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Research Article

On the boundedness of B-Riesz potential and its commutators on generalized weighted B-Morrey spaces

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Abstract

In the present paper, we shall investigate a characterization for the boundedness of the B-Riesz potential and its commutators on the generalized weighted B-Morrey spaces. We also give a characterization for the generalized weighted B-Morrey spaces via the boundedness of the Riesz potential and its commutators generated by generalized translate operators associated with Laplace-Bessel differential operator.

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1. Introduction

The classical Morrey spaces were introduced by Morrey [26] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. In [11, 25, 28], Guliyev, Mizuhara and Nakai introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$. In [22], Komori and Shirai defined weighted Morrey spaces $L^{p,k}(w)$. Guliyev [14] gave a concept of the generalized weighted Morrey spaces $M^{p,\varphi,w}(\mathbb{R}^n)$ which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,k}(w)$. Authors also studied the boundedness of the classical operators and their commutators in spaces $M^{p,\varphi,w}(\mathbb{R}^n)$.

The boundedness of Riesz potential operator and its commutators on certain function spaces and their characterizations play an important role in various area in harmonic analysis, etc. See for example [1–3, 5–7, 9, 15–18, 22, 25, 26, 28, 29, 31, 32] and the references therein.

Let us now present some of the studies obtained and considered in this study for the maximal operator and Riesz potential. Suppose that $f \in L_1^{loc}(\mathbb{R}^n)$. Let M be a maximal operator and I^α be Riesz potential operator on $\mathbb{R}^n$ defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

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$$I^{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0<\alpha<n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$ (see [30, 35, 36]). In order to reveal the importance and difference of the results obtained in this study, let us give information about the studies and some results obtained in Morrey spaces related to these operators: In [26], they were introduced by C. Morrey and defined as follows: For $0\le\lambda\le n$ and $1\le p<\infty$, $f\in\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ if $f\in L_p^{\rm loc}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,r))}<\infty.$$

If $\lambda=0$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)=L_p(\mathbb{R}^n)$, if $\lambda=n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)=L_\infty(\mathbb{R}^n)$, if $\lambda<0$ or $\lambda>n$, then $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Also by $W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we denote the weak Morrey space of all functions $f\in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}}\|f\|_{WL_p(B(x,r))}<\infty,$$

where $WL_p(\mathbb{R}^n)$ denotes the weak $L_p(\mathbb{R}^n)$ spaces (for detail see [6–8]). Here the results can be summarized as follows:

Theorem 1.1. *Let $0<\alpha<n$, $0\le\lambda<n$ and $1\le p<\infty$.*

i) *If $1<p<\infty$, then M is bounded from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$.*
ii) *If $p=1$, then M is bounded from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$.*

The classical result by Hardy-Littlewood-Sobolev states that if $1<p<q<\infty$, then I^α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha=\frac{n}{p}-\frac{n}{q}$ and for $p=1<q<\infty$, I^α is bounded from $L_1(\mathbb{R}^n)$ to $WL_q(\mathbb{R}^n)$ if and only if $\alpha=n-\frac{n}{q}$. In [1], D. R. Adams studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement (see, also [7])

Theorem 1.2. *Let $0<\alpha<n$, $0\le\lambda<n$ and $1\le p<\frac{n-\lambda}{\alpha}$.*

i) *If $1<p<\frac{n-\lambda}{\alpha}$, then condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness I^α from $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.*
ii) *If $p=1$, then condition $1-\frac{1}{q}=\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness I^α from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$.*

If $\alpha=\frac{n}{p}-\frac{n}{q}$, then $\lambda=0$ and the statement of Theorem 1.2 reduces to the above mentioned result by Hardy-Littlewood-Sobolev.

If in place of the power function r^λ in the definition of $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ we consider any positive measurable weight function $\omega(r)$, then it becomes generalized Morrey space $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.

