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Global Asymptotic Stability of a System of Difference Equations with Quadratic Terms

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Abstract
In this article, we discuss the global asymptotic stability of following system of difference equations with quadratic terms: $x_{i+1}=\alpha+\beta\frac{y_{i-1}}{y_i^2}, \quad y_{i+1}=\alpha+\beta\frac{x_{i-1}}{x_i^2}$ where α, β are positive numbers and the initial values are positive numbers. We also study the rate of convergence and oscillation behaviour of the solutions of related system. We will give also, some numerical examples to illustrate our results.

Keywords: Difference equations, Equilibrium, Globally asymptotically stable, Oscillates, Prime period two solution, Qualitative properties of solutions of difference equations, Rational difference equations.
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1. Introduction

The difference equations or systems have too many applications among many branches of science. over the last two decades, difference equations or their systems have been huge interest between scholars which are mathematicians . For example, in [22] discussed global dynamics of an one-dimensional discrete-time laser model. Further in [8] Din et al. discussed stability of a discrete ecological model. Studies of difference equations are increasing day by day and will continue to increase. Therefore, there are many papers related to applications of difference equations or systems. More specifically, some scientists studied the dynamics of solutions of difference equations or systems (for example, see [1]-[5],[7, 9, 12], [14]-[21], [23], [25]-[30]). Additionally, there are many results related to our study as follows:
In [31], Yang et al. studied the solutions, stability and asymptotic behaviour of the system of the two nonlinear difference equations

$$x_{n+1}=\frac{Ax_n}{1+y_n^p}, \quad y_{n+1}=\frac{By_n}{1+x_n^p}.$$

In [11], Elabbasy et al. investigated the global behaviour of following system of difference equations

$$x_{n+1}=\frac{a_1x_n}{a_2+a_3y_n^r}, \quad y_{n+1}=\frac{b_1y_n}{b_2+b_3x_n^r}.$$

In [6], Bacani et al. discussed solutions of the following two nonlinear difference equations

$$x_{n+1}=\frac{q}{p+x_n^v}, \quad y_{n+1}=\frac{q}{-p+y_n^v}.$$

In [24], Hadziabdic et al. examined the global behaviours of following system of difference equations

$$x_{n+1} = \frac{b_1 x_n^2}{A_1 + y_n^2}, \quad y_{n+1} = \frac{a_2 + c_2 y_n^2}{x_n^2}$$

In [8], Burgic et al. investigated the global stability properties and asymptotic behaviour of solutions for the system of difference equations

$$x_{n+1} = \frac{x_n}{a + y_n^2}, \quad y_{n+1} = \frac{y_n}{b + x_n^2}.$$

In [10], Beso et al. concentrates on discussing boundedness of solutions of following difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2}.$$

In [13], Tasdemir et al. discussed the global asymptotic stability of a system of difference equations with quadratic terms

$$x_{n+1} = A + B\frac{y_n}{y_{n-m}^2}, \quad y_{n+1} = A + B\frac{x_n}{x_{n-m}^2}$$

They also studied global asymptotic stability of related difference equation. Motivated by difference equations and their systems, we consider the following system of difference equations

$$x_{i+1} = \alpha + \beta\frac{y_{i-1}}{y_i^2}, \quad y_{i+1} = \alpha + \beta\frac{x_{i-1}}{x_i^2} \tag{1.1}$$

where α and β are positive numbers and the initial values are positive numbers. In this paper we study the stability, global behaviour and rate of convergence of solutions of system (1.1). We also discussed the oscillation behaviour of solutions of related system. In this here, we obtain two theorems which are used during this study.

Theorem 1.1. *(Linearized Stability Theorem [25]) Assume that*

$$X_{i+1} = F(X_i), i = 0, 1, \ldots$$

is a system of difference equations such that $\bar{X}$ is a fixed point of F.
(i) If all eigenvalues of the Jacobian matrix β about $\bar{X}$ lie inside the open unit disk $|\lambda| < 1$, that is, if all of them have absolute value less than one, then $\bar{X}$ is locally asymptotically stable.
(ii) If at least one of them has a modulus greater than one, then $\bar{X}$ is unstable.

