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Araştırma Makalesi -Research Article

Some Matrix Applications on the Special Integer Number Sequences

Özel Tam Sayı Dizilerinin Bazı Matris Uygulamaları

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ABSTRACT

In this paper, the matrices related to Fibonacci, Lucas, Pell, and Pell-Lucas numbers have been examined. By using these matrices new identities related to these integer sequences have been investigated.

Keywords- Matrix Method, Fibonacci Numbers, Pell Numbers, Lucas Numbers, Pell-Lucas Numbers

ÖZ

Bu çalışmada, Fibonacci, Lucas, Pell ve Pell-Lucas sayı dizileri ile ilgili matrisler incelendi. Bu matrisleri kullanarak bu tam sayı dizileri ile ilgili yeni özdeşlikler araştırıldı.

Anahtar Kelimeler- Matris Metodu, Fibonacci Sayıları, Pell Sayıları, Lucas Sayıları, Pell-Lucas Sayıları

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I. INTRODUCTION

The investigation of Fibonacci, Lucas, Pell, and Pell-Lucas numbers is one of the most important research areas in the number theory. Especially, the matrix applications of these numbers are used in many areas such as the coding theory, security systems, electric network theory. The authors have investigated applications of Fibonacci, Lucas, Pell and Pell-Lucas sequences by using matrices in [1-4]. The authors examine the relation between the suborbital graphs and Fibonacci numbers [5,6]. Then, the author finds new matrices and identities related to Fibonacci and Lucas numbers [7]. The relation between Pell and Pell- Lucas numbers and the suborbital graphs has been investigated in [8]. As a result, the authors produce new matrices related to these integer sequences.

In this paper, these matrices have been characterized. Firstly, the characteristic roots of these matrices have been found in terms of values of $\varphi, \beta, \gamma, \delta$. Then, by using λ-function method and the matrices, the new identities related to these integer sequences have been found. Let $A = (a_{ij})_{n\times n}$ and $A^* = (a_{ij} + k)_{n\times n}$, where $k \in \mathbb{Z}$. λ– function for matrix A is written as $\lambda(A) = |A^*| - |A|$. For example, for matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\lambda(A) = a + d - b - c$ and for matrix $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $\lambda(B) = \begin{vmatrix} a+e-b-d & b+f-c-e \\ d+h-g-e & e+i-h-f \end{vmatrix}$ are taken [9].

Let's introduce to the special integer sequences which we use in the paper.

For initial conditions $F_0 = 0$ and $F_1 = 1$ by recurrence relation $F_n = F_{n-1} + F_{n-2},\ n \geq 2$, $\{F_n\}$ is called a Fibonacci sequence. Here, n^{th} Fibonacci number is F_n. Similarly, the n^{th} Lucas number is $L_n$for recurrence relation $L_n = L_{n-1} + L_{n-2},\ n \geq 2$, where initial conditions are $L_0 = 2$ and $L_1 = 1$. For details, see [10]. From both integer sequences, there are a lot of identities which have been discovered. Let us give some of them.

$$F_{n-1}F_{n+1} - {F_n}^2 = (-1)^n, n \geq 1 (\text{Cassini Formula}), \tag{1}$$

$$F_{2n} = F_n L_n,\ n \geq 1, \tag{2}$$

$$L_n = F_{n-1} + F_{n+1} = F_{n+2} - F_{n-2}, \qquad n \geq 2, \tag{3}$$

$${L_n}^2 = 5{F_n}^2 + 4(-1)^n, \qquad n \geq 1, \tag{4}$$

$$F_{n-1}F_nF_{n+1} = F_n^3 + (-1)^{n-1}F_n, \qquad n \geq 1, \tag{5}$$

$$F_mF_n - F_{m+k}F_{n-k} = (-1)^{n-k}F_{m+k-n}F_k, \qquad n \geq 1. \tag{6}$$

Let $\varphi = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The characteristic equation for the recurrence relation of Fibonacci numbers is $x^2 - x - 1 = 0$. So, its solutions are φ and β, which are characteristic roots of this equation. Also, from the Binet formulas, $F_n = \frac{\varphi^n - \beta^n}{\sqrt{5}}$ and $L_n = \varphi^n + \beta^n$, where φ is known as golden ratio.