Definition 1.3. Let $\omega(r)$ positive measurable weight function on $(0,\infty)$ and $1\le p<\infty$. We denote by $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ generalized Morrey spaces, the spaces of all functions $f\in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{r^{-\frac{n}{p}}}{\omega(r)}\|f\|_{L_p(B(x,r))}.$$

T. Mizuhara [25], E. Nakai [28, 29] and V. S. Guliyev [11] obtained sufficient conditions on weights ω_1 and ω_2 ensuring the boundedness of integral operators T from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$. In [28], the following statement was proved, containing the result in [25] and in the general setting of metric measure spaces obtained in [31, 32]. In these studies, the authors obtained sufficient conditions on weights ω_1 and ω_2 for the boundedness of the singular integral operator T from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$. In [28], the following doubling conditions were imposed on $\omega(r)$:

$$c^{-1}\omega(r)\le\omega(t)\le c\,\omega(r), \tag{1.1}$$

whenever $r \leq t \leq 2r$, where $c \geq 1$ does not depend on t and r, jointly with the condition:

$$\int_r^\infty \omega^p(t)\frac{dt}{t} \leq C\,\omega^p(r) \tag{1.2}$$

for the maximal or singular integral operator and the condition

$$\int_r^\infty t^{\alpha p}\omega^p(t)\frac{dt}{t} \leq C\,r^{\alpha p}\omega^p(r) \tag{1.3}$$

for potential and fractional maximal operators, where $C > 0$ does not depend on r.

Theorem 1.4 ([28]). *Let $1 < p < \infty$ and $\omega(r)$ satisfy conditions (1.1)-(1.2). Then the operators M and singular integral operator T are bounded in $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$.*

Theorem 1.5 ([28]). *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $\omega(t)$ satisfy conditions (1.1) and (1.3). Then the operators M^α and I^α are bounded from $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega}(\mathbb{R}^n)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

Note that Theorems 1.6 and 1.7 do not require condition (1.1)

Theorem 1.6 ([11]). *Let $1 < p < \infty$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty \omega_1(t)\frac{dt}{t} \leq c_1\,\omega_2(r) \tag{1.4}$$

with $c_1 > 0$ not depending on $t > 0$. Then the operators M and singular integral operator T are bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$.

Theorem 1.7 ([11]). *Let $0 < \alpha < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty t^\alpha\omega_1(t)\frac{dt}{t} \leq c_1\,\omega_2(r). \tag{1.5}$$

Then the operators M^α and I^α are bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\omega_2}(\mathbb{R}^n)$.

The results given so far are obtained in Morrey space and generalized Morrey spaces of the maximal operator M^α and Riesz potential operator I^α. In this paper, we shall investigate the maximal operator $M_{\alpha,\gamma}$ (B-Maximal operator) and the Riesz potential operator $I_{\alpha,\gamma}$ (B-Riesz potential operator) related to the generalized translate operator associated with the Laplace-Bessel differential operator and its commutators on generalized weighted Morrey spaces $M^{p,\varphi,w}(\mathbb{R}^n)$. We also give a characterization for the B-BMO space via the boundedness of the commutator of the B-Riesz potential $I_{\alpha,\gamma}$. Our aim is to present these two different characterizations of generalized weighted B- Morrey spaces for I^α and $I_{\alpha,\gamma}$ and its commutators.

The maximal operator and potential operator related topics associated with the Laplace-Bessel differential operator have been investigated by many researchers, see B. Muckenhoupt and E. Stein [27], I. Kipriyanov [21], K. Trimeche [38], L. Lyakhov [24], K. Stempak [37], A.D. Gadjiev and I.A. Aliev [10], V.S. Guliyev [12,13], V.S. Guliyev and J.J. Hasanov [16], J.J. Hasanov [18], A. Serbetci, I. Ekincioglu [2,15,33] and others [19].

Now, let us introduce the Riesz potential operator $I_{\alpha,\gamma}$ (B-Riesz potential operator) related to the generalized translate operator associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^k \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \ldots, \gamma_k > 0.$$

At first, we prove that the B-Riesz potential operator $I_{\alpha,\gamma}$ and their commutators for $0 < \alpha < n+|\gamma|$ is bounded from the generalized weighted B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$

to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$, where $\alpha/(n+|\gamma|) = 1/p - 1/q$, $1 < p < (n+|\gamma|)/\alpha$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$, $\frac{1}{p} + \frac{1}{p'} = 1$.

We now consider the generalized translation operator generated by the Laplace-Bessel differential operator Δ_B. Then, the B-maximal operator and B-Riesz potential associated with this operator are investigated in generalized weighted B-Morrey spaces. We obtain for the operator $I_{\alpha,\gamma}$ to be bounded from generalized weighted B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$ and from generalized weighted B-Morrey space $\mathcal{M}_{1,\omega_1,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ to weak generalized weighted B-Morrey space $W\mathcal{M}_{q,\omega_2,\varphi,\gamma}(\mathbb{R}^n_{k,+})$.

