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GRADED S-NOETHERIAN MODULES

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Abstract. Let G be an abelian group and S a given multiplicatively closed subset of a commutative G-graded ring A consisting of homogeneous elements. In this paper, we introduce and study G-graded S-Noetherian modules which are a generalization of S-Noetherian modules. We characterize G-graded S-Noetherian modules in terms of S-Noetherian modules. For instance, a G-graded A-module M is G-graded S-Noetherian if and only if M is S-Noetherian, provided G is finitely generated and S is countable. Also, we generalize some results on G-graded Noetherian rings and modules to G-graded S-Noetherian rings and modules.

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1. Introduction

In the context of ring theory, the introduction of Noetherian rings and modules [20] gave a new motivation for studying the structure theory of commutative rings. Recall that a module over a ring is called Noetherian if it satisfies ascending chain condition on submodules, and a commutative ring is called Noetherian if it is a Noetherian module over itself. Due to its significance, Noetherian rings and their generalizations have been extensively studied by many authors (see [1], [4], [5], [6] and [18], for example). As one of its crucial generalizations, Anderson and Dumitrescu [1] introduced S-Noetherian rings and modules. Let A be a commutative ring with identity, S a multiplicatively closed subset (briefly, m.c.s.) of A, and M an A-module. Then M is called S-finite if there exist an $s \in S$ and a finitely generated submodule F of M such that $sM \subseteq F$. Also, M is called S-Noetherian if each submodule of M is S-finite. A ring A is called S-Noetherian if it is S-Noetherian as an A-module, [1]. They have extended many results on Noetherian rings and modules to S-Noetherian rings and modules. Noncommutative S-Noetherian rings and S-Noetherian modules were first introduced by Baeck, Lee and Lim in [3]. Thereafter S-Noetherian rings and modules were continuously studied by many authors (see [4], [6], [10], [11], [12], [13], [14], [15] and [16], for example). This notion has motivated many researchers to study S-version of known structures in ring and module theory (see [4], [6], [21] and [22], for example).

Theory of graded rings and modules extends the theory of rings and modules. Let G be an abelian group with identity element e and A be a commutative ring with identity. Then A is called G-graded if $A = \bigoplus_{g\in G} A_g$ for additive subgroups A_g and $A_gA_h \subseteq A_{gh}$ for every $g, h \in G$. An A-module M is called G-graded if $M = \bigoplus_{g\in G} M_g$ for additive subgroups M_g and $A_gM_h \subseteq M_{gh}$ for every $g, h \in G$. A submodule N of M is called graded if $N = \bigoplus_{g\in G}(N \cap M_g)$. Similarly, an ideal I of A is called graded if $I = \bigoplus_{g\in G}(I\cap A_g)$. A G-graded A-module M is called G-graded Noetherian if each graded submodule of M is finitely generated. A G-graded ring A is called G-graded Noetherian if it is G-graded Noetherian module over itself. Goto and Yamagishi [5] characterized G-graded Noetherian rings in terms of Noetherian rings. More precisely, they proved that a G-graded ring A is G-graded Noetherian if and only if A is Noethrian, provided G is finitely generated. Inspired by it, Kim and Lim [10] introduced the notion of G-graded S-Noetherian ring and extended previous result to this class. A G-graded ring $A = \bigoplus_{g\in G} A_g$ is called G-graded S-Noetherian, where S is a given m.c.s. of A_e, if each graded ideal of A is S-finite. They showed that a G-graded ring A is S-Noetherian if and only if A is G-graded S-Noetherian, if and only if A_e is S-Noetherian and A is an S-finite A_e-algebra, provided G is finitely generated and S is an anti-Archimedean subset of A_e (see [10, Theorem 1]). In [18], among other results, Năstăsescu and Van Oystaeyen obtained a characterization of G-graded Noetherian modules in terms of Noetherian modules. For instance, a G-graded A-module M is G-graded Noetherian if and only if M is Noetherian, provided G is finitely generated (see [18, Theorem 2.1]).

In this paper, we introduce and study the notion of G-graded S-Noetherian module as a generalization of both the S-Noetherian and the G-graded Noetherian modules. In view of the results in the previous paragraph, a natural question arises, under what conditions both the notions of G-graded S-Noetherian modules and S-Noetherian modules coincide. As an answer to this question, we prove two characterizations for a G-graded S-Noetherian module to be an S-Noetherian module. First, we prove that if G is a finitely generated abelian group and S a countable m.c.s. of A_e, then M is a G-graded S-Noetherian module if and only if M is an S-Noetherian module (Theorem 3.28). This result is the S-version of [18, Theorem

2.1]. Second, we prove that if G is an abelian group, A a strongly G-graded ring and S a m.c.s. of A_e, then M is a G-graded S-Noetherian module if and only if M_e is an S-Noetherian A_e-module (Theorem 3.32). Moreover, in Theorem 3.35 we obtain S-variant of [9, Theorem 2.38]. Finally, we characterize G-graded S-Noetherian rings as a generalization of [9, Theorem 2.41].

2. Preliminaries

Throughout this paper, G is an abelian group with identity e and all the rings are assumed to be commutative rings with identity unless otherwise stated.

Let G be a multiplicative abelian group, $A = \bigoplus_{g\in G} A_g$ a G-graded ring, and $M = \bigoplus_{g\in G} M_g$ a G-graded A-module. Then A_e is a subring of A containing 1_A and each M_g is an A_e-module. The elements of $h(M) = \bigcup_{g\in G} M_g$ are said to be homogeneous element of M, a nonzero $x \in M_g$ is said to be homogeneous of degree g, and we write deg(x)= g. Similarly, the elements of $h(A) = \bigcup_{g\in G} A_g$ are said to be homogeneous element of A. For any subset $X \subseteq M$, we denote $h(X)$ by the set of all homogeneous elements of X. A nonzero element a of A has a unique decomposition as $a = a_{g_1} + a_{g_2} + \cdots + a_{g_n}$ with $a_{g_i} \in A_{g_i}$. For $a \in h(A)$ and $x \in h(M)$, observe that $deg(ax) = deg(a)deg(x)$. M is said to be strongly G-graded if $A_g M_h = M_{gh}$ for all $g, h \in G$. A graded ideal P of A is said to be G-prime if $P \neq A$; and whenever $ab \in P$, we have $a \in P$ or $b \in P$, where $a, b \in h(A)$. A proper graded ideal I of A is said to be G-maximal if there is no proper graded ideal J of A such that $I \subset J$. A is called G-graded local if it has unique G-maximal ideal. A is called a G-graded field if each nonzero homogeneous element has a multiplicative inverse. The graded radical of a graded ideal I is a graded ideal $Gr(I) := \{a = \Sigma_{g\in G} a_g \in A : \textit{for every } g \in G, \textit{ there exists an integer } n_g \geq 1 \textit{ such that } a_g^{n_g} \in I\}$.

Let H be a subgroup of G. Then $A_H := \bigoplus_{h\in H} A_h$ is an H-graded ring. In fact A_H is a G-graded ring. Also, let $g \in G$ and gH be coset of H in G, then $M_{gH} := \bigoplus_{h\in H} M_{gh}$ is a G-graded A_H-submodule of M. In particular, M_H is a G-graded A_H-module. Write $M := \bigoplus_{g\in T} M_{gH}$, where T is a transversal of H in G. This defines a G/H-grading on M, and under this grading M is also a G/H-graded A-module.

For more details of the graded rings and modules, [17] and [19] are referred.

Proposition 2.1. [19, Lemma 5.4.1] *Let M be a G-graded A-module, H a subgroup of G and $g \in G$. Consider A_H as a G-graded ring, and let N be a graded A_H-submodule of M_{gH}. If AN is the graded A-submodule of M generated by N, then $AN \cap M_{gH} = N$.*

3. Characterizations of graded S-Noetherian rings and modules

In this section, we introduce the notions of G-graded S-Noetherian module and G-graded strong S-Noetherian module as a generalization of S-Noetherian module and obtain their characterizations. To do so, we begin this section by introducing their definitions.

Definition 3.1. Let M be a G-graded A-module and S be a m.c.s. of $h(A)$. Then M is called S-finite if there exist an $s \in S$ and a finitely generated graded submodule F of M such that $sM \subseteq F$. Also, M is called G-graded S-Noetherian if each graded submodule of M is S-finite.

Let M be an A-module and S a m.c.s. of A. Recall [6, Definition 2.1] that an ascending chain $\{N_n\}_{n\in\mathbb{N}}$ of submodules of M is called *S-stationary* if there exist a positive integer $j \geq 1$ and an $s \in S$ such that $sN_i \subseteq N_j$ for every $i \geq j$. We say that M is a *strong S-Noetherian module* if every ascending chain of submodules of M is S-stationary.