P_n is the n^{th} Pell number which satisfies the recurrence relation $P_n = 2P_{n-1} + P_{n-2},\ n \geq 2$ by initial conditions $P_0 = 0$ and $P_1 = 1$. Similarly, Q_n is the n^{th} Pell-Lucas number by $Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 2$ and initial conditions $Q_0 = 1$, $Q_1 = 1$. Binet-like formulas for P_n and Q_n are $P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$ *and* $Q_n = \frac{\gamma^n + \delta^n}{2},\ n \geq 2$, where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$. For Pell and Pell-Lucas numbers, the following identities hold [11]:

$$P_n + P_{n-1} = Q_n, \tag{7}$$

$$Q_n + Q_{n-1} = 2P_n, \tag{8}$$

$$P_{n+1} + P_{n-1} = 2Q_n, \tag{9}$$

$$P_{2n} = 2P_nQ_n, \tag{10}$$

$$Q_{n+1} + Q_{n-1} = 4P_n, \tag{10}$$

where $n \geq 1, n \in \mathbb{N}$.

II. MATRICES WITH INTEGER SEQUENCES

In this section, we introduce the matrices which are examined in the paper:

Theorem 2.1. [6] If F_n is the n^{th} Fibonacci number, then

$$S_n = \begin{pmatrix} (-1)^{n-1}F_{2n-2} & (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^n F_{2n+2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n, \tag{12}$$

where $n \geq 1$.

Lemma 2.2 [8]If L_n is the n^{th} Lucas number and $tr[S_n]$ is trace of matrix S_n, then

$$L_{2n} = (-1)^n tr[S_n], n \in \mathbb{N},$$

where $n \geq 1$.

By using matrix S_n, we have the following lemma:

Theorem 2.3 [8] If L_n is then^{th} Lucas number, then

$$T_n = \begin{pmatrix} -3 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}^n = \begin{pmatrix} (-1)^{n-1}L_{2n-2} & (-1)^n L_{2n} \\ (-1)^{n+1}L_{2n} & (-1)^n L_{2n+2} \end{pmatrix}, \tag{13}$$

$$det(T_n) = {L_{2n}}^2 - L_{2n-2}L_{2n+2} = -5, \tag{14}$$

where $n \geq 1$.

Theorem 2.4 [5]Let $F_n$be the n^{th} Fibonacci number, then we have the following equation as

$$P^n = \begin{pmatrix} (-1)^n F_{n-1} & 0 & (-1)^{n+1}F_n \\ 3[(-1)^{n+1}F_{n-2} - 1] & 1 & 3[(-1)^n F_{n-1} - 1] \\ (-1)^{n+1}F_n & 0 & (-1)^n F_{n+1} \end{pmatrix}, \tag{15}$$

where $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$ and $n > 1, n \in \mathbb{N}$.

Theorem 2.5 [8] Let $F_n$and $L_n$be the n^{th} Fibonacci number and the n^{th} Lucas number, respectively. So, we have the following matrix equation as

$$X_n = \begin{pmatrix} 2 & -3 \\ 3 & -7 \end{pmatrix}^n = \begin{cases} 5^{\frac{n-1}{2}}\begin{pmatrix} L_{2n-2} & -L_{2n} \\ L_{2n} & -L_{2n+2} \end{pmatrix}, & if \quad n \ is \ odd, \\ 5^{\frac{n}{2}}\begin{pmatrix} -F_{2n-2} & F_{2n} \\ -F_{2n} & F_{2n+2} \end{pmatrix}, & if \quad n \ is \ even, \end{cases} \tag{16}$$

where $n \in \mathbb{N}$ and $n \geq 1$.