Theorem 1.2. *[5] Let $i \in N_{i_0}^+$ and $g(i,u,v)$ be a decreasing function in u and v for any fixed n. Suppose that for $i \le i_0$,the inqualities*

$$y_{i+1} \le g(i, y_i, y_{i-1})$$

$$u_{i+1} \ge g(i, y_i, y_{i-1})$$

hold. Then

$$y_{i_0-1} \le u_{i_0-1}, y_{i_0} \le u_{i_0}$$

implies that

$$y_i \le u_i, i \ge i_0.$$

2. Linearized Stability of System (1.1)

First of all, we consider the change of the variables for system (1.1) as follows:

$$\zeta_i = \frac{x_i}{\alpha},\ \eta_i = \frac{y_i}{\alpha}.$$

From this, system (1.1) transform into following system:

$$\zeta_{i+1} = 1 + \mu\ \frac{\eta_{i-1}}{\eta_i^2}, \eta_{i+1} = 1 + \mu\ \frac{\zeta_{i-1}}{\zeta_i^2} \tag{2.1}$$

where $\mu = \frac{\beta}{\alpha^2} > 0$. From now on, we study the system (2.1).

Lemma 2.1. *Let $\mu > 0$. Unique positive equilibrium point of system (2.1) is*

$$(\bar{\zeta},\bar{\eta}) = \left(\frac{1+\sqrt{1+4\mu}}{2}, \frac{1+\sqrt{1+4\mu}}{2}\right).$$

Now, we consider a transformation as follows:

$$(\zeta_i,\zeta_{i-1},\eta_i,\eta_{i-1}) \to (t,t_1,z,z_1)$$

where $t = 1+\mu\ \frac{\eta_{i-1}}{\eta_i^2}, t_1 = \zeta_i, z = 1+\mu\ \frac{\zeta_{i-1}}{\zeta_i^2}, z_1 = \eta_i$. Thus we get the jacobian matrix about equilibrium point $(\bar{\zeta},\bar{\eta})$:

$$\beta(\bar{\zeta},\bar{\eta}) = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{\mu}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the linearized system of system (2.1) about the unique positive equilibrium point is given by $X_{I+1} = \beta(\zeta,\eta)X_I$, where

$$X_I = \begin{pmatrix} \zeta_i \\ \zeta_{i-1} \\ \eta_i \\ \eta_{i-1} \end{pmatrix},$$

$$\beta(\bar{\zeta},\bar{\eta}) = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{\mu}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, the characteristic equation of $\beta(\zeta,\eta)$ about the unique positive equilibrium point $(\bar{\zeta},\bar{\eta})$is

$$\lambda^4 - \frac{\mu^2}{\bar{\zeta}^2\bar{\eta}^2}\lambda^2 + \frac{4\mu^2}{\bar{\zeta}^2\bar{\eta}^2}\lambda - \frac{4\mu^2}{\bar{\zeta}^2\bar{\eta}^2} = 0.$$

Due to $\bar{\zeta} = \bar{\eta}$, we can rearrange the characteristic equation such that

$$\lambda^4 - \frac{\mu^2}{\bar{\zeta}^4}\lambda^2 + \frac{4\mu^2}{\bar{\zeta}^4}\lambda - \frac{4\mu^2}{\bar{\zeta}^4} = 0.$$

Therefore, we obtain the four roots of characteristic equation as follows:

$$\lambda_1 = \frac{\mu+\sqrt{\mu^2-8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2},$$

$$\lambda_2 = \frac{\mu-\sqrt{\mu^2-8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2},$$

$$\lambda_3 = \frac{-\mu+\sqrt{\mu^2+8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2},$$

$$\lambda_4 = \frac{-\mu-\sqrt{\mu^2+8\mu\ \bar{\zeta}^2}}{2\ \bar{\zeta}^2}.$$

Now, we calculate $\bar{\zeta}^2$ and write in λ_1. Then we have

$$\begin{aligned}\lambda_1 &= \frac{\mu+\sqrt{\mu^2-4\mu(1+2\mu+\sqrt{4\mu+1})}}{1+2\mu+\sqrt{4\mu+1}} \\ &= \frac{\mu+\sqrt{-7\mu^2-4\mu-4\mu\sqrt{1+4\mu}}}{1+2\mu+\sqrt{4\mu+1}} \\ &= \frac{\mu+\sqrt{7\mu^2+4\mu+4\mu\sqrt{1+4\mu}}i}{1+2\mu+\sqrt{4\mu+1}}.\end{aligned}$$