The structure of the paper is as follows. In first section, we present some definitions and auxiliary results. In second section, we introduce generalized B-Morrey spaces. In Section 3, the main results of the paper, the boundedness of the B-potential operator from B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi,\gamma}(\mathbb{R}^n_{k,+})$, is proved. In the last section, the boundedness of the commutators of the B-potential operator from generalized weighted B-Morrey space $\mathcal{M}_{p,\omega_1,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{p,\omega_2,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ is obtained.

2. Preliminaries

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. For $1 \le k \le n$, let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$ such that $x = (x', x'') \in \mathbb{R}^n$. Then, it is defined as

$$\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n;\, x_1 > 0, \ldots, x_k > 0\}.$$

for $n \ge 2$. Recall that $E(x,r) = \{y \in \mathbb{R}^n_{k,+}\,;\, |x-y| < r\}$ for a measurable subset $E \subset \mathbb{R}^n_{k,+}$. Let $E_r = E(0,r)$. If $\gamma = (\gamma_1, \ldots, \gamma_k)$ and $\gamma_1 > 0, \ldots, \gamma_k > 0$ then $|\gamma| = \gamma_1 + \ldots + \gamma_k$ and $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$. For $x' \in \mathbb{R}^k$, we define the measures on E by equality

$$|E|_\gamma = \int_E (x')^\gamma dx,$$

then $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, where $Q = n + |\gamma|$ and

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

First, we define the generalized translate operator (B-translate operator) T^x, $x \in \mathbb{R}^n$, on $L^p(\mathbb{R}^n, d\nu)$ by equality

$$T^x f(y) = C_{\gamma,k} \int_0^\pi \ldots \int_0^\pi f\left((x',y')_\beta, x''-y''\right)\, d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x',y')_\beta = ((x_1,y_1)_{\beta_1}, \ldots, (x_k,y_k)_{\beta_k})$, $d\nu(\beta) = \prod\limits_{i=1}^k \sin^{\gamma_i-1}\beta_i\, d\beta_1 \ldots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)} = \frac{2^k}{\pi^k}\omega(2k,k,\gamma).$$

It acts from $L^p(\mathbb{R}^n, d\nu)$ to$L^p(\mathbb{R}^n, d\nu)$ and $||T^x f||_p < ||f||_p$ and $T^x 1 = 1$ and L_p-boundedness.

We remark that the generalized translate operator T^x is closely connected with the Bessel differential operator B (for example, $n = k = 1$ see [23], $n > 1$, $k = 1$ see [21] and $n, k > 1$ see [24, 33, 34] for details).

Let $L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ be the space of Lebesgue measurable functions f such that

$$\|f\|_{L_{p,\varphi,\gamma}} = \|f\|_{L_{p,\varphi,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p \varphi^p(x) (x')^\gamma dx\right)^{1/p} < \infty$$

where $1 \leq p < \infty$. For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_{\infty,\varphi}} = \underset{x\in\mathbb{R}^n_{k,+}}{ess\ sup}\ \varphi(x)|f(x)|.$$

Definition 2.1. The weight function φ belongs to the class $A_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1 \leq p < \infty$, if

$$\sup_{x\in\mathbb{R}^n_{k,+},\, r>0} \Bigg(|E(x,r)|_\gamma^{-1} \int\limits_{E(x,r)} \varphi^p(y)(y')^\gamma dy\Bigg)^{\frac{1}{p}} \Bigg(|E(x,r)|_\gamma^{-1} \int\limits_{E(x,r)} \varphi^{-p'}(y)(y')^\gamma dy\Bigg)^{\frac{1}{p'}} < \infty$$

and φ belongs to $A_{1,\gamma}(\mathbb{R}^n_{k,+})$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n_{k,+}$ and $r > 0$

$$|E(x,r)|_\gamma^{-1} \int\limits_{E(x,r)} \varphi(y)(y')^\gamma dy \leq C_{y\in E(x,r)} \frac{1}{\varphi(y)}.$$

Definition 2.2. The weight function (φ_1, φ_2) belongs to the class $\widetilde{A}_{p,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \infty$, if

$$\sup_{x\in\mathbb{R}^n_{k,+},\, r>0} \Bigg(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi_2^p(y)(y')^\gamma dy\Bigg)^{\frac{1}{p}} \Bigg(\frac{1}{|E(x,r)|_\gamma} \int\limits_{E(x,r)} \varphi_1^{-p'}(y)(y')^\gamma dy\Bigg)^{\frac{1}{p'}} < \infty.$$

The generalized translate operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)[T^x g(y)](y')^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \leq \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \leq p, q \leq r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 2.3. *For all $x \in \mathbb{R}^n_{k,+}$ the following equality is holds*

$$\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E((x,0),t)} g\Big(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\Big) d\mu(z, \overline{z'}),$$

where $E((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \ :\ \Big|\Big(x - z, \overline{z'}\Big)\Big| < t\}$.