Let M be a G-graded A-module and S be a m.c.s. of $h(A)$. We say that M is a *G-graded strong S-Noetherian A-module* if for every ascending chain $\{N_n\}_{n\in\mathbb{N}}$ of graded submodules of M is S-stationary. Also, a G-graded ring A is called a *G-graded strong S-Noetherian ring* if it is a G-graded strong S-Noetherian module over itself.

Example 3.2. Let M be a G-graded A-module and S be a m.c.s. of $h(A)$. If M is an S-Noetherian A-module, then M is a G-graded S-Noetherian module.

The converse of Example 3.2 is not true in general. For this, consider the following example.

Example 3.3. Let $A = K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, \ldots]$ be a Laurent polynomial ring in infinitely many indeterminates over a field K. Consider the group $G = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ and m.c.s. $S = K \setminus \{0\}$ of A. Then A is a G-graded field with canonical G-grading (see [8, Example 1.1.22]). Consequently, A is a G-graded S-Noetherian A-module since it has only two graded submodules, namely 0 and A. However, A is not an S-Noetherian A-module.

Thus Example 3.3 shows that the notion of G-graded S-Noetherian module is a proper generalization of the notion of S-Noetherian modules. Now, the following example shows that the notion of G-graded S-Noetherian modules also generalizes the notion of G-graded Noetherian modules.

Example 3.4. Every G-graded Noetherian A-module is a G-graded S-Noetherian A-module for every m.c.s. $S \subseteq h(A)$. The converse is also true if $S \subseteq U(h(A))$, where $U(h(A))$ denotes the set of all units of $h(A)$.

The converse of Example 3.4 is not true in general since any G-graded A-module M with $Gr(Ann(M)) \cap S \neq \emptyset$ is trivially a G-graded S-Noetherian A-module. Thus, from now, we assume that $Gr(Ann(M)) \cap S = \emptyset$ in this work. The following example presents a G-graded S-Noetherian module which is not a G-graded Noetherian module satisfying the condition $Gr(Ann(M)) \cap S = \emptyset$.

Example 3.5. Let $G = \mathbb{Z}_2$, $A = \mathbb{Z} = A_{\bar{0}}$, and $M = \mathbb{Z}_k[x] \oplus L$ be a G-graded A-module with $M_{\bar{0}} = \mathbb{Z}_k[x] \oplus 0$, $M_{\bar{1}} = 0 \oplus L$, where $k \geq 1$ and L is a torsion-free Noetherian A-module. Notice that the graded submodules of M are the submodules of the form $N \oplus N'$, where N is a submodule of $\mathbb{Z}_k[x]$ and N' is a submodule of L. Consider the m.c.s. $S = \{k^n : n \geq 0\}$ of A. Here we note that $S \cap Gr(Ann(M)) = \emptyset$ since $k^n M \neq 0$ for every integer $n \geq 0$. Put $s = k$. Then $s(N \oplus N') \subseteq N' \subseteq N \oplus N'$. Since N' is finitely generated submodule of L, so $N \oplus N'$ is S-finite. Thus, M is a G-graded S-Noetherian A-module. However, M is not a G-graded Noetherian A-module since $\mathbb{Z}_k[x]$ is not a Noetherian A-module.

Remark 3.6.

1. Let M be a G-graded A-module and S be a m.c.s. of $h(A)$. If M is a G-graded S-Noetherian module, then M is a G-graded strong S-Noetherian module. Indeed, let M be a G-graded S-Noetherian module and let $\{N_n\}_{n\in\mathbb{N}}$ be an ascending chain of graded submodules of M. Put $N = \bigcup_{i\geq 1} N_i$. Then N is S-finite, and so there exist an $s \in S$ and $x_1, x_2, \ldots, x_n \in h(M)$ such that $sN \subseteq Ax_1 + Ax_2 + \cdots + Ax_n \subseteq N$. Suppose $x_i \in N_{j_i}$ for $i = 1, 2, \ldots, n$. Take $j = max\{j_1, j_2, \ldots, j_n\}$. Then $x_i \in N_j$ for all $i = 1, 2, \ldots, n$. This implies that $sN \subseteq Ax_1 + Ax_2 + \cdots + Ax_n \subseteq N_j$. Consequently, $sN_i \subseteq N_j$ for all $i \geq 1$, and so $\{N_n\}_{n\in\mathbb{N}}$ is S-stationary. Hence M is a G-graded strong S-Noetherian module.
2. Let M be a G-graded A-module and S be a m.c.s. of $h(A)$. Suppose each submodule of M is countably generated. If M is a G-graded strong S-Noetherian module, then M is a G-graded S-Noetherian module. Indeed, let N be a graded submodule of M. Since N is countably generated, there

exist a countable set of homogeneous generators $\{x_n\}_{n\in\mathbb{N}}$ for N. Consider an ascending chain $Ax_1 \subseteq Ax_1+Ax_2 \subseteq \cdots \subseteq Ax_1+Ax_2+\cdots+Ax_n \subseteq \cdots$ of graded submodules of M. Since M is G-graded strong S-Noetherian, there exist an $s \in S$ and $k \geq 1$ such that $s(Ax_1+Ax_2+\cdots+Ax_i) \subseteq Ax_1+Ax_2+\cdots+Ax_k$ for all $i \geq k$. Consequently, $sN \subseteq Ax_1 + Ax_2 + \cdots + Ax_k \subseteq N$ which implies that N is S-finite. Hence M is a G-graded S-Noetherian module.

(3) Let M be a G-graded A-module and S be a countable m.c.s. of $h(A)$. If M is a G-graded strong S-Noetherian module, then M is a G-graded S-Noetherian module. The proof is on the same line by replacing ring to module and ideals to submodules in [7, Remark 2.2] and needs only minor modifications to work in the graded case.

Let A be a G-graded ring and $a \in h(A)$. Then $S_a := \{a^n : n \geq 0\}$ is a m.c.s. of $h(A)$. Also, $U(A)$ denotes the set of all units of A. In the following result, we obtain a characterization of G-graded Noetherian modules in terms of G-graded S-Noetherian modules.

Proposition 3.7. *Let A be a G-graded ring which is not a G-graded local ring and M be a G-graded A-module. Then M is a G-graded Noetherian A-module if and only if M is a G-graded S_a-Noetherian A-module for every $a \in h(A) \setminus U(A)$.*

Proof. The proof is on the same line by replacing descending chains to ascending chains of graded submodules in [2, Theorem 3.21]. □

The following proposition is a graded analogous to result of S-Noetherian modules discussed in [1], and the proof needs only minor modifications to work in the graded case.

Proposition 3.8. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be a short exact sequence of G-graded A-modules and S be a m.c.s. of $h(A)$. Then M is a G-graded S-Noetherian A-module if and only if M' and M'' are G-graded S-Noetherian A-modules. In particular, if A is G-graded S-Noetherian, then so is every finitely generated A-module*

Corollary 3.9. *Let M_1 and M_2 be G-graded S-Noetherian A-modules, where S is a m.c.s. of $h(A)$. Then $M_1 \oplus M_2$ is a G-graded S-Noetherian A-module.*

Proposition 3.10. *Let M be a G-graded A-module, N a graded submodule of M and S a m.c.s. of $h(A)$. Then M is a G-graded strong S-Noetherian A-module* *if and only if N and M/N are G-graded strong S-Noetherian A-modules. In particular, direct sum of two G-graded strong S-Noetherian A-modules is a G-graded strong S-Noetherian A-modules.*

Proof. The proof is on the same line by replacing descending chains to ascending chains of graded submodules in [2, Theorem 3.13]. □

The following theorem provides a characterization of G-graded S-Noetherian modules.

