \\ ... & ... & ... & ... & ... \\ -2 & -2 & ... & 2p & -2 \\ -2 & -2 & ... & -2 & p-1 \end{pmatrix},$$

0 denotes the zero matrix of suitable dimensions and the matrices $\mathcal{H}_2$, $\mathcal{H}_3$ are ones having the following diagonal forms

$$\begin{aligned} \mathcal{H}_2 &= diag(4(p+1),4(p+1),...,4(p+1)), \\ \mathcal{H}_3 &= diag(2(p+1),2(p+1),...,2(p+1)). \end{aligned}$$

Then a standard computation shows that the eigenvalues of $\mathcal{H}(\mathcal{Q}^{ker\sigma_*})$ are

$$\xi_{11}=0,\ \xi_{22}=p+3,\ \xi_{33}=...=\xi_{pp}=2(p+1),\ \xi_{ks}=4(p+1),$$
$$\xi_{kb_1}=2(p+1),\ \forall k,s\in\{1,2,...,p-1\},\ k\neq s.$$

Also it follows that $\mathcal{Q}^{ker\sigma_*}$ is parabolic and achieves a global minimum value $\mathcal{Q}^{ker\sigma_*}(c)$ for T^c- the solution of (4.43). However we obtain $\mathcal{Q}^{ker\sigma_*}(c) = 0$ and we get $\mathcal{Q}^{ker\sigma_*} \geq 0$. Thus,

$$2\tau^{ker\sigma_*} \leq \frac{1}{2}[(p^2-p)\mathcal{C}^{ker\sigma_*} + (p^2-1)\mathcal{C}^{ker\sigma_*}(\mathrm{L}^{ker\sigma_*})] + \frac{\nu}{4}(p^2-p) + \frac{3\nu}{2}(k_2\cos^2\theta) \tag{4.44}$$

and using (4.44) we obtain

$$\kappa^{ker\sigma_*} \leq [\frac{1}{2}\mathcal{C}^{ker\sigma_*} + \frac{p+1}{2p}\mathcal{C}^{ker\sigma_*}(\mathrm{L}^{ker\sigma_*})] + \frac{\nu}{4} + \frac{3\nu}{2p(p-1)}(k_2\cos^2\theta) \tag{4.45}$$

for all hyperplane $\mathrm{L}^{ker\sigma_*}$ of M_1. Now, taking the infimum in (4.45) over every hyperplane $\mathrm{L}^{ker\sigma_*}$, we get (i)

$$\kappa^{ker\sigma_*} \leq \varphi_{\mathcal{C}}^{ker\sigma_*}(p-1) + \frac{\nu}{4} + \frac{3\nu}{2p(p-1)}(k_2\cos^2\theta) \tag{4.46}$$

Besides, simply we can check that the equality sign holds in the (4.46) if and only if

$$\mathcal{T}_{ks}^{\beta} = 0,\ \forall k,s \in \{1,2,...,p\},\ k \neq s,\ \beta \in \{p+1,...,b_1\},$$

and

$$\mathcal{T}_{pp}^{\beta} = 2\mathcal{T}_{11}^{\beta} = ... = 2\mathcal{T}_{p-1\,p-1}^{\beta},\ \forall k,s \in \{p+1,p+2,...,b_1\}.$$

In a similar way we have (ii).

□

Using the Theorem 4.12, we obtain the following results:

Corollary 4.13. *Let σ be a $\mathcal{PHSRM}$ from a complex space form $(M_1^{b_1}(\nu), g_1, J)$ to a Riemannian manifold $(M_2^{b_2}, g_2)$ with $(range\sigma_*)^{\perp} = \{0\}$ and $3 \leq p$. Then the normalized $\sigma-$ Casorati curvatures $\sigma_{\mathcal{C}}^{ker\sigma_*}$ and $\bar{\sigma}_{\mathcal{C}}^{ker\sigma_*}$ on $(ker\sigma_*)_q$ satisfy*

$$(i)\ \kappa^{ker\sigma_*} \leq \varphi_{\mathcal{C}}^{ker\sigma_*}(p-1) + \frac{\nu}{4} \tag{4.47}$$

$$(ii)\ \kappa^{ker\sigma_*} \leq \bar{\varphi}_{\mathcal{C}}^{ker\sigma_*}(p-1) + \frac{\nu}{4} \tag{4.48}$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ if and only if with respect to suitable orthonormal basis $\{X_1,...,X_p\}$ on $(ker\sigma_*)_q$ and $\{X_{p+1},...,X_{b_1}\}$ on $((ker\sigma_*)_q)^{\perp}$, the components of $\mathcal{T}$ satisfy

$$\mathcal{T}_{11}^{\beta} = \mathcal{T}_{22}^{\beta} = ... = \mathcal{T}_{p-1p-1}^{\beta} = \frac{1}{2}\mathcal{T}_{pp}^{\beta},\ \ \beta \in \{p+1,p+2,...,b_1\},$$
$$\mathcal{T}_{ks}^{\beta} = 0,\ \ k,s \in \{1,,...,p\}(k \neq s),\ \ \beta \in \{p+1,p+2,...,b_1\}.$$

Acknowledgements

Both authors would like to thank Professor Bayram Şahin for his helpful suggestions and his valuable comments which helped to improve the manuscript. The authors also would like to thank the referees for many invaluable comments and suggestions that helped to clarify and improve the original manuscript.

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. This work is financially supported by 1001-Scientific and Technological Research Projects Funding Program of The Scientific and Technological Research Council of Turkey (TUBITAK) under project number 121F277.

Data availability. No data was used for the research described in the article.

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