Lemma 2.4. *For all $x \in \mathbb{R}^n_{k,+}$ the following equality is holds*

$$\int_{\mathbb{R}^n_{k,+}} T^y g(x)\varphi(y) M_\gamma \chi_{E_r}(y)(y')^\gamma dy$$

$$= \int_{\mathbb{R}^n \times (0,\infty)^k} g\Big(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z''\Big)\varphi(z, \overline{z'}) M_\nu \chi_{E((x,0),r)}(z, \overline{z'}) d\nu(z, \overline{z'}),$$

where $E((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \ :\ \Big|\Big(x - z, \overline{z'}\Big)\Big| < t\}$.

Lemmas 2.3 and 2.4 are straightforward via the following substitutions

$$z'' = x'', z_i = y_i \cos\alpha_i, \quad \overline{z_i} = y_i \sin\alpha_i, \quad 0 \leq \alpha_i < \pi, \ i = 1, \ldots, k,$$
$$y \in \mathbb{R}^n_{k,+}, \ \overline{z'} = (\overline{z}_1, \ldots, \overline{z}_k), \ (z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k, \ 1 \leq k \leq n.$$

Definition 2.5 ([13])**.** Let $1 \leq p < \infty$ and $0 \leq \lambda \leq Q$. We denote by $\mathcal{M}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ Morrey space ($\equiv$ B-Morrey space), associated with the Laplace-Bessel differential operator the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm

$$\|f\|_{\mathcal{M}_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \Big(t^{-\lambda} \int_{E_t} T^y[|f|\ ]^p(x)(y')^\gamma dy\Big)^{1/p}.$$

Define the B-maximal operator of f by

$$M_\gamma f(x) = \sup_{r>0} |E_r|_\gamma^{-1} \int\limits_{E_r} T^y[|f|\ ](x)(y')^\gamma dy,$$

and the fractional B-maximal operator by

$$M_{\alpha,\gamma} f(x) = \sup_{r>0} |E_r|_\gamma^{\frac{\alpha}{Q}-1} \int\limits_{E_r} T^y[|f|\ ](x)(y')^\gamma dy, \quad 0 \le \alpha < Q,$$

and the B-Riesz potential by

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}^n_{k,+}} T^y[|f|\ ](x)|y|^{\alpha-Q}(y')^\gamma dy, \quad 0 < \alpha < Q.$$

We write $M_{0,\gamma} f(x) = M_\gamma f(x)$ in the case $\alpha = 0$.

Let ω and φ positive measurable weight functions. The norm in the spaces $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and $\mathcal{M}_{p,\omega,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ defined in two forms,

$$\|f\|_{\mathcal{M}_{p,\omega,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \frac{t^{-\frac{Q}{p}}}{\omega(t)} \left(\int_{E_t} T^y[|f|\ ]^p(x)(y')^\gamma dy \right)^{1/p},$$

and

$$\|f\|_{\mathcal{M}_{p,\omega,\varphi,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \frac{1}{\omega(t)\|\varphi\|_{L_{p,\gamma}(E(0,t))}} \left(\int_{E_t} T^y[|f|\ ]^p(x)\varphi(y)(y')^\gamma dy \right)^{1/p}.$$

If $\omega(t) \equiv r^{-\frac{Q}{p}}$ then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$, if $\omega(t) \equiv t^{\frac{\lambda-Q}{p}}$, $0 \le \lambda < Q$, then $\mathcal{M}_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv \mathcal{M}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Denote by $M^\sharp_\gamma$, the sharp B-maximal function defined by

$$M^\sharp_\gamma f(x) = \sup_{t>0} |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|(y')^\gamma dy,$$

where $f_{E(0,t)}(x) = |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} T^y f(x)(y')^\gamma dy$.