Theorem 3.11. *Let A be a G-graded ring, H be a subgroup of G and S be a m.c.s. of $h(A_H)$. If M is a G-graded S-Noetherian A-module, then M_{gH} is a G-graded S-Noetherian A_H-module for every $g \in G$. Conversely, if $[G : H] < \infty$ and M_{gH} is a G-graded S-Noetherian A_H-module for every $g \in G$, then M is a G-graded S-Noetherian A-module.*

Proof. Suppose M is a G-graded S-Noetherian A-module. Let N be a graded A_H-submodule of M_{gH} for some $g \in G$. Then AN is an S-finite graded A-submodule of M. This implies that there exist an $s \in S$ and a finitely generated graded A-submodule F of M such that $sAN \subseteq F \subseteq AN$. By Proposition 2.1, $sN = sAN \cap M_{gH} \subseteq F \cap M_{gH} \subseteq AN \cap M_{gH} = N$, i.e., $sN \subseteq F \cap M_{gH} \subseteq N$. Write $F = Ax_1 + Ax_2 + \cdots + Ax_n$, for some $x_1, x_2, \ldots, x_n \in h(M)$. Since $F \subseteq AN$, we may assume that each x_i is a homogeneous element of N. Suppose $deg(x_i) = gh_i$ for some $h_i \in H$. Let $y \in F \cap M_{gH}$ be a G-homogeneous element of degree gh for some $h \in H$. Write $y = \sum_{i=1}^{n} a_i x_i$ for some $a_i \in A$. Since $deg(y) = gh$, we can assume that each a_i is homogeneous of degree hh_i^{-1} for $i = 1, 2, \ldots, n$. Thus each $a_i \in A_H$, and so $y \in A_H x_1 + A_H x_2 + \cdots + A_H x_n$. Consequently, $F \cap M_{gH} \subseteq A_H x_1 + A_H x_2 + \cdots + A_H x_n$. Thus we have $sN \subseteq A_H x_1 + A_H x_2 + \cdots + A_H x_n \subseteq N$. Hence N is S-finite, and so M_{gH} is a G-graded S-Noetherian A_H-module. For the converse, write $M = \bigoplus_{g \in G} M_{gH}$. This direct sum is finite as $[G : H] < \infty$. Now since each M_{gH} is a G-graded S-Noetherian A_H-module, so by Corollary 3.9, M is a G-graded S-Noetherian A_H-module. Hence M is a G-graded S-Noetherian A-module. □

Corollary 3.12. *Let G be a finite abelian group, A be a G-graded ring, $S \subseteq A_e$ be a m.c.s. and $M = \bigoplus_{g \in G} M_g$ be a G-graded A-module. Then the following are equivalent:*

(1) *M is a G-graded S-Noetherian A-module.*
(2) *M_g is an S-Noetherian A_e-module for every $g \in G$.*

(3) *M is an S-Noetherian A_e-module.*

(4) *M is an S-Noetherian A-module.*

Proof. (1) $\Longrightarrow$ (2): Follows from Theorem 3.11 for $H = \{e\}$.

(2) $\Longrightarrow$ (3): Follows from Corollary 3.9 since G is finite.

(3) $\Longrightarrow$ (4): Obvious.

(4) $\Longrightarrow$ (1): Follows from Example 3.2. $\square$

Corollary 3.13. *Let A be a G-graded ring, H be a subgroup of G and S be a m.c.s. of $h(A_H)$. If A is a G-graded S-Noetherian ring, then A_H is an H-graded S-Noetherian ring.*

Proof. Follows from Theorem 3.11 for $M = A$. $\square$

Theorem 3.14. *Let A be a G-graded ring, H be a subgroup of G and S be a m.c.s. of $h(A_H)$. If M is a G-graded strong S-Noetherian A-module, then M_{gH} is a G-graded strong S-Noetherian A_H-module for every $g \in G$. Conversely, if $[G : H] < \infty$ and M_{gH} is a G-graded strong S-Noetherian A_H-module for every $g \in G$, then M is a G-graded strong S-Noetherian A-module.*

Proof. The proof is on the same line by replacing descending chains to ascending chains of graded submodules in [2, Theorem 3.17]. $\square$

Corollary 3.15. *Let G be a finite abelian group, A be a G-graded ring, $S \subseteq A_e$ be a m.c.s. and $M = \bigoplus_{g\in G} M_g$ be a G-graded A-module. Then the following are equivalent:*

(1) *M is a G-graded strong S-Noetherian A-module.*

(2) *M_g is a strong S-Noetherian A_e-module for every $g \in G$.*

(3) *M is a strong S-Noetherian A_e-module.*

(4) *M is a strong S-Noetherian A-module.*

Proof. (1) $\Longrightarrow$ (2): Follows from Theorem 3.14 for $H = \{e\}$.

(2) $\Longrightarrow$ (3): Follows from Proposition 3.10 since G is finite.

(3) $\Longrightarrow$ (4): Obvious.

(4) $\Longrightarrow$ (1): Follows from the fact that a strong S-Noetherian module is a G-graded strong S-Noetherian module. $\square$

Let M be a $\mathbb{Z}$-graded A-module. Following [19], $M_{\geq 0} = \bigoplus_{n\geq 0} M_n$ and $M_{\leq 0} = \bigoplus_{n\leq 0} M_n$. Clearly, $A_{\geq 0}$ and $A_{\leq 0}$ are graded subrings of A. Then $M_{\geq 0}$ (resp., $M_{\leq 0}$) is a $\mathbb{Z}$-graded $A_{\geq 0}$-module (resp., $A_{\leq 0}$-module). Let $N \subseteq M$ be an A-submodule of M and $x \in N$. Write $x = x_{n_1} + x_{n_2} + \cdots + x_{n_r}$ with $n_1 < n_2 < \ldots < n_r$ and

$x_{n_i} \in h(M)$. Denote $N^{\sim}$ (resp., $N_{\sim}$) by the submodule of M generated by x_{n_r} (resp., x_{n_1}) for all $x \in N$, i.e. $N^{\sim}$ (resp., $N_{\sim}$) is generated by the homogeneous components of largest (resp., smallest) degree present in the elements of N. Clearly, both $N^{\sim}$ and $N_{\sim}$ are graded submodules of M.

Now our aim is to characterize G-graded S-Noetherian modules in terms of S-Noetherian modules. For this, we need the following lemmas which are S-version of the similar results given in [17].

Lemma 3.16. *Let M be a $\mathbb{Z}$-graded A-module and S be a m.c.s. of A_0. Consider A-submodules $N \subseteq K \subseteq M$. Then the following are equivalent:*

(1) *$s'K \subseteq N$ for some $s' \in S$.*
(2) *$sK^{\sim} \subseteq N^{\sim}$ and $s(K \cap M_{\leq 0}) \subseteq N \cap M_{\leq 0}$ for some $s \in S$.*
(3) *$sK_{\sim} \subseteq N_{\sim}$ and $s(K \cap M_{\geq 0}) \subseteq N \cap M_{\geq 0}$ for some $s \in S$.*

Proof. (1) $\iff$ (2): If $s'K \subseteq N$ for some $s' \in S$, then $s'K^{\sim} \subseteq N^{\sim}$ and $s'(K \cap M_{\leq 0}) \subseteq N \cap M_{\leq 0}$ hold trivially. For the converse, let $y \in K$. Write $y = y_{n_1} + y_{n_2} + \cdots + y_{n_r}$ with $n_1 < n_2 < \ldots < n_r$ and $y_{n_i} \in h(M)$. If $n_r > 0$, then $sK^{\sim} \subseteq N^{\sim}$ yields that there exists $x_1 \in N$ such that $x_1 = u_{m_1} + u_{m_2} + \cdots + u_{m_{p-1}} + sy_{n_r}$ with $m_1 < m_2 < \ldots < m_{p-1} < n_r$ and $u_{m_i} \in h(M)$. This implies that $sy - x_1 \in K$ has a homogeneous decomposition containing highest degree less than n_r. Write $sy - x_1 = z_{k_1} + z_{k_2} + \cdots + z_{k_t}$ with $k_1 < k_2 < \ldots < k_t < n_r$ and $z_{k_i} \in h(M)$. But $sK^{\sim} \subseteq N^{\sim}$ implies that there exists $x_2 \in N$ such that $x_2 = w_{l_1} + w_{l_2} + \cdots + w_{l_{q-1}} + sz_{k_t}$ with $l_1 < l_2 < \ldots < l_{q-1} < k_t$ and $w_{l_i} \in h(M)$. This implies that $s^2y - sx_1 - x_2 \in K$ has a homogeneous decomposition containing highest degree less than k_t. Similarly, we can find $x_3, x_4, \ldots, x_k \in N$ such that $s^ky - s^{k-1}x_1 - s^{k-2}x_2 - \cdots - sx_{k-1} - x_k \in K \cap M_{\leq 0}$. Consequently, $s^{k+1}y - s^kx_1 - s^{k-1}x_2 - \cdots - s^2x_{k-1} - sx_k \in s(K \cap M_{\leq 0}) \subseteq N \cap M_{\leq 0}$; hence $s^{k+1}y \in N$. Also, if $n_r \leq 0$, then obviously $y \in K \cap M_{\leq 0}$ which implies that $s^{k+1}y \in s(K \cap M_{\leq 0}) \subseteq N \cap M_{\leq 0} \subseteq N$. Thus $s'K \subseteq N$, where $s' = s^{k+1}$, as desired. Similarly (1) $\iff$ (3) can be proved. $\square$

As a consequence of the Lemma 3.16, we have the following result.