Theorem 2.6 [8] If P_n is the n^{th} Pell number, then

$$V_n = \begin{pmatrix} \frac{(-1)^{n-1}}{2}P_{2n-2} & \frac{(-1)^n}{2}P_{2n} \\ \frac{(-1)^{n+1}}{2}P_{2n} & \frac{(-1)^n}{2}P_{2n+2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n \tag{17}$$

Theorem 2.7 [8] If Q_n is the n^{th} Pell-Lucas number, then we get

$$W_n = \begin{pmatrix} -6 & 2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n = \begin{pmatrix} 2(-1)^{n-1}Q_{2n-2} & 2(-1)^n Q_{2n} \\ 2(-1)^{n+1}Q_{2n} & 2(-1)^n Q_{2n+2} \end{pmatrix}, \tag{18}$$

where $n \geq 2, n \in \mathbb{N}$.

Now, the characteristic roots of S_n,T_n,P^n and X_n matrices are examined.. To find the characteristic roots, we use same motivation in [4]. Since $|Q^n - \lambda I| = \lambda^2 - L_n\lambda + (-1)^n = 0$, then the characteristic roots $\frac{L_n+\sqrt{5}F_n}{2} = \varphi^n$ and $\frac{L_n-\sqrt{5}F_n}{2} = \beta^n$ are obtained by using identity $F_{n-1}F_nF_{n+1} = F_n^3 + (-1)^{n-1}F_n$, where $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$.

Theorem 2.8 [10] The characteristic roots of Q^n are φ^n and β^n.

Firstly, to determine the characteristic roots of $\mathrm{S_n}$, let us find the characteristic equation. So,

$$|S_n - \lambda I| = \begin{vmatrix} (-1)^{n-1}F_{2n-2} - \lambda & (-1)^nF_{2n} \\ (-1)^{n+1}F_{2n} & (-1)^nF_{2n+2} - \lambda \end{vmatrix}$$

$$= ((-1)^{n-1}F_{2n-2} - \lambda)((-1)^nF_{2n+2} - \lambda) - (-1)^nF_{2n}(-1)^{n+1}F_{2n}$$

$$= -F_{2n-2}F_{2n+2} - \lambda[(-1)^{n-1}F_{2n-2} + (-1)^nF_{2n+2}] + \lambda^2 + F_{2n}^2$$

$$= \lambda^2 - (-1)^nL_{2n}\lambda + 1.$$

Hence, the characteristic equation is $\lambda^2 - (-1)^nL_{2n}\lambda + 1 = 0$. Then, characteristic roots as$\lambda_{1,2} = \frac{(-1)^nL_{2n}\pm\sqrt{L_{2n}^2-4}}{2} = \frac{(-1)^nL_{2n}\pm\sqrt{5}F_{2n}}{2}$ are found by using identity$L_n{}^2 = 5F_n{}^2 + 4(-1)^n$. Thus, the following corollary is written:

Corollary 2.9 The characteristic roots of S_n are,

$$\begin{cases} \varphi^{2n} \ and \ \beta^{2n}, & if \ n \ is \ even; \\ -\beta^{2n} \ and \ -\varphi^{2n}, & if \ n \ is \ odd. \end{cases} \tag{19}$$

Secondly, the characteristic roots for the matrix T_n with even terms of Lucas numbers are examined. The characteristic equation of T_n is,

$$|T_n - \lambda I| = \begin{vmatrix} (-1)^{n-1}L_{2n-2} - \lambda & (-1)^nL_{2n} \\ (-1)^{n+1}L_{2n} & (-1)^nF_{2n+2} - \lambda \end{vmatrix}$$

$$= ((-1)^{n-1}L_{2n-2} - \lambda)((-1)^nL_{2n+2} - \lambda) - (-1)^{2n+1}L_{2n}^2$$

$$= -5 - \lambda(-1)^n(L_{2n+2} - L_{2n-2}) - \lambda^2$$

$$= \lambda^2 - (-1)^n5F_{2n}\lambda - 5 = \lambda^2 - (-1)^n5F_{2n}\lambda - 5 = 0.$$

By using identity $F_{n-1}F_nF_{n+1} = F_n^3 + (-1)^{n-1}F_n$, the characteristic roots are given as

$$\lambda_{1,2} = \frac{(-1)^n5F_{2n} \pm \sqrt{5}L_{2n}}{2}.$$

Therefore, the following corollary is written:

Corollary 2.10 The characteristic roots of T_n are,

$$\begin{cases} \sqrt{5}\varphi^{2n} \ and \ -\sqrt{5}\beta^{2n}, & if \ n \ is \ even; \\ \sqrt{5}\beta^{2n} \ and \ -\sqrt{5}\varphi^{2n}, & if \ n \ is \ odd. \end{cases} \tag{20}$$

Finally, to find the characteristic roots of P^n matrix, let us give the characteristic equation:

$$|Z^n - \lambda I| = \begin{vmatrix} (-1)^nF_{n-1} - \lambda & 0 & (-1)^{n+1}F_n \\ k[(-1)^{n+1}F_{n-2} - 1] & 1-\lambda & k[(-1)^nF_{n-1} - 1] \\ (-1)^{n+1}F_n & 0 & (-1)^nF_{n+1} - \lambda \end{vmatrix}$$

$$= (F_{n-1}F_{n+1} - F_n^2 - \lambda(-1)^n[F_{n-1} + F_{n+1}] + \lambda^2)(1 - \lambda)$$

$$= (\lambda^2 - \lambda(-1)^nL_n + (-1^n))(1 - \lambda).$$

Then, the characteristic equation is$\lambda^2 - \lambda(-1)^nL_n + (-1^n) = 0$. By using identity $L_n{}^2 = 5F_n{}^2 + 4(-1)^n$, the characteristic roots are given as

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_{2,3} = \frac{(-1)^n L_n \pm \sqrt{5} F_n}{2}.$$

Consequently, the following corollary is given:

Corollary 2.11 The characteristic roots of Z^n are,

$$\begin{cases} 1, \varphi^n \ \ and \ \ \beta^n, & if \ \ n \ \ is \ even; \\ 1, \beta^n \ \ and \ \ -\varphi^n, & if \ \ n \ \ is \ odd. \end{cases} \tag{21}$$

By the same motivation, if the characteristic roots of X_n, V_n and W_n are examined,the following corollary is written:

Corollary 2.12 The characteristic roots of X_n, V_n and W_n are,

$$\begin{cases} 5^{\frac{n}{2}}\varphi^{2n} \ \ and \ \ 5^{\frac{n}{2}}\beta^{2n}, & if \ \ n \ \ is \ even; \\ 5^{\frac{n}{2}}\beta^{2n} \ \ and \ \ -5^{\frac{n}{2}}\varphi^{2n}, & if \ \ n \ \ is \ odd, \end{cases} \tag{22}$$

$$\begin{cases} \gamma^{2n} \ \ and \ \ \delta^{2n}, & if \ \ n \ \ is \ even; \\ -\delta^{2n} \ \ and \ \ -\gamma^{2n}, & if \ \ n \ \ is \ odd, \end{cases} \tag{23}$$

$$\begin{cases} 4\sqrt{2}\gamma^{2n} \ \ and \ \ -4\sqrt{2}\delta^{2n}, & if \ \ n \ \ is \ even; \\ 4\sqrt{2}\delta^{2n} \ \ and \ \ -4\sqrt{2}\gamma^{2n}, & if \ \ n \ \ is \ odd, \end{cases} \tag{24}$$

respectively.

III. FINDING IDENTITIES WITH FIBONACCİ AND LUCAS NUMBERS

In this section, some new identities of Fibonacci, Lucas, Pell, and Pell-Lucas numbers are found.