$B-BMO$ space, $BMO_\gamma(\mathbb{R}^n_{k,+})$, defined as the space of locally integrable functions f with finite norm

$$\|f\|_{BMO_\gamma} = \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|(y')^\gamma dy < \infty,$$

or

$$\|f\|_{BMO_\gamma} = \inf_C \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - C|(y')^\gamma dy < \infty.$$

The following theorem was proved in [4].

Theorem 2.6. i) *If Let $f \in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+})$.*

$$\sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} \left(|E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|^p (y')^\gamma dy \right)^{1/p} = \|f\|_{BMO_{p,\gamma}} < \infty,$$

then for any $1 < p < \infty$

$$\|f\|_{BMO_\gamma} \le \|f\|_{BMO_{p,\gamma}} \le A_p \|f\|_{BMO_\gamma}$$

where the constant A_p depends only on p.

ii) *Let $f \in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then, there is a constant $C>0$ such that*

$$|f_{E(0,r)} - f_{E(0,t)}| \le C\|f\|_{BMO_\gamma} \ln \frac{t}{r}, \qquad 0<2r<t$$

where C is independent of f, x, r and t.

Lemma 2.7 ([20]). *Let $1<p<\infty$, $\varphi \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$. Then*

$$\|b\|_{BMO_\gamma} \approx \sup_{x\in\mathbb{R}^n_{k,+}, r>0} \frac{\left\| T^{\cdot} b(x) - b_{E(0,r)} \right\|_{L_{p,\varphi,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}}.$$

3. B-Riesz potentials on generalized weighted B-Morrey spaces

Theorem 3.1. *Let $0<\alpha<Q$, $1<p<\frac{Q}{\alpha}$, $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$, $(\varphi_1,\varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition*

$$\int_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}} \frac{dr}{r} \le C\omega_2(t). \tag{3.1}$$

Then $I_{\alpha,\gamma}$ is bounded from $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. Let $f \in \mathcal{M}_{p,\omega_1,\varphi_1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then

$$I_{\alpha,\gamma} f(x) = I_{\alpha,\gamma} f_1(x) + I_{\alpha,\gamma} f_2(x). \tag{3.2}$$

Firstly, we estimate $I_{\alpha,\gamma} f_1(x)$. By using the Hölder's inequality, we have

$$\begin{aligned}
|I_{\alpha,\gamma} f_1(x)| &\le \int_{E(0,t)} T^y|f(x)|\,|y|^{\alpha-Q}(y')^\gamma dy \\
&\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y|f(x)|(y')^\gamma dy \\
&\le \sum_{j=-\infty}^{-1} \left(2^j t\right)^{\alpha-Q} \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y \left|f(x)\right|^p \varphi_1^p(y)(y')^\gamma dy \right)^{1/p} \\
&\times \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} \varphi_1^{-p'}(y)(y')^\gamma dy \right)^{1/p'} \\
&\le C\|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \|\varphi_2\|^{-1}_{L_{q,\gamma}(E(0,t))}.
\end{aligned}$$

By the inequality (3.1), we obtain

$$\begin{aligned}
&\|I_{\alpha,\gamma} f_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le C\|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \\
&\le C\|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \frac{\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}}{\omega_1(t)\|\varphi_1\|_{L_{p,\gamma}(E(0,t))}} \int_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}} \frac{dr}{r} \\
&\le C\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t) \|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.
\end{aligned}$$

Hence, we have

$$\|I_{\alpha,\gamma} f_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le C\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \omega_2(t) \|\varphi_2\|_{L_{q,\gamma}(E(0,t))}. \tag{3.3}$$

Now we estimate $I_{\alpha,\gamma}f_2(x)$. By using the Hölder's inequality, we get

$$\begin{aligned}
|I_{\alpha,\gamma}f_2(x)| &\leq \int\limits_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^y|f(x)||y|^{\alpha-Q}(y')^\gamma dy \\
&\leq \sum_{j=0}^{\infty}\left(2^j t\right)^{\alpha-Q} \int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y|f(x)|(y')^\gamma dy \\
&\leq \sum_{j=0}^{\infty}\left(2^j t\right)^{\alpha-Q} \Bigg(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} \varphi_1^{-p'}(y)(y')^\gamma dy\Bigg)^{1/p'} \\
&\times \Bigg(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y\,|f(x)|^p\,\varphi_1^p(y)(y')^\gamma dy\Bigg)^{1/p} \\
&\leq C\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}} \int\limits_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}}\frac{dr}{r}.
\end{aligned}$$

Thus, by the inequality (3.1), we obtain

$$|I_{\alpha,\gamma}f_2(x)| \leq C\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t). \tag{3.4}$$