Corollary 3.17. *Let M be a $\mathbb{Z}$-graded A-module and S be a m.c.s. of A_0. Then*

(1) *$M_{\geq 0}$ is a $\mathbb{Z}$-graded strong S-Noetherian $A_{\geq 0}$-module if and only if $M_{\geq 0}$ is a strong S-Noetherian $A_{\geq 0}$-module.*
(2) *$M_{\leq 0}$ is a $\mathbb{Z}$-graded strong S-Noetherian $A_{\leq 0}$-module if and only if $M_{\leq 0}$ is a strong S-Noetherian $A_{\leq 0}$-module.*

Proof. (1) Suppose $M_{\geq 0}$ is a $\mathbb{Z}$-graded strong S-Noetherian $A_{\geq 0}$-module. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of $A_{\geq 0}$-submodule of $M_{\geq 0}$. Then $N_1^{\sim} \subseteq N_2^{\sim} \subseteq \cdots \subseteq N_n^{\sim} \subseteq \cdots$ is an ascending chain of graded $A_{\geq 0}$-submodule of $M_{\geq 0}$. Then there exist an $s_1 \in S$ and $k_1 \geq 0$ such that $s_1 N_i^{\sim} \subseteq N_{k_1}^{\sim}$ for all $i \geq k_1$. Also consider the chain $N_1 \cap M_{\leq 0} \subseteq N_2 \cap M_{\leq 0} \subseteq \cdots \subseteq N_n \cap M_{\leq 0} \subseteq \cdots$ of A_0-submodule of M_0 as $N_i \cap M_{\leq 0} \subseteq M_0$ for each i. Since $M_{\geq 0}$ is a $\mathbb{Z}$-graded S-Noetherian $A_{\geq 0}$-module, so by Theorem 3.14, M_0 is a strong S-Noetherian A_0-module. This implies that there exist an $s_2 \in S$ and $k_2 \geq 1$ such that $s_2(N_i \cap M_{\leq 0}) \subseteq N_{k_2} \cap M_{\leq 0}$ for all $i \geq k_2$. Put $s = s_1 s_2$ and $k = max\{k_1, k_2\}$. Then $sN_i^{\sim} \subseteq N_k^{\sim}$ and $s(N_i \cap M_{\leq 0}) \subseteq N_k \cap M_{\leq 0}$ for all $i \geq k$. So by Lemma 3.16, there exists an $s' \in S$ such that $s' N_i \subseteq N_k$ for all $i \geq k$. Hence $M_{\geq 0}$ is a strong S-Noetherian $A_{\geq 0}$-module.

(2) Similar to (1). □

Lemma 3.18. *Let M be a $\mathbb{Z}$-graded S-Noetherian A-module, where S is a m.c.s. of A_0. Then*

(1) *$M_{\geq 0}$ is a strong S-Noetherian $A_{\geq 0}$-module.*
(2) *$M_{\leq 0}$ is a strong S-Noetherian $A_{\leq 0}$-module.*

Proof. (1) Let $N = \bigoplus_{i \geq 0} N_i$ be a graded $A_{\geq 0}$-submodule of $M_{\geq 0}$. Since M is a $\mathbb{Z}$-graded S-Noetherian A-module, so AN is an S-finite graded submodule of M. This implies that there exist $x_1, x_2, \ldots, x_r \in h(N)$ and $s_1 \in S$ such that $s_1 AN \subseteq Ax_1 + Ax_2 + \cdots + Ax_r \subseteq AN$. Suppose $deg(x_i) = n_i \geq 0$. Put $n = max(n_1, n_2, \ldots, n_r)$. Let $y \in h(N)$ with $deg(y) = m \geq n$. Then there exist $a_1, a_2, \ldots, a_r \in h(A)$ such that $s_1 y = a_1 x_1 + a_2 x_2 + \cdots + a_r x_r$. Consequently, $deg(a_i) = m - n_i \geq 0$ as $m \geq n$, and so $a_i \in A_{\geq 0}$ for all $i = 1, 2, \ldots, r$. Then we have $s_1(\bigoplus_{i \geq n} N_i) \subseteq F_1 \subseteq \bigoplus_{i \geq n} N_i$, where F_1 is a finitely generated $A_{\geq 0}$-submodule of $M_{\geq 0}$ generated by the set $\{x_1, x_2, \ldots, x_r\}$. On the other hand by Theorem 3.11, each M_i is an S-Noetherian A_0-module which implies that each N_i is an S-Noetherian A_0-module. Consequently, by Corollary 3.9, $L = \bigoplus_{i=1}^{n} N_i$ is an S-finite A_0-module. This implies that there exist an $s_2 \in S$ and a finitely generated A_0-submodule F_2 of N generated by the set $\{y_1, y_2, \ldots, y_k\}$ such that $s_2 L \subseteq F_2 \subseteq L$. Put $s = s_1 s_2$. Consider the $A_{\geq 0}$-submodule F of N generated by the set $\{x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_k\}$. Then $sN \subseteq F_1 + F_2 \subseteq F \subseteq N$ since $N = L \oplus (\oplus_{i \geq n} N_i)$ and $F_2 \subseteq F$. Thus N is S-finite, and so $M_{\geq 0}$

is a $\mathbb{Z}$-graded S-Noetherian $A_{\geq 0}$-module. By Remark 3.6(1), $M_{\geq 0}$ is a $\mathbb{Z}$-graded strong S-Noetherian $A_{\geq 0}$-module. Hence by Corollary 3.17, $M_{\geq 0}$ is a strong S-Noetherian $A_{\geq 0}$-module.

(2) Similar to (1). $\square$

Corollary 3.19. *Let M be a $\mathbb{Z}$-graded S-Noetherian A-module, where S is a countable m.c.s. of A_0. Then*

(1) *$M_{\geq 0}$ is an S-Noetherian $A_{\geq 0}$-module.*
(2) *$M_{\leq 0}$ is an S-Noetherian $A_{\leq 0}$-module.*

Proof. Follows from Lemma 3.18 and Remark 3.6(3). $\square$

Lemma 3.20. *Let M be an S-finite $\mathbb{Z}$-graded A-module, where S is a m.c.s. of A_0. If M is a $\mathbb{Z}$-graded strong S-Noetherian A-module. Then*

(1) *$M_{\geq 0}$ is a strong S-Noetherian $A_{\geq 0}$-module.*
(2) *$M_{\leq 0}$ is a strong S-Noetherian $A_{\leq 0}$-module.*

Proof. (1) Since M is S-finite, there exist an $s \in S$ and $x_1, x_2, \ldots, x_r \in h(M)$ such that $sM \subseteq Ax_1 + Ax_2 + \cdots + Ax_r$. Suppose $deg(x_i) = k_i$ for $i = 1, 2, \ldots, r$. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of graded $A_{\geq 0}$-submodules of $M_{\geq 0}$. Then $AN_1 \subseteq AN_2 \subseteq \cdots \subseteq AN_n \subseteq \cdots$ is an ascending chain of graded A-submodules of M. Since M is a $\mathbb{Z}$-graded strong S-Noetherian A-module, there exist an $s' \in S$ and $m \geq 0$ such that $s'AN_i \subseteq AN_m$ for all $i \geq m$. Let $x \in h(N_i)$ with $deg(x) = j \geq 0$. Then $s'x \in AN_m$ is a homogeneous element. Write $ss'x = a_1x_1 + a_2x_2 + \cdots + a_rx_r$ for some $a_i \in h(A)$. Since $ss'x$ is homogeneous, we may assume that each a_ix_i is a homogeneous element of AN_m. Then we have $j = deg(a_i) + k_i$, i.e., $deg(a_i) = j - k_i$. Put $k = max\{k_1, k_2, \ldots, k_r\}$. So if $j \geq k$, then $a_i \in A_{\geq 0}$ for $i = 1, 2, \ldots, r$. This implies that $ss'x \in N_m$. Suppose $j < k$, then $j \in \{0, 1, 2, \ldots, k-1\}$. Since M is $\mathbb{Z}$-graded strong S-Noetherian, so by Theorem 3.14, each M_i is a $\mathbb{Z}$-graded strong S-Noetherian A_0-module for $i = 0, 1, 2, \ldots, k-1$. Also, since N_i is a $\mathbb{Z}$-graded $A_{\geq 0}$-submodule of $M_{\geq 0}$ and $A_0 \subseteq A_{\geq 0}$, so each N_i is a $\mathbb{Z}$-graded A_0-submodule of $\mathbb{Z}$-graded A_0-module $M_{\geq 0}$. Consider finitely many ascending chains $N_1 \cap M_t \subseteq N_2 \cap M_t \subseteq \cdots \subseteq N_n \cap M_t \subseteq \cdots$ of $\mathbb{Z}$-graded A_0-submodules of M_t for $t = 0, 1, 2, \ldots, k-1$. This implies that there exist $s_t \in S$ and $j_t \geq 1$ such that $s_t(N_i \cap M_t) \subseteq N_{j_t} \cap M_t$, for all $i \geq j_t$, for $t = 0, 1, 2, \ldots, k-1$. Put

$s'' = s_0s_1s_2 \dots s_{k-1}$ and $j' = max\{j_0, j_1, j_2, \dots, j_{k-1}\}$. Then $s''(N_i \cap M_t) \subseteq N_{j'} \cap M_t$ for all $i \geq j'$, for $t = 0, 1, 2, \dots, k-1$. Thus if $x \in M_t$ for $t < k$, then $s''x \in s''(N_i \cap M_t) \subseteq N_{j'} \cap M_t$. This implies that $s''x \in N_{j'}$. Put $s''' = ss's''$ and $p = max\{m, j'\}$, then $s'''x \in N_p$ for all $x \in h(N_i)$, and so $s'''N_i \subseteq N_p$ for all $i \geq p$. Thus $M_{\geq 0}$ is a $\mathbb{Z}$-graded strong S-Noetherian $A_{\geq 0}$-module. Hence by Corollary 3.17, $M_{\geq 0}$ is a strong S-Noetherian $A_{\geq 0}$-module.