Theorem 3.1. For all $n \geq 1$, let $F_n$be the n^{th} Fibonacci number. Then,

$$F_{2n-1}[F_{2n+2} + F_{2n}] = 1 + F_{4n}. \tag{25}$$

Proof. To prove the equation, we will use λ- function of a matrix [9]. According to paper, when we write λ-function for S_n matrix; we get $\lambda(S_n) = (-1)^{n-1}F_{2n-2} + (-1)^n F_{2n+2} - (-1)^n F_{2n} - (-1)^{n-1}F_{2n} = (-1)^n L_{2n}$ by using recurrence relation of Fibonacci numbers and identity $L_n = F_{n-1} + F_{n+1} = F_{n+2} - F_{n-2}$and $|S_n{}^*| = |S_n| + k\lambda(S_n)$, where $k = (-1)^n F_{2n}$ and $S_n{}^* = S_n + k$;

$$S_n{}^* = \begin{pmatrix} (-1)^{n-1}F_{2n-2} + (-1)^n F_{2n} & (-1)^n F_{2n} + (-1)^n F_{2n} \\ (-1)^{n+1}F_{2n} + (-1)^n F_{2n} & (-1)^n F_{2n+2} + (-1)^n F_{2n} \end{pmatrix}.$$

The determinant of matrixS_n is $|S_n| = F_{2n}{}^2 - F_{2n-2}F_{2n+2} = 1$ and the determinant of matrix $S_n{}^*$is

$$|S_n{}^*| = |(-1)^n| \begin{vmatrix} -F_{2n-2} + F_{2n} & 2F_{2n} \\ 0 & F_{2n+2} + F_{2n} \end{vmatrix} = F_{2n-1}[F_{2n+2} + F_{2n}].$$

If we write them in the equation $|S_n{}^*| = |S_n| + k\lambda(S_n)$ and use the identity $F_{2n} = F_n L_n$, we get $F_{2n-1}[F_{2n+2} + F_{2n}] = 1 + (-1)^n F_{2n}(-1)^n L_{2n} = 1 + F_{4n}$.

Theorem 3.2 For all $n \geq 1$,$n \in \mathbb{N}$, let $F_n$and $L_n$ben^{th}Fibonacci number and n^{th} Lucas number, respectively, then

$$L_{2n-1}[L_{2n+2} + L_{2n}] = 5[-1 + F_{4n}]. \tag{26}$$

Proof. For the proof, we will use the same motivation as proof of Theorem 3.1. When we take matrix T_n, then we get $|T_n{}^*| = |T_n| + k\lambda(T_n)$ and $\lambda(T_n) = (-1)^{n-1}L_{2n-2} + (-1)^n L_{2n+2} - (-1)^n L_{2n} - (-1)^{n-1}L_{2n}$.After simplification, that is $\lambda(T_n) = 5(-1)^n F_{2n}$. From equation (2.1), we get $|T_n| = -5$. Therefore, $|T_n{}^*| = 1 + k(-1)^n L_{2n}$. Now, if $k = (-1)^n L_{2n}$, then for $T_n{}^* = T_n + k$,

$$T_n{}^* = \begin{pmatrix} (-1)^{n-1}L_{2n-2} + (-1)^n L_{2n} & (-1)^n L_{2n} + (-1)^n L_{2n} \\ (-1)^{n+1}L_{2n} + (-1)^n L_{2n} & (-1)^n L_{2n+2} + (-1)^n L_{2n} \end{pmatrix}$$

and

$$|T_n{}^*| = |(-1)^n| \begin{vmatrix} -L_{2n-2} + L_{2n} & 2L_{2n} \\ 0 & L_{2n+2} + L_{2n} \end{vmatrix}$$

are obtained. If we write them in the equation $|T_n{}^*| = |T_n| + k\lambda(T_n)$, $L_{2n-1}[L_{2n+2} + L_{2n}] = -5 + (-1)^n L_{2n} 5(-1)^n F_{2n}$ is obtained. By using identity $F_{2n} = F_n L_n$, that is $L_{2n-1}[L_{2n+2} + L_{2n}] = 5[-1 + F_{4n}]$.

Theorem 3.3 For all $n \geq 1$, let $F_n$be the n^{th} Fibonacci number. Then,

$$F_{2n} + 2F_n{}^2 = F_n F_{n+3} \tag{27}$$

holds.