So, from (3.3) and (3.4), we have

$$\begin{aligned}
\|I_{\alpha,\gamma}f\|_{L_{q,\varphi_2,\gamma}(E(0,t))} &\leq \|I_{\alpha,\gamma}f_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} + \|I_{\alpha,\gamma}f_2\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \\
&\leq C\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.
\end{aligned}$$

Finally $I_{\alpha,\gamma}f \in \mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|I_{\alpha,\gamma}f\|_{\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}} \leq C\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}.$$

□

Corollary 3.2. *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$. Then the operator $I_{\alpha,\gamma}$ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n)$ to $L_{q,\varphi_2,\gamma}(\mathbb{R}^n)$.*

Corollary 3.3. *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$. Then the operator $M_{\alpha,\gamma}$ is bounded from $L_{p,\varphi_1,\gamma}(\mathbb{R}^n)$ to $L_{q,\varphi_2,\gamma}(\mathbb{R}^n)$.*

4. Commutators of B-Riesz potential on generalized weighted B-Morrey spaces

In this section, we consider commutators of the B-Riesz potential defined as the following equality

$$[b, I_{\alpha,\gamma}]f(x) = \int\limits_{\mathbb{R}^n_{k,+}} (b(x) - b(y))|y|^{\alpha-Q}\,T^y f(x)(y')^\gamma dy, \quad 0 < \alpha < Q.$$

Given a measurable function b the operator $|b, I_{\alpha,\gamma}|$ is defined by

$$|b, I_{\alpha,\gamma}|f(x) = \int\limits_{\mathbb{R}^n_{k,+}} |b(x) - b(y)||y|^{\alpha-Q}\,T^y f(x)(y')^\gamma dy, \quad 0 < \alpha < Q.$$

Theorem 4.1. *Let $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$, $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$ and $\omega_1(r)$, $\omega_2(r)$ be positive measurable functions satisfying the condition (3.1). Then $|b, I_{\alpha,\gamma}|$ is bounded from $\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+})$ to $\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$.*

Proof. Let $f \in \mathcal{M}_{p,\omega_1,\varphi_1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then

$$\begin{aligned}|b, I_{\alpha,\gamma}|f(x) &= \Bigg(\int\limits_{E(0,t)} + \int\limits_{\mathbb{R}^n_{k,+}\setminus E(0,t)}\Bigg) T^y|[b-b(x)]f(x)||y|^{\alpha-Q}(y')^\gamma dy \\ &= F_1(x,t) + F_2(x,t).\end{aligned}$$

Firstly, we estimate $F_1(x,t)$. By using the Hölder's inequality, we have

$$\begin{aligned}F_1(x,t) &= \int\limits_{E(0,t)} T^y|[b-b(x)]f(x)||y|^{\alpha-Q}(y')^\gamma dy \\ &\le \sum_{j=-\infty}^{-1}(2^j t)^{\alpha-Q} \int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y|[b-b(x)]f(x)|(y')^\gamma dy \\ &\le \sum_{j=-\infty}^{-1}\left(2^j t\right)^{\alpha-Q}\Bigg(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} |T^y b(x)-b|^{p'}\varphi_1^{-1}(y)(y')^\gamma dy\Bigg)^{1/p'} \\ &\times \Bigg(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y\left|f(x)\right|^p \varphi_1(y)(y')^\gamma dy\Bigg)^{1/p} \\ &\le C\left\|b\right\|_{BMO_\gamma}\left\|f\right\|_{L_{p,\varphi_1,\gamma}(E(0,t))}\|\varphi_2\|^{-1}_{L_{q,\gamma}(E(0,t))}.\end{aligned}$$

By the inequality (3.1), we obtain

$$\begin{aligned}&\|F_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le C\left\|b\right\|_{BMO_\gamma}\left\|f\right\|_{L_{p,\varphi_1,\gamma}(E(0,t))} \\ &\le C\left\|b\right\|_{BMO_\gamma}\left\|f\right\|_{L_{p,\varphi_1,\gamma}(E(0,t))}\frac{\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}}{\omega_1(t)\|\varphi_1\|_{L_{p,\gamma}(E(0,t))}}\int\limits_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}}\frac{dr}{r} \\ &\le C\left\|b\right\|_{BMO_\gamma}\left\|f\right\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.\end{aligned}$$