(2) Similar to (1). □

Lemma 3.21. *Let M be a $\mathbb{Z}$-graded A-module and S be a m.c.s. of A_0. If M is a $\mathbb{Z}$-graded S-Noetherian A-module, then M is a strong S-Noetherian A-module.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of A-submodules of M. Then $N_1 \cap M_{\leq 0} \subseteq N_2 \cap M_{\leq 0} \subseteq \cdots \subseteq N_n \cap M_{\leq 0} \subseteq \cdots$ is an ascending chain of $A_{\leq 0}$-submodules of $M_{\leq 0}$. By Lemma 3.18, $M_{\leq 0}$ is a strong S-Noetherian $A_{\leq 0}$-module, so there exist an $s_1 \in S$ and $j \geq 1$ such that $s_1(N_i \cap M_{\leq 0}) \subseteq N_j \cap M_{\leq 0}$ for all $i \geq j$. Also, $N_1^{\sim} \subseteq N_2^{\sim} \subseteq \cdots \subseteq N_n^{\sim} \subseteq \cdots$ is an ascending chain of graded A-submodules of M. Since M is $\mathbb{Z}$-graded S-Noetherian, by Remark 3.6(1), M is a $\mathbb{Z}$-graded strong S-Noetherian module. So there exist an $s_2 \in S$ and $k \geq 1$ such that $s_2N_i^{\sim} \subseteq N_k^{\sim}$ for all $i \geq k$. Put $s = s_1s_2$ and $r = max(j, k)$. Then $s(N_i \cap M_{\leq 0}) \subseteq M_{\leq 0} \cap N_r$ and $sN_i^{\sim} \subseteq N_r^{\sim}$ for all $i \geq r$. So by Lemma 3.16, $s'N_i \subseteq N_r$ for some $s' \in S$ and for all $i \geq r$. Hence M is a strong S-Noetherian A-module. □

Corollary 3.22. *Let M be a $\mathbb{Z}$-graded A-module and S be a countable m.c.s. of A_0. Then the following are equivalent:*

(1) *M is a $\mathbb{Z}$-graded S-Noetherian A-module.*
(2) *M is an S-Noetherian A-module.*

Proof. Follows from Lemma 3.21 and Remark 3.6(3). □

Lemma 3.23. *Let M be a $\mathbb{Z}$-graded A-module and S be a m.c.s. of A_0. If M is a $\mathbb{Z}$-graded strong S-Noetherian A-module, then M is a strong S-Noetherian A-module.*

Proof. Similar to the proof of Lemma 3.21 by using Lemma 3.20. □

Let G be a finitely generated abelian group. Then $G \cong \mathbb{Z}^r \oplus T$, where T is torsion part of G. Consider the subgroup $H = \mathbb{Z}^{r-1} \oplus T$ of G. Then $G/H \cong \mathbb{Z}$, and so there exists $\xi \in G$ such that $G/H =< \xi H >= \{\xi^n H : n \in \mathbb{Z}\}$. Suppose M is a G-graded A-module. Following [17], define

$$M^+ = \bigoplus_{m\in\mathbb{Z}_+}(\bigoplus_{h\in H} M_{\xi^m h}) = \bigoplus_{m\geq 0} M_{\xi^m H}$$
$$M^- = \bigoplus_{m\in\mathbb{Z}_-}(\bigoplus_{h\in H} M_{\xi^m h}) = \bigoplus_{m\leq 0} M_{\xi^m H}.$$

Then M^+ (resp., M^-) is a G-graded A^+-module (resp., A^--module). Let $h \in H$. Write $M_{(h)} = \bigoplus_{n\in\mathbb{Z}} M_{\xi^n h}$. Then $A_{(e)} = \bigoplus_{n\in\mathbb{Z}} A_{\xi^n}$ is a $\mathbb{Z}$-graded ring and each $M_{(h)}$ is a $\mathbb{Z}$-graded $A_{(e)}$-module. Also, $A = \bigoplus_{h\in H} A_{(h)}$ is an H-graded ring and $M = \bigoplus_{h\in H} M_{(h)}$ is an H-graded A-module. Hence M is G-graded as well as H-graded A-module. It is clear that each G-graded submodule of M is also H-graded but the converse may not true. Let $N = \bigoplus_{h\in H} N_{(h)}$ be an H-graded A-submodule of M, where $N_{(h)} = M_{(h)} \cap N$. Define

$$N^{\sim} = \bigoplus_{h\in H} N^{\sim}_{(h)} \text{ (resp., } N_{\sim} = \bigoplus_{h\in H} N_{(h)\sim}),$$

where $N^{\sim}_{(h)}$ (resp., $N_{(h)\sim}$) is a $\mathbb{Z}$-graded $A_{(e)}$-submodule of $M_{(h)}$ generated by the homogeneous components of largest (resp., smallest) degree present in the elements of $N_{(h)}$ as defined before. If $M = M^+$ (resp., M^-), then $N^{\sim}$ (resp., $N_{\sim}$) is a G-graded A-submodule of M, [18, Lemma 1.4].

With these notations, we have the following lemmas which are S-version of similar results given in [18].

Lemma 3.24. *If M is a G-graded S-Noetherian A-module, where S is a m.c.s. of A_e. Then M^+ is a G-graded S-Noetherian A^+-module.*

Proof. Let N be a graded A^+-submodule of M^+. Then the graded A-submodule AN of M is S-finite, i.e. there exist an $s \in S$ and homogeneous elements $x_1, x_2, \dots, x_r$ in AN such that $sAN \subseteq Ax_1 + Ax_2 + \cdots + Ax_r \subseteq AN$. We may assume that each x_i is a homogeneous element of N, say $x_i \in N_{\xi^{k_i} h_i} = N \cap M_{\xi^{k_i} h_i}$ for some $h_i \in H$ and some $k_i \geq 0$. Put $k = max(k_1, k_2, \dots, k_r)$ and $L = \bigoplus_{i=1}^{k}(\bigoplus_{h\in H} N_{\xi^i h}) = \bigoplus_{i=1}^{k} N_{\xi^i H}$. Since M is a G-graded S-Noetherian A-module, so by Theorem 3.11, each $M_{\xi^i H}$ is a G-graded S-Noetherian A_H-module. But then each $N_{\xi^i H}$ is a G-graded S-Noetherian A_H-module, and so by Corollary 3.9, L is a G-graded S-Noetherian A_H-module which implies that there exists an $s' \in S$ such that $s'L \subseteq F \subseteq L$ for some finitely generated graded A_H-submodule F of L generated by a set of homogeneous elements $\{y_1, y_2, \dots, y_p\}$. Let $x \in h(N)$. Then $x \in N_{\xi^j h}$ for some $j \geq 0$ and some $h \in H$. Write $sx = a_1x_1 + a_2x_2 + \cdots + a_rx_r$, where $a_i \in h(A)$. Then we have $\xi^j h = deg(a_i)\xi^{k_i} h_i$ which implies that $deg(a_i) = \xi^{j-k_i} h h_i^{-1}$ for all $i = 1, 2, \dots, r$. So if $j \geq k$, then $a_i \in A^+$ for $i = 1, 2, \dots, r$. This implies that

$sx \in A^+x_1 + A^+x_2 + \cdots + A^+x_r$. Also if $j < k$, then $s'x \in F$. Therefore we conclude that $ss'x \in A^+x_1 + A^+x_2 + \cdots + A^+x_r + A^+y_1 + A^+y_2 + \cdots + A^+y_p$ since $A_H \subseteq A^+$. Consequently, $ss'N \subseteq A^+x_1 + A^+x_2 + \cdots + A^+x_r + A^+y_1 + A^+y_2 + \cdots + A^+y_p \subseteq N$, and so N is S-finite. Hence M^+ is a G-graded S-Noetherian A^+-module. □

Lemma 3.25. *Let M be an S-finite G-graded A-module, where S is a m.c.s. of A_e. If M is a G-graded strong S-Noetherian A-module, then M^+ is a G-graded strong S-Noetherian A^+-module.*