Proof. Here, we will use λ- function for 3×3 types of matrices. If we take P^n matrix, we have $|P^{n*}| = |P^n| + k\lambda(P^n)$and

$$\lambda(P^n) = \begin{vmatrix} 3(-1)^n F_{n-2} + (-1)^n F_{n-1} + 4 & 3(-1)^n F_{n-1} + (-1)^n F_n - 4 \\ 3(-1)^{n+1} F_{n-2} - (-1)^{n+1} F_n - 4 & -3(-1)^n F_{n-1} + (-1)^n F_{n+1} + 4 \end{vmatrix}.$$

When we use identity $F_{n+h}F_{n+k} - F_n F_{n+h+k} = (-1)^n F_h F_k$, then $\lambda(P^n) = (-1)^n[4F_{n+3} - 1]$

is obtained and from Cassini formula$|P^n| = (-1)^n$ is written. Therefore,$|P^{n*}| = (-1)^n + k(-1)^n[4F_{n+3} - 1]$.

Now, if $k = (-1)^n F_n$, then from $P^{n*} = P^n + k$,

$$P^{n*} = \begin{pmatrix} (-1)^n F_{n-1} + (-1)^n F_n & (-1)^n F_n & 0 \\ 3(-1)^{n+1} F_{n-2} + (-1)^n F_n - 3 & 1 + (-1)^n F_n & 3(-1)^n F_{n-1} + (-1)^n F_n - 3 \\ 0 & (-1)^n F_n & (-1)^n F_{n+1} + (-1)^n F_n \end{pmatrix}.$$

By using identities $L_n = F_{n-1} + F_{n+1} = F_{n+2} - F_{n-2}$, $P_n + P_{n-1} = Q_{n,} Q_n + Q_{n-1} = 2P_n$and Cassini formula; that is $|P^{n*}| = 4F_{2n} + 8F_n{}^2 + (-1)^n - 5F_n$. So, $4F_{2n} + 8F_n{}^2 + (-1)^n - 5F_n = (-1)^n + (-1)^n F_n (-1)^n [4F_{n+3} - 5]$. After simplification, that is $F_{2n} + 2F_n{}^2 = F_n F_{n+3}$.

IV. SUM OF MATRICES RELATED TO INTEGER SEQUENCES

Byusingsum of thematricesS_n and X^n, new identitiescan be obtained;

Theorem 4.1 The following equalities are provided for $n, m \in \mathbb{Z}^+$,

i. $F_{2m+2n} = F_{2n}F_{2m+2} - F_{2n-2}F_{2m}$,
ii. $F_{2n-2m} = F_{2n}F_{2m+2} - F_{2n+2}F_{2m}$.

Proof. By using matrix S_n, matrix $S^{n+m}$is written as

$$S^{n+m} = \begin{pmatrix} (-1)^{n+m-1} F_{2n+2m-2} & (-1)^{n+m} F_{2n+2m} \\ (-1)^{n+m+1} F_{2n+2m} & (-1)^{n+m} F_{2n+2m+2} \end{pmatrix}. \tag{28}$$

Now, let us take multiplication of the matrices S^n and S^m,

$$S^n S^m = (-1)^{m+n} \begin{pmatrix} F_{2n-2}F_{2m-2} - F_{2n}F_{2m} & F_{2n}F_{2m+2} - F_{2n-2}F_{2m} \\ F_{2n}F_{2m+2} - F_{2n+2}F_{2m} & F_{2n+2}F_{2m+2} - F_{2n}F_{2m} \end{pmatrix}. \tag{29}$$

By comparing equation (4.1) with equation (4.2), a new identity $F_{2m+2n} = F_{2n}F_{2m+2} - F_{2n-2}F_{2m}$ is obtained.

Now, let us take the inverse of the matrix S^{-m},

$$S^{-m} = \begin{pmatrix} (-1)^m F_{2m+2} & (-1)^{m+1} F_{2m} \\ (-1)^{m+2} F_{2m+2} & (-1)^{m-1} F_{2m-2} \end{pmatrix}.$$

When we compute the equality $S^{n-m} = S^n S^{-m}$, by using the same motivation as above, the following identity is obtained;$F_{2n-2m} = F_{2n}F_{2m+2} - F_{2n+2}F_{2m}$.