Thus straightforward calculations show that

$$|\lambda_1| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}}.$$

Additionally, we obtain similarly calculations that

$$|\lambda_2| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}}.$$

On the other hand, we consider λ_3 as follows:

$$\begin{aligned}
\lambda_3 &= \frac{-\mu+\sqrt{9\mu^2+4\mu+4\mu\sqrt{4\mu+1}}}{1+2\mu+\sqrt{4\mu+1}} \\
&= \frac{-\mu+\sqrt{(3\mu+\sqrt{1+4\mu})^2-1-2\mu\sqrt{4\mu+1}}}{1+2\mu+\sqrt{4\mu+1}} \\
&< \frac{-\mu+\sqrt{(3\mu+\sqrt{1+4\mu})^2}}{1+2\mu+\sqrt{4\rho+1}} \\
&= \frac{2\mu+\sqrt{1+4\mu}}{1+2\mu+\sqrt{4\mu+1}} < 1.
\end{aligned}$$

Moreover, we clearly see that $\lambda_3 > 0$. So $0 < \lambda_3 < 1$ for all $\mu > 0$. Similar calculations we have that $-1 < \lambda_4 < 0$ for all $\mu > 0$.

Theorem 2.2. *Suppose that $\mu > 0$.Then the following cases hold for system (2.1):*
(i) If $\mu < 2$ then the equilibrium point of system (2.1) is locally asymptotically stable.
(ii) If $\mu = 2$ then the equilibrium point of system (2.1) is a non-hyperbolic equilibrium .
(iii) If $\mu > 2$ then the equilibrium point of system (2.1) is a repeller.

Proof. Firstly we know that $|\lambda_3|, |\lambda_4| < 1$ for all $\mu > 0$. Now we consider

$$|\lambda_1| = |\lambda_2| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}}.$$

If the equilibrium point of system (2.1) is locally asymptotically stable, then all roots of characteristic equation must lie the unit disk. Therefore,we must show that $|\lambda_1|, |\lambda_2| < 1$. Hence

$$|\lambda_1| = |\lambda_2| = \frac{2\sqrt{2\mu}}{1+\sqrt{1+4\mu}} < 1.$$

Thus, we have $2\sqrt{2\mu} < 1+\sqrt{1+4\mu}$. From this, we obtain that $\mu < 2$. The proofs of other cases can be obtained in a similar way. □

3. An Oscillation Result of Solutions of System (2.1)

In this here, we investigate the oscillation behaviour of solutions of system (2.1).

Theorem 3.1. *Assume $\{(\zeta_i, \eta_i)\}$ be a positive solution of system (2.1) $\mu > 0$. Then for any $i \geq 0$ the following cases are true.*

(i) if $\zeta_{i+1}, \eta_i < \bar{\zeta} = \bar{\eta} < \zeta_i, \eta_{i+1}$ then

$$\begin{aligned}
&(\zeta_{i+2k-1})_{k=1}^{\infty} < \bar{\zeta} < (\zeta_{i+2k})_{k=1}^{\infty}, \\
&(\eta_{i+2k})_{k=1}^{\infty} < \bar{\eta} < (\eta_{i+2k-1})_{k=1}^{\infty}.
\end{aligned} \tag{3.1}$$

(ii) if $\zeta_i, \eta_{i+1} < \bar{\zeta} = \bar{\eta} < \zeta_{i+1}, \eta_i$ then

$$\begin{aligned}
&(\zeta_{i+2k})_{k=1}^{\infty} < \bar{\zeta} < (\zeta_{i+2k-1})_{k=1}^{\infty}, \\
&(\eta_{i+2k-1})_{k=1}^{\infty} < \bar{\eta} < (\eta_{i+2k})_{k=1}^{\infty}.
\end{aligned} \tag{3.2}$$

Proof. Firstly we consider case (3.1). Assume that $\zeta_{i+1}, \eta_i < \bar{\zeta} = \bar{\eta} < \zeta_i, \eta_{i+1}$. Then we obtain that

$$\zeta_{i+2} = 1 + \mu\ \frac{\eta_i}{\eta_{i+1}^2} > 1 + \mu\ \frac{\bar{\eta}}{\bar{\eta}^2} = \bar{\eta} = \bar{\zeta},$$

$$\eta_{i+2} = 1 + \mu\ \frac{\zeta_i}{\zeta_{i+1}^2} < 1 + \mu\ \frac{\bar{\zeta}}{\bar{\zeta}^2} = \bar{\zeta} = \bar{\eta},$$

$$\zeta_{i+3} < \bar{\zeta}, \eta_{i+3} > \bar{\eta}, \zeta_{i+4} > \bar{\zeta}, \eta_{i+4} < \bar{\eta}.$$

Therefore we have by using induction

$$\zeta_i, \zeta_{i+2}, \ldots, \zeta_{i+2k}, \ldots > \bar{\zeta} > \zeta_{i+1}, \zeta_{i+3}, \ldots, \zeta_{i+2k-1}, \ldots$$

$$\eta_{i+1}, \eta_{i+3}, \ldots, \eta_{i+2k-1}, \ldots > \bar{\eta} > \eta_i, \eta_{i+2}, \ldots, \eta_{i+2k}, \ldots$$

Thus the proof of (3.1) is completed as desired. The proof of (3.2) is similar to proof of (3.1). □

4. Boundedness of System (2.1)

Lemma 4.1. *Let* $\{(\zeta_i, \eta_i)\}$ *be a positive solution of system* (2.1) *and* $\mu > 0$ *. Then* $\zeta_i > 1$ *and* $\eta_i > 1$ *for* $i \geq 1$.