Proof. Since M is S-finite, there exist $s \in S$ and $x_1, x_2, \ldots, x_r \in h(M)$ such that $sM \subseteq Ax_1 + Ax_2 + \cdots + Ax_r$. Suppose $deg(x_i) = \xi^{k_i}h_i$, i.e., $x_i \in M_{\xi^{k_i}h_i}$ for some $h_i \in H$ and some $k_i \geq 0$. Let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of G-graded A^+-submodules of M^+. Then $AN_1 \subseteq AN_2 \subseteq \cdots \subseteq AN_n \subseteq \cdots$ is an ascending chain of G-graded A-submodules of M. Since M is a G-graded strong S-Noetherian A-module, there exist an $s' \in S$ and $m \geq 1$ such that $s'AN_i \subseteq AN_m$ for all $i \geq m$. Now let $x \in N_i$ be a homogeneous element. Then $s'x \in s'AN_i \subseteq AN_m$. Write $ss'x = a_1x_1 + a_2x_2 + \cdots + a_rx_r$, for some $a_i \in h(A)$. Since $ss'x$ is homogeneous, we can assume that each a_ix_i is a homogeneous element of AN_m. Also since $x \in N_i$ is a homogeneous element of M^+, then $x \in N_i \cap M_{\xi^j h}$ for some $j \geq 0$ and some $h \in H$. Consequently, $\xi^j h = deg(a_i)\xi^{k_i}h_i$ which implies that $deg(a_i) = \xi^{j-k_i}hh_i^{-1}$ for $i = 1, 2, \ldots, r$. Put $k = max\{k_1, k_2, \ldots, k_r\}$. So if $j \geq k$, then $a_i \in A^+$ for $i = 1, 2, \ldots, r$. This implies that $ss'x \in N_m$. Suppose $j < k$, then $j \in \{0, 1, 2, \ldots, k-1\}$. Since M is G-graded strong S-Noetherian, so by Theorem 3.14, $M_{\xi^i H}$ is a G-graded strong S-Noetherian A_H-module for $i = 0, 1, 2, \ldots, k-1$. Also, since N_i is a G-graded A^+-submodule of M^+ and $A_H \subseteq A^+$, so each N_i is a G-graded A_H-submodule of G-graded A_H-module M^+. Consider finitely many ascending chains $N_1 \cap M_{\xi^t H} \subseteq N_2 \cap M_{\xi^t H} \subseteq \cdots \subseteq N_n \cap M_{\xi^t H} \subseteq \cdots$ of G-graded A_H-submodules of $M_{\xi^t H}$ for $i = 0, 1, 2, \ldots, k-1$. This implies that there exist $s_t \in S$ and $j_t \geq 1$ such that $s_t(N_i \cap M_{\xi^t H}) \subseteq N_{j_t} \cap M_{\xi^t H}$, for all $i \geq j_t$, for $t = 0, 1, 2, \ldots, k-1$. Put $s'' = s_0s_1s_2\ldots s_{k-1}$ and $j' = max\{j_0, j_1, \ldots, j_{k-1}\}$. Then $s''(N_i \cap M_{\xi^t H}) \subseteq N_{j'} \cap M_{\xi^t H}$, for all $t = 0, 1, 2, \ldots, k-1$, for all $i \geq j'$. Thus if $x \in M_{\xi^j h}$ for $j < k$, then $s''x \in s''(N_i \cap M_{\xi^j H}) \subseteq N_{j'} \cap M_{\xi^j H}$. This implies that $s''x \in N_{j'}$. Put $s''' = ss's''$ and $p = max\{m, j'\}$, then $s'''x \in N_p$ for all $x \in h(N_i)$ and so $s'''N_i \subseteq N_p$ for all $i \geq p$. This implies that M^+ is a G-graded strong S-Noetherian A^+-module. □

Lemma 3.26. *Let $M = M^+$ be a G-graded A-module and S be a m.c.s. of A_e. If $N \subseteq K \subseteq M$ are H-graded A-submodules of M such that $sK^{\sim} \subseteq N^{\sim}$ for some $s \in S$, then $s'K \subseteq N$ for some $s' \in S$.*

Proof. Since N and K are H-graded submodules of $M = \bigoplus_{h\in H} M_{(h)}$, so we can write $N = \bigoplus_{h\in H} N_{(h)}$ and $K = \bigoplus_{h\in H} K_{(h)}$, where $N_{(h)} = M_{(h)} \cap N$ and $K_{(h)} = M_{(h)} \cap K$ with $N_{(h)} \subseteq K_{(h)}$ for all $h \in H$. Now $sK^{\sim} \subseteq N^{\sim}$ yields that $sK^{\sim}_{(h)} \subseteq N^{\sim}_{(h)}$ for all $h \in H$. Also, since $M = M^+$ and $N_{(h)}, K_{(h)}$ are submodules of $\mathbb{Z}$-graded $A_{(e)}$-module $M_{(h)} = \bigoplus_{i\geq 0} M_{\xi^i h}$, so by Lemma 3.16, $s'K_{(h)} \subseteq N_{(h)}$ for some $s' \in S$ and for all $h \in H$. Consequently, $s'K \subseteq N$, as desired. □

Lemma 3.27. *Let M be a G-graded A-module and S be a m.c.s. of A_e. If $N \subseteq K \subseteq M$ are H-graded A-submodules of M such that $s(K \cap M^+) \subseteq N \cap M^+$ and $s'K_{\sim} \subseteq N_{\sim}$ for some $s, s' \in S$, then $s''K \subseteq N$ for some $s'' \in S$.*

Proof. Since $M = \bigoplus_{h\in H} M_{(h)}$ is an H-graded A-module, so we can write $M^+ = \bigoplus_{h\in H} M^+_{(h)}$, where $M^+_{(h)}$ is a $\mathbb{Z}$-graded $A^+_{(e)}$-module. Consequently, $N \cap M^+ = \bigoplus_{h\in H}(N_{(h)} \cap M^+_{(h)})$ and $K \cap M^+ = \bigoplus_{h\in H}(K_{(h)} \cap M^+_{(h)})$. Now $s(K\cap M^+) \subseteq N \cap M^+$ yields that $s(K_{(h)} \cap M^+_{(h)}) \subseteq N_{(h)} \cap M^+_{(h)}$ for every $h \in H$. Also, $s'K_{\sim} \subseteq N_{\sim}$ yields that $s'K_{(h)_{\sim}} \subseteq N_{(h)_{\sim}}$ for every $h \in H$. Now since $N_{(h)}$ and $K_{(h)}$ are $A_{(e)}$-submodule of the $\mathbb{Z}$-graded $A_{(e)}$-module $M_{(h)}$ such that $ss'K_{(h)_{\sim}} \subseteq N_{(h)_{\sim}}$ and $ss'(K_{(h)} \cap M^+_{(h)}) \subseteq N_{(h)} \cap M^+_{(h)}$ for all $h \in H$, so by Lemma 3.16, $s''K_{(h)} \subseteq N_{(h)}$ for some $s'' \in S$ and for all $h \in H$. Consequently, $s''K \subseteq N$, as desired. □

Now we are in a position to characterize G-graded S-Noetherian modules in terms of S-Noetherian modules. This result is a generalization of [18, Theorem2.1].

Theorem 3.28. *Let G be a finitely generated abelian group, M a G-graded A-module and S a countable m.c.s. of A_e. Then M is a G-graded S-Noetherian A-module if and only if M is an S-Noetherian A-module.*

Proof. If M is an S-Noetherian A-module, then by Example 3.2, M is a G-graded S-Noetherian A-module. For the converse, assume that M is a G-graded S-Noetherian A-module. Write $G = \mathbb{Z}^r \oplus T$, where T denotes the torsion part of G. We use induction on r to prove this theorem. If $r = 0$ or 1, then theorem follows from Corollary 3.12 and Corollary 3.22. Assume theorem is true for groups of the type $\mathbb{Z}^r \oplus T$, where $r > 1$. Consider the group $G = \mathbb{Z}^{r+1} \oplus T'$ and its subgroup $H = \mathbb{Z}^r \oplus T'$, where T' denotes the torsion part of G. Then $M = \bigoplus_{h\in H} M_{(h)}$ is an H-graded A-module. We claim that M is an H-graded S-Noetherian A-module.