Theorem 4.2 The following equalities are provided for $n, m \in \mathbb{Z}^+$;

i. $5F_{2m+2n} = L_{2m}L_{2n+2} - L_{2m-2}L_{2n}$,

ii. $L_{2m+2n} = L_{2m+2}F_{2n} - L_{2m}F_{2n-2}$,
iii. $L_{2m-2n} = L_{2m-2}F_{2n} - L_{2m}F_{2n-2}$,
iv. $5F_{2m-2n} = L_{2m}L_{2n-2} - L_{2m-2}L_{2n}$.

Proof. For the proof we use the matrix X_n. Here, we represent that matrix as X. Firstly, let's write the matrix X^{m+n}:

$$X^{m+n} = \begin{cases} 5^{\frac{m+n-1}{2}} \begin{pmatrix} L_{2m+2n-2} & -L_{2m+2n} \\ L_{2m+2n} & -L_{2m+2n+2} \end{pmatrix}, & if \quad m+n \ is \ odd, \\ 5^{\frac{m+n}{2}} \begin{pmatrix} -F_{2m+2n-2} & F_{2m+2n} \\ -F_{2m+2n} & F_{2m+2n+2} \end{pmatrix}, & if \quad m+n \ is \ even. \end{cases} \tag{30}$$

When we take odd n and m values, we get the following matrix for $X^n X^m$,

$$5^{\frac{m+n}{2}} 5^{-1} \begin{pmatrix} L_{2m-2}L_{2n-2} - L_{2m}L_{2n} & -L_{2m-2}L_{2n} + L_{2m}L_{2n+2} \\ L_{2m}L_{2n-2} - L_{2m+2}L_{2n} & -L_{2m-2}L_{2n} + L_{2m+2}L_{2n+2} \end{pmatrix}. \tag{31}$$

By comparing the matrices in (4.3) and (4.4), the following equation is given:

$$5F_{2m+2n} = L_{2m}L_{2n+2} - L_{2m-2}L_{2n}.$$

For even m and odd n values or odd m and even n values, we write for $X^n X^m$,

$$5^{\frac{m+n-1}{2}} \begin{pmatrix} -L_{2m-2}F_{2n-2} + L_{2m}F_{2n} & L_{2m-2}F_{2n} - L_{2m}F_{2n+2} \\ -L_{2m}F_{2n-2} + L_{2m+2}F_{2n} & L_{2m}F_{2n} - L_{2m+2}F_{2n+2} \end{pmatrix} \tag{32}$$

When we compare the matrices in (4.3) and (4.5), the following equation is obtained;$L_{2m+2n} = L_{2m+2}F_{2n} - L_{2m}F_{2n-2}$.Now, let us take inverse of the matrix X^n,

$$X^{-n} = \begin{cases} 5^{\frac{-n-1}{2}} \begin{pmatrix} L_{2n+2} & -L_{2n} \\ L_{2n} & -L_{2n-2} \end{pmatrix}, & if \quad n \ is \ odd, \\ 5^{-\frac{n}{2}} \begin{pmatrix} -F_{2n+2} & F_{2n} \\ -F_{2n} & F_{2n-2} \end{pmatrix}, & if \quad n \ is \ even. \end{cases}.$$

So,

$$X^{m-n} = \begin{cases} 5^{\frac{m-n-1}{2}} \begin{pmatrix} L_{2m-2n-2} & -L_{2m-2n} \\ L_{2m+2n} & -L_{2m+2n+2} \end{pmatrix}, & if \quad m-n \ is \ odd, \\ 5^{\frac{m-n}{2}} \begin{pmatrix} -F_{2m-2n-2} & F_{2m-2n} \\ -F_{2m-2n} & F_{2m-2n+2} \end{pmatrix}, & if \quad m-n \ is \ even. \end{cases}$$

is written. When we examine for odd m and even n, we obtain $L_{2m-2n} = L_{2m-2}F_{2n} - L_{2m}F_{2n-2}$,

for odd m, n values, we get $5F_{2m-2n} = L_{2m}L_{2n-2} - L_{2m-2}L_{2n}$.

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