Proof. Assume $\{(\zeta_i, \eta_i)\}$ be a positive solution of system (2.1). Then we have from system (2.1):

$$\zeta_1 = 1 + \mu\ \frac{\eta_{-1}}{\eta_0^2} > 1,$$

$$\eta_1 = 1 + \mu\ \frac{\zeta_{-1}}{\zeta_0^2} > 1.$$

Therefore, we obtain by induction

$$\zeta_{i+1} = 1 + \mu\ \frac{\eta_{i-1}}{\eta_i^2} > 1,$$

$$\eta_{i+1} = 1 + \mu\ \frac{\zeta_{i-1}}{\zeta_i^2} > 1.$$

So, the proof of lemma is completed. □

Theorem 4.2. *If* $0 < \mu < 1$ *then every solution of system* (2.1) *is bounded.*

Proof. Firstly we have from system (2.1) $\zeta_i > 1$ and $\eta_i > 1$ for $i \geq 1$ and $\mu > 0$. Moreover, every solution of system (2.1) satisfies

$$\zeta_{i+1} \leq 1 + \mu + \mu^2\ \zeta_{i-1},\ i \geq 1, \tag{4.1}$$

which due to Theorem 1.2, means that $\zeta_i \leq q_i$, $i = 0, 1, \ldots$, where $\{u_i\}$ satisfy

$$u_{i+1} = 1 + \mu + \mu^2\ u_{i-1}, i \geq 1, \tag{4.2}$$

such that

$$u_s = \zeta_s, u_{s+1} = \zeta_{s+1}, s \in \{-1, 0, 1, \ldots\}, i \geq s.$$

Hence the solution u_i of the difference equation (4.2) is

$$u_i = \frac{1}{1-\mu} + \mu^i C_1 + (-\mu)^i C_2. \tag{4.3}$$

Actually, we have from (4.2)

$$u_{i+1} = 1 + \mu + \mu^2 u_{i-1} \Rightarrow \lambda^2 - \mu^2 = 0 \Rightarrow \lambda_{1,2} = \pm\mu.$$

From this, the homogeneous solution of difference equation (4.2) is

$$u_n = \mu^i C_1 + (-\mu)^i C_2.$$

In additon, from (4.2), the equilibrium solution of difference equation (4.2) is

$$\bar{u} = 1+\mu+\mu^2\bar{u} \Rightarrow \bar{u} = \frac{1}{1-\mu}.$$

Additionally, relations (4.1) and (4.2) imply that

$$\zeta_{i+1} - u_{i+1} \leq \mu^2 \left(\zeta_{i-1} - u_{s-1}\right),\ i > s, \mu \in (0,1).$$

Therefore we have

$$\zeta_i \leq u_i, i > s \tag{4.4}$$

Hence, we obtain from (4.3), (4.4) and Lemma 4.1,

$$1 < \zeta_i \leq \frac{1}{1-\mu} + \mu^i C_1 + (-\mu)^i C_2 = N_1,$$

where

$$C_1 = \frac{1}{2\mu}\left(\mu\zeta_0 + \zeta_1 - \frac{1+\mu}{1-\mu}\right),$$
$$C_2 = \frac{1}{2\mu}\left(\mu\zeta_0 - \zeta_1 + 1\right).$$

Similarly we can write that

$$1 < \eta_i \leq \frac{1}{1-\mu} + \mu^i C_3 + (-\mu)^i C_4 = N_2$$

where

$$C_3 = \frac{1}{2\mu}\left(\mu\zeta_0 + \zeta_1 - \frac{1+\mu}{1-\mu}\right),$$
$$C_4 = \frac{1}{2\mu}\left(\mu\zeta_0 - \zeta_1 + 1\right).$$