For this, let $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n \subseteq \cdots$ be an ascending chain of H-graded A-submodules of M. Then we obtain an ascending chain $N_1 \cap M^+ \subseteq N_2 \cap M^+ \subseteq \cdots \subseteq N_n \cap M^+ \subseteq \cdots$ of H-graded A^+-submodules of M^+. Here we note that M^+ is both G-graded and H-graded A^+-module. Consequently, we have an ascending chain $(N_1 \cap M^+)^{\sim} \subseteq (N_2 \cap M^+)^{\sim} \subseteq \cdots \subseteq (N_n \cap M^+)^{\sim} \subseteq \cdots$ of G-graded A^+-submodules of M^+. Now since M is a G-graded S-Noetherian A-module, by Lemma 3.24, M^+ is a G-graded S-Noetherian A^+-module. So by Remark 3.6(1), M^+ is a G-graded strong S-Noetherian A^+-module. This implies that there exist an $s_1 \in S$ and an index $j \geq 1$ such that $s_1(N_i \cap M^+)^{\sim} \subseteq (N_j \cap M^+)^{\sim}$ for all $i \geq j$. But then by Lemma 3.26, there exists an $s_2 \in S$ such that $s_2(N_i \cap M^+) \subseteq N_j \cap M^+$ for all $i \geq j$. Also, we have another ascending chain $N_{1_\sim} \subseteq N_{2_\sim} \subseteq \cdots \subseteq N_{n_\sim} \subseteq \cdots$ of G-graded A-submodules of M. Since M is G-graded S-Noetherian, there exist an $s_3 \in S$ and an index $k \geq 1$ such that $s_3 N_{i_\sim} \subseteq N_{k_\sim}$ for all $i \geq k$. Put $t = max(j, k)$. Then we have $s_2(N_i \cap M^+) \subseteq N_t \cap M^+$ and $s_3 N_{i_\sim} \subseteq N_{t_\sim}$ for all $i \geq t$. Therefore by Lemma 3.27, there exists an $s \in S$ such that $sN_i \subseteq N_t$ for all $i \geq t$. Thus M is an H-graded strong S-Noetherian A-module, and so by Remark 3.6(3), M is an H-graded S-Noetherian A-module. Hence by induction hypothesis, M is an S-Noetherian A-module, as desired. □

For the case $S = \{1\}$ of the result above, we obtain the following result.

Corollary 3.29. (*cf.* [18, Theorem 2.1]) *Let G be a finitely generated abelian group. Then M is a G-graded Noetherian A-module if and only if M is a Noetherian A-module.*

The next theorem provides a characterization of G-graded strong S-Noetherian modules in terms of strong S-Noetherian modules.

Theorem 3.30. *Let G be a finitely generated abelian group, M an S-finite G-graded A-module, where S is a m.c.s. of A_e. Then M is a G-graded strong S-Noetherian A-module if and only if M is a strong S-Noetherian A-module.*

Proof. The proof is on the same line by changing the use of Corollary 3.12, Lemma 3.24 and Corollary 3.22 by Corollary 3.15, Lemma 3.25 and Lemma 3.23, respectively in Theorem 3.28. □

As an immediate consequence, we have the following corollary.

Corollary 3.31. *Let G be a finitely generated abelian group, M a G-graded A-module, where S is a m.c.s. of A_e. If M is a G-graded S-Noetherian A-module, then M is a strong S-Noetherian A-module.*

For arbitrary abelian group G, the next theorem provides another characterization of G-graded S-Noetherian modules in terms of S-Noetherian modules.

Theorem 3.32. *Let A be a strongly G-graded ring, S a m.c.s. of A_e and M a G-graded A-module. Then $M = \bigoplus_{g\in G} M_g$ is a G-graded S-Noetherian A-module if and only if M_e is an S-Noetherian A_e-module.*

Proof. Suppose M_e is an S-Noetherian A_e-module. Consider the tensor product $A \underset{A_e}{\otimes} M_e$ which is a G-graded A-module with grading $A \underset{A_e}{\otimes} M_e = \bigoplus_{g\in G}(A \underset{A_e}{\otimes} M_e)_g$, where $(A \underset{A_e}{\otimes} M_e)_g = A_g \underset{A_e}{\otimes} M_e$ for all $g \in G$. Then we have a short exact sequence $0 \longrightarrow A_e \underset{A_e}{\otimes} M_e \longrightarrow A \underset{A_e}{\otimes} M_e \longrightarrow A_e \underset{A_e}{\otimes} M_e \longrightarrow 0$ of G-graded A_e-modules. Now since $A_e \underset{A_e}{\otimes} M_e \cong M_e$ as an A_e-module which is S-Noetherian, so by Proposition 3.8, $A \underset{A_e}{\otimes} M_e$ is a G-graded S-Noetherian A_e-module, and so G-graded S-Noetherian A-module. Also since A is strongly G-graded, so $A \underset{A_e}{\otimes} M_e \cong M$ as a G-graded A-module by [17, Theorem 1.3.4]. Hence M is a G-graded S-Noetherian A-module. The converse part follows from Theorem 3.11. $\square$

Corollary 3.33. *If G is finitely generated, A is a strongly G-graded ring, S is a countable m.c.s. of A_e and M is a G-graded A-module, then the following are equivalent:*

(1) *M is a G-graded S-Noetherian A-module.*
(2) *M is an S-Noetherian A-module.*
(3) *M_e is an S-Noetherian A_e-module.*

We next present an example which shows that the condition strongly graded in Theorem 3.32 is not superfluous.

Example 3.34. Let $G = \mathbb{Z}$, $A = \mathbb{Z} = A_0$ and $M = \mathbb{Z}_4^{(\mathbb{N})}$ (Direct sum of countable copies of $\mathbb{Z}_4$) be a naturally G-graded A-module. Take the m.c.s. $S = \{3^n : n \geq 0\}$. Then $M_0 = \mathbb{Z}_4$ is a G-graded S-Noetherian A_0-module but M is not a G-graded S-Noetherian A-module.

Let A be a ring, S a m.c.s. of A, and B an A-algebra. Following [10], B is said to be an S-finite A-algebra if there exist $s \in S$ and $b_1, b_2, \ldots, b_n \in B$ such that $sB \subseteq A[b_1, b_2, \ldots, b_n]$. Also, recall from [1] that a m.c.s. S of a ring A is called anti-Archimedean if $\cap_{k=1}^{\infty} s^k A \cap S \neq \emptyset$ for all $s \in S$. The following theorem is a generalization of [9, Theorem 2.38].

Theorem 3.35. *Let A be a G-graded ring, H a subgroup of G such that G/H is finitely generated, and S an anti-Archimedean m.c.s. of $h(A_H)$. If A is G-graded S-Noetherian, then A is an S-finite A_H-algebra.*

Proof. Assume G/H is generated by n elements. We prove this theorem by induction on n. If $n = 1$, then $G/H =< gH >= \{g^k H : k \in \mathbb{Z}\}$ for some $g \in G$; hence we can write $A = \bigoplus_{k\in\mathbb{Z}} A_{g^k H}$, where $A_{g^k H} = \bigoplus_{h\in H} A_{g^k h}$. By Theorem 3.11, each $A_{g^k H}$ is a G-graded S-Noetherian A_H-module, so if order of gH is finite then by Corollary 3.9, A is a G-graded S-Noetherian A_H-module. Consequently, A is an S-finite A_H-module, and therefore A is an S-finite A_H-algebra. Now suppose order of gH is infinite. Consider the ideal $I =< \{\bigcup_{k>0} A_{g^k H}\} >$ of A. Then I is a G-graded ideal of A, and therefore I is S-finite. This implies that there exist $s \in S$ and homogeneous elements $x_1, x_2, \dots, x_j \in I$ such that $sI \subseteq Ax_1 + Ax_2 + \cdots + Ax_j \subseteq I$. Suppose $deg(x_i) = g^{k_i} h_i$ for some $h_i \in H$ and some $k_i > 0$ for all $i = 1, 2, \dots, j$. Put $K := max\{k_1, k_2, \dots, k_j\}$. By Theorem 3.11, each $A_{g^k H}$ is a G-graded S-Noetherian A_H-module, and so by Corollary 3.9, $L = \bigoplus_{i=1}^{K} A_{g^i H}$ is an S-finite A_H-module; hence there exist $s' \in S$ and homogeneous elements $y_1, y_2, \dots, y_m \in L$ such that $s'L \subseteq A_H y_1 + A_H y_2 + \cdots + A_H y_m \subseteq L$. Consider the G-graded subring $B = A_H[x_1, x_2, \dots, x_j, y_1, y_2, \dots, y_m]$ of A. Now we induct on k to show $(ss')^k A_{g^k H} \subseteq B$ for every $k \geq 0$. Obviously this holds for every $k \leq K$. Suppose $k > K$ and $(ss')^i A_{g^i H} \subseteq B$ for all $i < k$. Let $x \in h(A_{g^k H})$, then $x \in I$, in fact $x \in A_{g^k h}$ for some $h \in H$. This implies that $sx \in Ax_1 + Ax_2 + \cdots + Ax_j$, and so we can write $sx = a_1x_1 + a_2x_2 + \cdots + a_jx_j$ for some $a_1, a_2, \dots, a_j \in A$. Suppose $deg(s) = h'$ for some $h' \in H$. Since sx is a homogeneous element of degree $g^k hh'$, we may assume that each a_i is a homogeneous element of A. Then we have $deg(a_i) = deg(x_i)^{-1} deg(sx) = g^{-k_i} h_i^{-1} g^k hh' = g^{k-k_i} hh' h_i^{-1}$; hence $a_i \in A_{g^{k-k_i} H}$. By induction hypothesis, $(ss')^{k-k_i} a_i \in (ss')^{k-k_i} A_{g^{k-k_i} H} \subseteq B$ for $i = 1, 2, \dots, j$. Consequently, $(ss')^k x = s'(ss')^{k-1} a_1 x_1 + ss'^2 (ss')^{k-2} a_2 x_2 + \cdots + s^{k_j - 1} s'^{k_j} (ss')^{k-k_j} a_j x_j \in B$ since $s, s' \in S \subseteq h(A_H) \subseteq B$. Hence by induction, $(ss')^k A_{g^k H} \subseteq B$ for all $k \geq 0$. Also since S is an anti-Archimedean subset of A_H, there exists an element $t \in \bigcap_{k=1}^{\infty} (ss')^k A_H \cap S$ such that $tA_{g^k H} \subseteq B$ for all $k \geq 0$. Similarly if we take $I =< \{\bigcup_{k<0} A_{g^k H}\} >$, then using similar argument, there exits $t' \in S$ such that $t' A_{g^k H} \subseteq B$ for all $k < 0$. Thus we conclude that $tt'A \subseteq B = A_H[x_1, x_2, \dots, x_l, y_1, y_2, \dots, y_m]$; hence A is an S-finite A_H-algebra which implies that statement of the theorem is true for $n = 1$. Now assume theorem is true for $n-1$, where $n > 1$. Since G is abelian, we can obtain a subgroup H' of G containing H such that G/H' is cyclic and H'/H is generated by $n-1$ elements.

