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Fixed Points for Functions of Different Variables

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Abstract : The stady can be seen from this example that the conditions for the existence and uniqueness of a Fixed point are sufficient, but not necessary.

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1.Intradaction

Previously, we know how to use fixed -point iteration to solve a single nonlinear equation of the for f(x) = 0 by first transforming the equation into one of the form

$$x = g(x)$$

Then, after choosing an initial guess $x^{(0)}$, we compute a sequence of iterates by,

$$x^{(k+1)} = g(x^k) \text{ ; k = 0; 1; 2; : : : ;}$$

that, hopefully, converges to a solution of the original equation.

We have also learned that if the function g is a continuous function that maps an interval D into

itself, then g has a fexed point (also called a stationary point) x_* in D, which is a point that satisfies

$x_* = g(x_*)$ That is, a solution to f(x) = 0 exists within I. Furthermore, if there is a constant q < 1 such

$$|g'(x)| < q \text{ , } x \in D;$$

then this fixed point is unique.

It is worth noting that the constant q, which can be used to indicate the speed of convergence of fixed point iteration, corresponds to the spectral radius q(T) of the iteration matrix

$$T = S^{-1}N$$

used in a stationary iterative method of the form

$$x^{(k+1)} = Tx^k + S^{-1}d$$

for solving Ax = d , where A = S^{-1}N .

We now generalize fixed-point iteration to the problem of solving a system of n nonlinear equations in n unknowns,

$$f_1(x_1 x_2 \dots x_n) = 0$$
$$f_2(x_1 x_2 \dots x_n) = 0$$
$$\dots$$
$$f_n(x_1 x_2 \dots x_n) = 0$$

For simplicity, we express this system of equations in vector form,

$$F(x) = 0 ,$$

where

F : E ⊆ $R^n \to R^n$ is a vector-valued function of n variables represented by the vector

x = $(x_1, x_2, \dots, x_n)$ and $f_1, f_2, \dots, f_n$ are the component functions, or coordinate functions of F.

The notions of limit and continuity generalize to vector-valued functions and functions of several

variables in a straightforward way. Given a function f : E ⊆ $R^n \to$ R and a point $x_0 \in$ E, we write

$$\lim_{X \to x_0} f(x) = L$$

if, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon$$

whenever x ∈ E and

$$0 < \|x - x_0\| < \delta$$

In this definition, we can use any appropriate vector norm ‖ ‖. We also say that f is continuous at a

point $x_0 \in$ E if,

$$\lim_{X \to x_0} f(x) = f(x_0)$$

It can be shown f is continuous at x_0 if its partial derviatives are bounded near x_0.
Having defined limits and continuity for scalar-valued functions of several variables, we can now
define these concepts for vector-valued functions. Given F : E ⊆ $R^n \rightarrow R^n$ and x_0 ∈ E, we say that

$$\lim_{X \rightarrow x_0} f(x) = L$$

if and only if

$$\lim_{X \rightarrow x_0} f_i\,(x) = L_i\,, \qquad i=1,2,3,\ldots,n$$

Similarly, we say that F is continuous at x_0 if and only if each coordinate function f_i is continuous at x_0. Equivalently, F is continuous at x_0 if

$$\lim_{X \rightarrow x_0} F(x) = F(x_0)$$

Now, we can define fixed-point iteration for solving a system of nonlinear equations

$$F(x) = 0$$

First, we transform this system of equations into an equivalent system of the form

$$x = G(x)$$

2. Displayed mathematical equations

One approach to doing this is to solve the ith equation in the original system for x_i. This is analogous to the derivation of the Jacobi method for solving systems of linear equations.
Next, we choose an initial guess $x^{(0)}$. Then, we compute subsequent iterates by

$$x^{(k+1)} = G(x^k)\,, \qquad k = 0; 1; 2; : :$$

The existence and uniqueness of fixed points of vector-valued functions of several variables can
be described in an analogous manner to how it is described in the single-variable case. The function
G has a fixed point in a domain E ⊆ R^n if G maps E into E. Furthermore, if there exists a constant
q < 1 such that, in some natural matrix norm,

$$\|J_G(x)\| \leq q \quad , x \in \mathrm{E}$$

where JG(x) is the Jacobian matrix of first partial derivatives of G evaluated at x, then G has a unique fixed point x^* in E, and fixed-point iteration is guaranteed to converge to x^* for any initial

guess chosen in E. This can be seen by computing a multivariable Taylor expansion of the error $x^{(k+1)} \rightarrow x^*$ around x^*

. The constant q measures the rate of convergence of fixed-point iteration, as the error approxi-
mately decreases by a factor of q at each iteration. It is interesting to note that the convergence
of fixed-point iteration for functions of several variables can be accelerated by using an approach
similar to how the Jacobi method for linear systems is modified to obtain the Gauss-Seidel method.

That is, when computing x_i^{k+1} by evaluating $f_i^{x^k}$, we replace x_j^k , for j < i, by x_j^{k+1} , since it has already been computed (assuming all components of x(k+1) are computed in order). There fore, as in Gauss-Seidel, we are using the most up-to-date information available when computing each iterate.

For Example Consider the system of equations

$$x_1 = x_2^2$$

$$x_1^2 + x_2^2 = 1$$

The first equation describes a parabola, while the second describes the unit circle. By graphing
both equations, it can easily be seen that this system has two solutions, one of which lies in the
first quadrant ($x_1 > 0$ *and* $x_2 > 0$).

To solve this system using fixed-point iteration, we solve the second equation for x_2 and obtain the equivalent system

$$x_2 = \sqrt{1 - x_1^2} \ , \qquad x_1 = x_2^2$$

If we consider the rectangle

$$E = \{(x_1, x_2); 0 \le x_1 \le 1 \; and \; 0 \le x_2 \le 1 \}$$

we see that the function

$$G(x_1 , x_2) = (x_2^2 , \sqrt{1 - x_1^2})$$

maps E into itself. Because G is also continuous on E, it follows that G has a fixed point in E. However, G has the Jacobian matrix

$$J_G(x) = \begin{bmatrix} 0 & 2 x_2 \\ - x_1/\sqrt{1 - x_1^2} & 0 \end{bmatrix}$$

which cannot satisfy $\|J_G(x)\| < 1$ on E. Therefore, we cannot guarantee that fixed-point iteration

with this choice of G will converge, and, in fact, it can be shown that it does not converge. Instead,

the iterates tend to approach the corners of E, at which they remain. In an attempt to achieve convergence, we note that

$$\frac{\partial g_2}{\partial x_2} = 2x_2 > 1$$

near the fixed point. Therefore, we modify G as follows:

$$G(x_1 , x_2) = (x_2^2, \sqrt{1 - x_1^2})$$

For this choice of G , JG still has partial derivatives that are greater than 1 in magnitude near the fixed point. However, there is one crucial distinction: near the fixed point, q(JG) < 1, whereas with the original choice of G, q(JG) > 1. Attempting fixed-point iteration with the new G we see that convergence is actually achieved, although it is slow.

3. Result

It can be seen from this example that the conditions for the existence and uniqueness of a fixed point are sufficient, but not necessary.

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