□

5. Convergence Results of Solutions of System (2.1)

Theorem 5.1. *If $\zeta_i \geq \bar{\zeta}$ and $\eta_i \geq \bar{\eta}$ (resp., $\zeta_i \geq \bar{\zeta}$ and $\eta_i \geq \bar{\eta}$) for $i \geq s$ and $s \in \{-1,0,\dots\}$ then the solution $\{(\zeta_i,\eta_i)\}$ of system (2.1) tends to equilibrium point $\left\{\left(\bar{\zeta},\bar{\eta}\right)\right\}$ as $i \to \infty$.*

Proof. Let $\{(\zeta_i,\eta_i)\}$ be a positive solution of system (2.1) such that

$$\zeta_i \geq \bar{\zeta},\ \eta_i \geq \bar{\eta},\ i \geq s, \tag{5.1}$$

where $s \in \{-1,0,\dots\}$. Hence, we obtain from (5.1), system (2.1) and Lemma 4.1:

$$\zeta_{i+1} \leq 1+\mu+\mu^2\zeta_{i-1}. \tag{5.2}$$

$$u_{i+1} = 1+\mu+\mu^2 u_{i-1}, \tag{5.3}$$

$$u_s = \zeta_s, u_{s+1} = \zeta_{s+1}, s \in \{-1,0,\dots\},\ i \geq s. \tag{5.4}$$

Therefore, we get from the solution of the difference equation (5.3):

$$u_i = \frac{1}{1-\mu} + \mu^i C_1 + (-\mu)^i C_2 \tag{5.5}$$

where C_1, C_2 depent on ζ_s ,ζ_{s+1}. Moreover, we have from (5.2) and (5.3):

$$\zeta_{i+1} - u_{s+1} \leq \mu^2 \left(\zeta_{i-1} - u_{s-1}\right), \ i > s \tag{5.6}$$

Thus we obtain from (5.4), (5.6) and by induction

$$\zeta_i \leq u_i, \ i \geq s. \tag{5.7}$$

From (5.1), (5.5) and (5.7), we obtain that

$$\lim_{i\to\infty} \zeta_i = \bar{\zeta}.$$

Then we similarly obtain that $\lim\limits_{i\to\infty} \eta_i = \bar{\eta}$. The proof of the other case of this theorem is similar to this case, so we leave it to readers. $\square$

Theorem 5.2. *Suppose that* $0 < \mu < \frac{1}{2}$. *Then the positive equilibrium point of system* (2.1) *is globally asymptotically stable.*

Proof. We have from Theorem 4.2,

$$\begin{aligned}
&1 < m_1 = \liminf_{i\to\infty} \zeta_i \leq N_1,\\
&1 < m_2 = \liminf_{i\to\infty} \eta_i \leq N_2,\\
&1 < U_1 = \limsup_{i\to\infty} \zeta_i \leq N_1,\\
&1 < U_2 = \limsup_{i\to\infty} \eta_i \leq N_2.
\end{aligned}$$

By system (2.1), we can write

$$\begin{aligned}
&U_1 \leq 1 + \mu \frac{U_2}{m_2^2}, m_1 \geq 1 + \mu \frac{m_2}{U_2^2},\\
&U_2 \leq 1 + \mu \frac{U_1}{m_1^2}, m_2 \geq 1 + \mu \frac{m_1}{U_1^2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&U_1 + \mu \frac{m_1}{U_1} \leq U_1 m_2 \leq m_2 + \mu \frac{U_2}{m_2},\\
&U_2 + \mu \frac{m_2}{U_2} \leq U_2 m_1 \leq m_1 + \mu \frac{U_1}{m_1}.
\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}
&U_1 + \mu \frac{m_1}{U_1} + U_2 + \mu \frac{m_2}{U_2} \leq m_2 + \mu \frac{U_2}{m_2} + m_1 + \mu \frac{U_1}{m_1},\\
&U_1 + \mu \frac{m_1}{U_1} + U_2 + \mu \frac{m_2}{U_2} - m_2 - \mu \frac{U_2}{m_2} - m_1 - \mu \frac{U_1}{m_1} \leq 0,\\
&(U_1 - m_1)\left(1 - \mu\left(\frac{1}{m_1} + \frac{1}{U_1}\right)\right) + (U_2 - m_2)\left(1 - \mu\left(\frac{1}{m_2} + \frac{1}{U_2}\right)\right) \leq 0.
\end{aligned}$$

In this here if $\mu \in (0, \frac{1}{2})$ than

$$\begin{aligned}
&1 - \mu\left(\frac{1}{m_1} + \frac{1}{U_1}\right) > 0,\\
&1 - \mu\left(\frac{1}{m_2} + \frac{1}{U_2}\right) > 0.
\end{aligned}$$