Then by Corollary 3.13, $A_{H'}$ is H'-graded S-Noetherian as $S \subseteq h(A_H) \subseteq h(A_{H'})$. By induction hypothesis, $A_{H'}$ is an S-finite A_H-algebra. So by the case $n = 1$, A is an S-finite $A_{H'}$-algebra. Hence by [10, Lemma 5], A is an S-finite A_H-algebra. $\square$

In the next result, we prove Hilbert's basis theorem for G-graded S-Noetherian rings. This result is a generalization of [1, Proposition 9].

Proposition 3.36. *Let A be a G-graded ring and $S \subseteq h(A)$ an anti-Archimedean multiplicatively closed subset. If A is G-graded S-Noetherian, then so is the polynomial ring $A[x]$.*

Proof. Since $A = \oplus_{g\in G} A_g$ is G-graded, $A[x]$ is also a G-graded ring with grading $(A[x])_g = A_{gh}[x]$ for every $g \in G$, where $h = deg(x)$. Let I be a graded ideal of $A[x]$. Let J be the ideal of A consisting of zero and leading coefficients of polynomials in I. First we show J is a graded ideal of A. For this, let $\alpha \in J$. Then there exists an element $f(x) \in I$ such that $f(x) = \alpha x^n+$ (lower terms). Write $\alpha = \sum_{g\in G} \alpha_g$, where $\alpha_g \in h(A)$. Then $\alpha_g x^n+$ (lower terms) $\in I$ since I is a graded ideal of $A[x]$. Consequently, $\alpha_g \in J$ for each $g \in G$ and so J is a graded ideal of A. Since A is G-graded S-Noetherian, $sJ \subseteq (a_1, a_2, \ldots, a_n)$ for some $s \in S$ and homogeneous elements $a_1, a_2, \ldots, a_n \in J$. For each a_i, let f_i be an element of the form $a_i x^{d_i}+$ (lower terms) $\in I$. Since a_i is homogeneous, we can assume each f_i is homogeneous. Then the ideal I' generated by the set $\{f_1, f_2, \ldots, f_n\}$ is a graded ideal of $A[x]$. Let $d = max(d_i)$. Let T be the A-submodule of $A[x]$ generated by the set $\{1, x, x^2, \ldots, x^{d-1}\}$. Clearly, T is a finitely generated graded A-module and so by Proposition 3.8, T is a G-graded S-Noetherian A-module. In particular T is S-finite; whence there exist $t \in S$ and homogeneous elements $g_1, g_2, \ldots, g_m \in T$ such that $tT \subseteq Ag_1 + Ag_2 + \cdots + Ag_m$.

Let $h(x) = ax^k+$ (lower terms) be any homogeneous elements of I. Then $a \in J$, and so we can write $sa = \sum_{i=1}^{n} u_i a_i$, where $u_i \in A$. If $k \geq d$, then $sh(x) - \sum u_i f_i x^{k-d_i} \in I$ and has degree less than k. Proceeding in this way, we can go on subtracting elements of I' from $s^i h(x)$ until we get a polynomial $g(x)$ of degree less than d, that is, we get $r \geq 1$ such that $s^r h(x) = g(x) + p(x)$ where $p(x) \in I'$. Consequently, $s^r h(x) \in I' + T$. Now since S is anti-Archimedean, there exists $w \in \cap_i s^i A$. This implies that $twh(x) \in I' + L$, where L is the graded ideal of $A[x]$ generated by the set $\{g_1, g_2, \ldots, g_m\}$. Thus $twI \subseteq F \subseteq I$, where $F = I' + L$ is a finitely generated graded ideal of $A[x]$. Hence $A[x]$ is a G-graded S-Noetherian ring. $\square$

Now we end this section by giving a characterization of G-graded S-Noetherian rings which is a generalization of [9, Theorem 2.41].

Theorem 3.37. *Let A be a G-graded ring, S an anti-Archimedean m.c.s. of A_0 and H a finitely generated subgroup of G. Then A is a G-graded S-Noetherian ring if and only if A is a G/H-graded S-Noetherian ring.*

Proof. Suppose A is G-graded S-Noetherian. Notice that $A = \bigoplus_{g \in T} A_{gH}$ is also a G/H-graded ring, where T is a transversal of H in G. Let I be a G/H-graded ideal of A. Suppose H is generated by n elements. We use the induction on n. If $n = 1$, then $H =< h >= \{h^i : i \in \mathbb{Z}\}$ for some $h \in H$. Let $b \in I$ be a G/H-homogeneous element of degree gH for some $g \in G$. Write $b = b_{k_1} + b_{k_2} + \cdots + b_{k_m}$ where $0 \neq b_{k_i} \in A_{gh^{k_i}}$ and $k_1 < k_2 < \ldots < k_m$. Suppose order of h is infinite, then $gh^{k_i} \neq gh^{k_j}$ for $i \neq j$. Consider the G-grading on the polynomial ring $A[x]$ with $deg(x) = h$. Consider the polynomial $f_b(x) = b_{k_1}x^{k_m-k_1} + b_{k_2}x^{k_m-k_2} + \cdots + b_{k_m}$. Since $deg(b_{k_i}x^{k_m-k_i}) = gh^{k_i}h^{k_m-k_i} = gh^{k_m}$, so each $b_{k_i}x^{k_m-k_i}$ is a G-homogeneous element of same degree which implies that $f_b(x)$ is a G-homogeneous element of $A[x]$ of degree gh^{k_m}. Consider the G-graded ideal $\bar{I}$ of $A[x]$ generated by the set $\{f_b(x) :$ *b is G/H-homogeneous element of I* $\}$. Now since A is G-graded S-Noetherian, so by Proposition 3.36, $A[x]$ is a G-graded S-Noetherian ring which yields that $\bar{I}$ is S-finite. Then there exist $s \in S$ and G-homogeneous elements $f_{a_1}(x), f_{a_2}(x), \ldots, f_{a_r}(x) \in \bar{I}$ such that $s\bar{I} \subseteq A[x]f_{a_1}(x) + A[x]f_{a_2}(x) + \cdots + A[x]f_{a_r}(x) \subseteq \bar{I}$; hence there exists $\beta_i(x) \in A[x]$ such that $sf_b(x) = \beta_1(x)f_{a_1}(x) + \beta_2(x)f_{a_2}(x) + \cdots + \beta_r(x)f_{a_r}(x)$. Put $x = 1$, then $f_b(1) = b$, $\beta_i(1) \in A$, and $a_i := f_{a_i}(1) \in h(A)$. Then we have $sb = \beta_1(1)a_1 + \beta_2(1)a_2 + \cdots + \beta_r(1)a_r$. Consequently, $sI \subseteq Aa_1 + Aa_2 + \cdots + Aa_r \subseteq I$, and therefore I is S-finite in this case. Now suppose order of h is finite, say p. Then there exist unique $q_i, r_i \in \mathbb{Z}$ such that $k_i = q_ip + r_i$ where $0 \leq r_i < p$; hence we get $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, m\}$ such that $r_{i_1} < r_{i_2} < \ldots < r_{i_k} < p$. We can write $b = b_{r_{i_1}} + b_{r_{i_2}} + \cdots + b_{r_{i_k}}$ and we can proceed as above to show I is S-finite in this case also. Thus A is a G/H-graded S-Noetherian ring. □

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