Hence we have

$$\|F_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \le C\left\|b\right\|_{BMO_\gamma}\left\|f\right\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}. \tag{4.1}$$

Now we estimate $F_2(x,t)$. By using the Hölder's inequality, we get

$$\begin{aligned}F_2(x,t) &\le \int\limits_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^y(|b-b(x)||f(x)|)|y|^{\alpha-Q}(y')^\gamma dy \\ &\le \sum_{j=0}^{\infty}\left(2^j t\right)^{\alpha-Q}\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y(|b-b(x)||f(x)|)(y')^\gamma dy \\ &\le \sum_{j=0}^{\infty}(2^j t)^{\alpha-Q}\Bigg(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} |T^y b(x)-b|^{p'}\varphi_1^{-1}(y)(y')^\gamma dy\Bigg)^{1/p'} \\ &\times \Bigg(\int\limits_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y\left|f(x)\right|^p \varphi_1(y)(y')^\gamma dy\Bigg)^{1/p} \\ &\le C\left\|b\right\|_{BMO_\gamma}\left\|f\right\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}\int\limits_t^\infty \frac{\omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))}}{\|\varphi_2\|_{L_{q,\gamma}(E(0,r))}}\frac{dr}{r}.\end{aligned}$$

Thus by the inequality (3.1), we have

$$F_2(x,t) \le C\left\|b\right\|_{BMO_\gamma}\left\|f\right\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t). \tag{4.2}$$

Therefore, from (4.1) and (4.2), we obtain

$$\begin{aligned}\|[b, I_{\alpha,\gamma}]f\|_{L_{q,\varphi_2,\gamma}(E(0,t))} &\leq \|F_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} + \|F_2\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \\ &\leq C\,\|b\|_{BMO_\gamma}\,\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.\end{aligned}$$

Finally, we get $[b, I_{\alpha,\gamma}]f \in \mathcal{M}_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|[b, I_{\alpha,\gamma}]f\|_{\mathcal{M}_{q,\omega_2,\varphi_2,\gamma}} \leq C\,\|b\|_{BMO_\gamma}\,\|f\|_{\mathcal{M}_{p,\omega_1,\varphi_1,\gamma}}.$$

$\square$

Corollary 4.2. *Let* $0 < \alpha < Q$, $1 < p < \frac{Q}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$, $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$, $(\varphi_1, \varphi_2) \in \widetilde{A}_{1+\frac{q}{p'},\gamma}(\mathbb{R}^n_{k,+})$ *and* $\varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_{k,+})$. *Then the operator* $[b, I_{\alpha,\gamma}]$ *is bounded from* $L_{p,\varphi_1,\gamma}(\mathbb{R}^n)$ *to* $L_{q,\varphi_2,\gamma}(\mathbb{R}^n)$.

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References

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math. J. **42**, 765-778, 1975.

[2] A. Akbulut, I. Ekincioglu, A. Serbetci and T. Tararykova, *Boundedness of the anisotropic fractional maximal operator in anisotropic local Morrey-type spaces,* Eurasian Math. J. **2** (2), 5-30, 2011.

[3] R. Ayazoglu and J.J. Hasanov, *On the boundedness of B-Riesz potential in the generalized weighted B-Morrey spaces*, Georgian Math. J. **23** (2), 143-155, 2016.

[4] C. Aykol and J.J. Hasanov, *On the boundedness of B-maximal commutators, commutators of B-Riesz potentials and B-singular integral operators in modified B-Morrey spaces*, Acta Sci. Math. (Szeged) **86** (3-4), 521-547, 2020.

[5] M. Bramanti and M.C. Cerutti, *Commutators of singular integrals on homogeneous spaces*, oll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. **4**, 843-884, 1996.

[6] V.I. Burenkov and H.V. Guliyev, *Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces*, Studia Math. **163** (2), 157-176, 2004.

[7] V.I. Burenkov and V.S. Guliyev, *Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces*, Potential Anal. **31** (2), 1-39, 2009.

[8] F. Chiarenza and M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function,* Rend. Sem. Mat. Univ. Padova **7**, 273-279, 1987.

[9] G. Di Fazio and M.A. Ragusa, *Commutators and Morrey spaces*, Boll. Unione Mat. Ital. Sez. A Mat. Soc. Cult. **5** (3), 323-332, 1991.

[10] A.D. Gadjiev and I.A. Aliev, *On classes of operators of potential types, generated by a generalized translate*, Reports of enlarged Session of the Seminars of I.N.Vekua Inst. of Applied Mathematics Tbilisi **3** (2), 21-24, 1988 (Russian).