Thus, we get that

$$U_1 - m_1 = 0, \quad U_2 - m_2 = 0.$$

So, $U_1 = m_1$ and $U_2 = m_2$. The proof is completed as desired. $\square$

6. Rate of Convergence of System (2.1)

Now we study the rate of convergence of system (2.1). Hence, we consider the following system:

$$E_{i+1} = (\alpha + \beta(i))E_i, \tag{6.1}$$

where E_i is a k-dimensional vector, $\alpha \in C^{k\times k}$ is a constant matrix, and $\beta : \mathbf{Z}^+ \to C^{k\times k}$ is a matrix function satisfying

$$\|\beta(i)\| \to 0, \tag{6.2}$$

as $i \to \infty$, where $\|\cdot\|$ denotes any matrix norm that is associated with the vector norm

$$\|(x,y)\| = \sqrt{x^2+y^2}.$$

Theorem 6.1. *(Perronas Theorem, [24]) Assume that condition (6.2) holds. If E_i is a solution of (6.1), then either $E_i = 0$ for all as $i \to \infty$,or*

$$\lim_{i\to\infty} \sqrt[i]{\|E_i\|},$$

or

$$\lim_{i\to\infty} \frac{\|E_{i+1}\|}{\|E_i\|},$$

exists and is equal to modulus of one of the eigenvalues of matrix α.

Theorem 6.2. *Suppose that $0 < \mu < \frac{1}{2}$ and $\{(\zeta_i, \eta_i)\}$ be a solution of the system (2.1) such that $\lim_{i\to\infty} \zeta_i = \bar{\zeta}$ and $\lim_{i\to\infty} \eta_i = \bar{\eta}$. Then the error vector*

$$E_i = \begin{pmatrix} e_i^1 \\ e_{i-1}^1 \\ e_i^2 \\ e_{i-1}^2 \end{pmatrix} = \begin{pmatrix} \zeta_i - \bar{\zeta} \\ \zeta_{i-1} - \bar{\zeta} \\ \eta_i - \bar{\eta} \\ \eta_{i-1} - \bar{\eta} \end{pmatrix}$$

of every solution of system (2.1) satisfies both of the following asymptotic relations:

$$\lim_{i\to\infty} \sqrt[i]{\|E_i\|} = \left|\lambda_{1,2,3,4}\ F_J(\bar{\zeta},\bar{\eta})\right|,$$
$$\lim_{i\to\infty} \frac{\|E_{i+1}\|}{\|E_i\|} = \left|\lambda_{1,2,3,4}\ F_J(\bar{\zeta},\bar{\eta})\right|.$$

where $\lambda_{1,2,3,4}\ F_J(\bar{\zeta},\bar{\eta})$ are the characteristic roots of the Jacobian matrix $F_J(\bar{\zeta},\bar{\eta})$.

Proof. To find the error terms, we set

$$\zeta_{i+1} - \bar{\zeta} = \sum_{n=0}^{1} A_n \left(t_{i-n} - \bar{\zeta}\right) + \sum_{n=0}^{1} B_n \left(z_{i-n} - \bar{\eta}\right),$$
$$\eta_{i+1} - \bar{\eta} = \sum_{n=0}^{1} D_n \left(\zeta_{i-n} - \bar{\zeta}\right) + \sum_{n=0}^{1} G_n \left(\eta_{i-n} - \bar{\eta}\right),$$

and $e_i^1 = \zeta_i - \bar{\zeta}, e_i^2 = \eta_i - \bar{\eta}$. Thus we have

$$e_{i+1}^1 = \sum_{n=0}^{1} A_n e_{i-n}^1 + \sum_{n=0}^{1} B_n e_{i-n}^2,$$

$$e_{i+1}^1 = \sum_{n=0}^{1} D_n e_{i-n}^1 + \sum_{n=0}^{1} G_n e_{i-n}^2,$$

where

$$
\begin{aligned}
&A_0 = A_1 = 0,\\
&B_0 = \frac{\mu}{\eta_i^2}, B_1 = \frac{-\mu\left(\bar{\eta}+\eta_i\right)}{\bar{\eta}\eta_i^2},\\
&D_0 = \frac{\mu}{\zeta_i^2}, D_1 = \frac{-\mu\left(\bar{\zeta}+\zeta_i\right)}{\bar{\zeta}\zeta_i^2},\\
&G_0 = G_1 = 0.
\end{aligned}
$$

Now we take the limits

$$
\begin{aligned}
&\lim_{i\to\infty} A_0 = \lim_{i\to\infty} A_1 = 0,\\
&\lim_{i\to\infty} B_0 = \frac{\mu}{\bar{\eta}^2}, \quad \lim_{i\to\infty} B_1 = \frac{-2\mu}{\bar{\eta}^2},\\
&\lim_{i\to\infty} D_0 = \frac{\mu}{\bar{\zeta}^2}, \quad \lim_{i\to\infty} D_1 = \frac{-2\mu}{\bar{\zeta}^2},\\
&\lim_{i\to\infty} G_0 = \lim_{i\to\infty} G_1 = 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&B_0 = \frac{\mu}{\bar{\eta}^2} + b_i, \quad B_1 = \frac{-2\mu}{\bar{\eta}^2} + r_i,\\
&D_0 = \frac{\mu}{\bar{\zeta}^2} + d_i, \quad D_1 = \frac{-2\mu}{\bar{\zeta}^2} + t_i,
\end{aligned}
$$

where $b_i \to 0,\ r_i \to 0,\ d_i \to 0,\ t_i \to 0$ as $i \to \infty$. Therefore, we obtain the system of the form (6.1)

$$E_{i+1} = (\alpha + \beta(i))E_i$$

where

$$
\begin{aligned}
&\alpha = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{\mu}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\\
&\beta(i) = \begin{pmatrix} 0 & 0 & b_i & r_i \\ 1 & 0 & 0 & 0 \\ d_i & t_i & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\end{aligned}
\tag{6.3}
$$

and $\|\beta(i)\| \to 0$ as $i \to \infty$. So, the limiting system of error terms about the equilibrium point can be written as follows:

$$
\begin{pmatrix} e_i^1 \\ e_i^1 \\ e_{i+1}^2 \\ e_i^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{\mu}{\bar{\eta}^2} & \frac{-2\mu}{\bar{\eta}^2} \\ 1 & 0 & 0 & 0 \\ \frac{p}{\bar{\zeta}^2} & \frac{-2\mu}{\bar{\zeta}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_i^1 \\ e_{i-1}^1 \\ e_i^2 \\ e_{i-1}^2 \end{pmatrix}
$$

which is same as linearized system of system (2.1) about equilibrium point$(\bar{\zeta}, \bar{\eta})$. $\square$

7. Numerical Examples

In this section, we give two examples which include three figures to verify our theoretical results.

Example 7.1. *We consider system* (2.1) *for* $\mu = 0.43$. *With the initial values* $\zeta_{-1} = 1$, $\zeta_0 = 1.2$, $\eta_{-1} = 3$ *and* $\eta_0 = 0.95$ *positive equilibrium point of system* (2.1) *is globally asymptotically stable. Figures 7.1, 7.2 verify our theoretical results.*

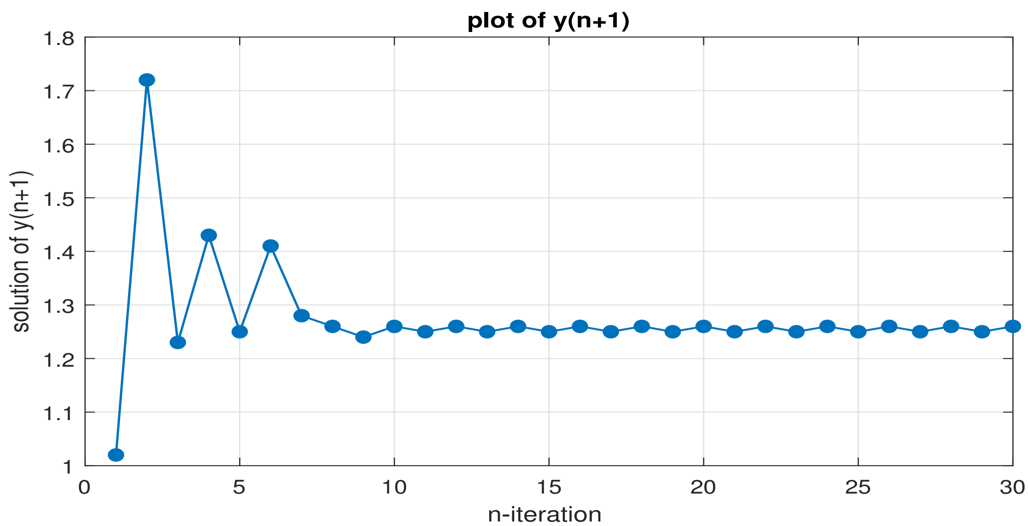

Figure 7.1

Example 7.2. *We consider system* (2.1) *for* $\mu = 2.2$. *With the initial values* $\zeta_{-1} = 2.08$, $\zeta_0 = 2.02$, $\eta_{-1} = 2.03$ *and* $\eta_0 = 2.08$, *solutions of system* (2.1) *oscillate about positive equilibrium point* $(\bar{\zeta},\ \bar{\eta} = (0.0652, 0.0652)$. *Figure 7.3 verifies our theoretical results.*