[11] V.S. Guliyev, Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$, PhD thesis, Mat. Inst. Steklov, 1994.

[12] V.S. Guliyev, *Sobolev theorems for the Riesz B-potentials*, Dokl. Akad. Nauk **358** (4), 450-451, 1998. (Russian)

[13] V.S. Guliyev, *Sobolev theorems for anisotropic Riesz-Bessel potentials on Morrey-Bessel spaces*, Dokl. Akad. Nauk Russia **367** (2), 155-156, 1999.

[14] V.S. Guliyev, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (3), 3361, 2012.

[15] V. Guliyev, I. Ekincioglu, E. Kaya and Z. Safarov, *Characterizations for the fractional maximal commutator operator in generalized Morrey spaces on Carnot group*, Integral Transforms Spec. Funct. **30** (6), 453-470, 2019.

[16] V.S. Guliyev and J.J. Hasanov, *Sobolev-Morrey type inequality for Riesz potentials, associated with the Laplace-Bessel differential operator*, Fract. Calc. Appl. Anal. **9** (1), 17-32, 2006.

[17] V.S. Guliyev and J.J. Hasanov, *Necessary and sufficient conditions for the boundedness of Riesz potential associated with the Laplace-Bessel differential operator in Morrey spaces,* J. Math. Anal. Appl. **347**, 113-122, 2008.

[18] J.J. Hasanov, *A note on anisotropic potentials, associated with the Laplace-Bessel differential operator*, Oper. Matrices **2** (4), 465-481, 2008.

[19] J.J. Hasanov, R. Ayazoglu and S. Bayrakci, *B-maximal commutators, commutators of B-singular integral operators and B-Riesz potentials on B-Morrey spaces*, Open Mathematics, **18**, 715-730, 2020.

[20] J.J. Hasanov, I. Ekincioglu and C. Keskin, *A characterization for B-singular integral operator and its commutators on generalized weighted B-Morrey spaces*, (accepted for publication)

[21] I.A. Kipriyanov, *Fourier-Bessel transformations and imbedding theorems*, Trudy Math. Inst. Steklov **89**, 130-213, 1967.

[22] Y. Komori and S. Shirai, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282** (2), 219-231, 2009.

[23] B.M. Levitan, *Bessel function expansions in series and Fourier integrals*, Uspekhi Mat. Nauk. (Russian), **6** (2), 102-143, 1951.

[24] L.N. Lyakhov, *Multipliers of the Mixed Fourier-Bessel transform*, Proc. Steklov Inst. Math. **214**, 234-249, 1997.

[25] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces* Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, 183-189, Springer-Verlag, Tokyo 1991.

[26] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166, 1938.

[27] B. Muckenhoupt and E.M. Stein, *Classical expansions and their relation to conjugate harmonic functions*, Trans. Amer. Math. Soc. **118**, 17-92 1965.

[28] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces* , Math. Nachr. **166**, 95-103, 1994.

[29] E. Nakai, *Generalized fractional integrals on generalized Morrey spaces* , Math. Nachr. **287** (2-3), 339-351, 2014.

[30] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivative. Theory and Applications*, Gordon and Breach Sci. Publishers, 1993.

[31] Y. Sawano, *Generalized Morrey spaces for non-doubling measures*, NoDEA Nonlinear Differential Equations Appl. **12** (4-5), 413-425, 2008.

[32] Y. Sawano, S.Sugano and H. Tanaka, *Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces*, Trans. Amer. Math. Soc. **363** (12), 6481-6503, 2011.

[33] A. Serbetci and I. Ekincioglu, *On Boundedness of Riesz potential generated by generalized translate operator on Ba spaces*, Czech. Math. J. **54** (3), 579-589, 2004.

[34] E.L. Shishkina and S.M. Sitnik, *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics.* Elsevier, 2020.

[35] E.M. Stein, *Singular Integrals And Differentiability Properties of Functions*, Princeton New Jersey, Princeton Uni. Press, 1970.

[36] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton, NJ, Princeton Univ. Press, 1971.

[37] K. Stempak, *Almost everywhere summability of Laguerre series*, Studia Math. **100** (2), 129-147, 1991.

[38] K. Trimeche, *Inversion of the Lions transmutation operators using generalized wavelets*, Appl. Comput. Harmon. Anal. **4**, 97-112, 1997.