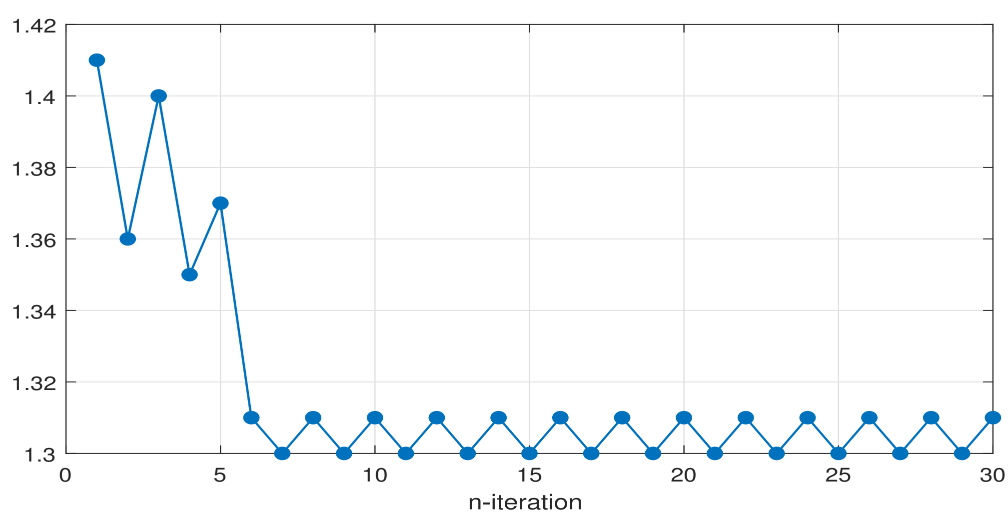

Figure 7.2

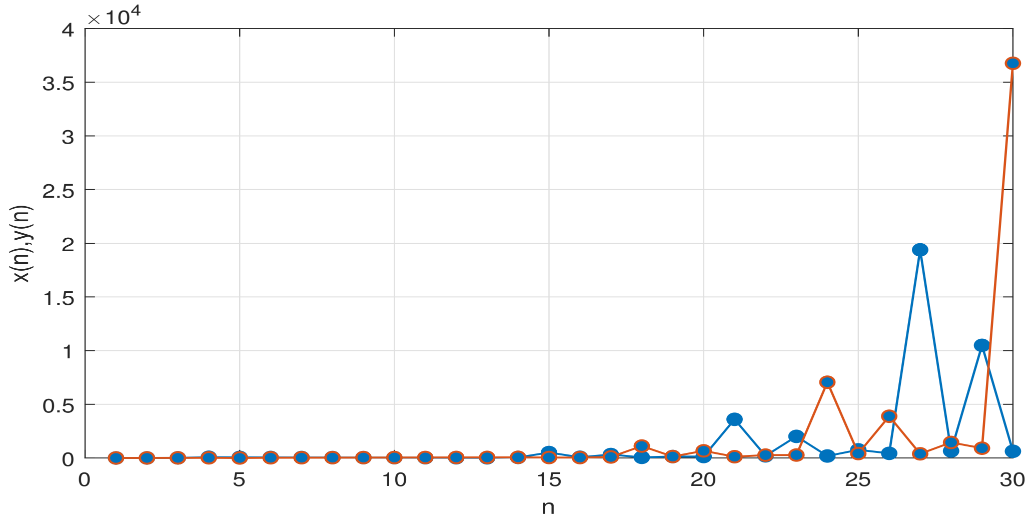

Figure 7.3

8. Conclusions

In this paper we studied convergence results of a system of second order difference equations . Firstly we deal with the unique positive equilibrium point of system(2.1). Then we analyse the bounded solutions of system (2.1). We also investigate the oscillation of solutions of system. More specifically, we focus on the convergence results of solutions of system. According to our results, if $0 < \mu < \frac{1}{2}$ then the positive equilibrium point of system (2.1) is globally asymptotically stable. After this we concentrates on discussing the rate of convergence of solutions of system(2.1). Moreover to this we give two numerical examples to verify our theoretical results.

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