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Rituals and explorations in students' mathematical discourses: The case of polynomial inequalities

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The study is a qualitative case study that seeks to determine whether students' mathematical discourses in solving polynomial inequalities are more ritualistic or explorative. A comprehensive analysis of students' routines was conducted through the observations of what they said and did (write, draw, and so on) around task situations in a small group. This study's participants were five 11th-grade students from a public high school. These participants were chosen using the maximum diversity method of sampling. The data for this study were obtained through small-group work. The small-group interactions lasted 80 minutes and were video-recorded with two cameras. The commognitive approach was used to analyze the student routines in this study. The criteria for analyzing routines were the performers' agentivity /external authority, focus on the goal or the procedure, and flexibility. The findings of this study revealed that the students' routines were neither purely ritualistic nor sheer explorative. Even those whose routines were ritualistic in all task situations thought about the procedure and asked logical questions about the task. In addition, the findings indicate that teachers can play an important role in encouraging students to engage in more exploratory mathematical discourse. This study contributes to the future research on students' discourse in the context of inequality.

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Introduction

While learning is usually viewed as a change process, theories attempting to explain the nature of learning differ in their answers to the issue of what changes when learning occurs. Learning, according to behaviorists, is a change in the learner's behavior. It is characterized in cognitive theories as a mental shift that occurs as a result of learning, receiving, or producing mental entities such as concepts, knowledge, or mental schemes. One prominent drawback of such "acquisitionist" methods is their failure to comprehend how historical and societal change in human behavior patterns has occurred (Sfard, 2020). The acquisitionist position was challenged in the second half of the twentieth century by the idea that individuals participate in well-defined and historically evolved kinds of activity in cognitive processes (Vygotsky, 1987; Cole, 1996). As a next step, other fields, including mathematics, embraced this "participationist" approach to learning (Sfard, 2020). The commognitive approach (Sfard, 2008) can explain how human activities have changed over time by concentrating on individual and communal discursive processes. This redefines the relationship between thinking and communication by allowing thoughts and ideas to live in a social environment (as communication) rather than being isolated (as something in one's head) (Wood, 2016). Like other socio-cultural methods, the commognitive method sees mathematics learning as participation in a certain community's discourse. According to the discursive definition, learning mathematics becomes equivalent to being able to participate in historically established discourses on quantities and space (Sfard, 2017). Thus, mathematical thinking is characterized as participating in a historically evolved discourse called mathematical discourse (Sfard, 2020).

Communication and language have been the focus of contemporary research in mathematics education. Therefore, the analysis of learning has grown more discursive (Nardi, 2005). These studies, which emphasize the relevance of the context in which learning occurs, are largely based on Sfard's (2008) commognitive approach (Emre-Akdoan, Güçler, & Argün, 2018; Heyd-Metzuyanim & Graven, 2019; Nachlieli & Katz, 2017; Nachlieli & Tabach, 2019; Tabach & Nachlieli, 2016; Viirman & Nardi, 2019; Sfard, 2017). Classroom communication, according to these studies' findings, is equal to thinking. Commognitive theory assumes that learning is not primarily a process by which an individual changes certain cognitive structures in his/her mind, but rather a process of change in routines of participation in a certain community (Heyd-Metzuyanim, Smith, Bill & Resnick, 2019). As a result, discourse analysis in classroom learning contexts includes hints about how learning occurs. The literature on mathematics education studies exploring the mathematical discourses of students and instructors has grown in recent years (Baccaglini-Frank, 2021; Heyd-Metzuyanim & Graven, 2019; Heyd-Metzuyanim & Shabtay, 2019; Heyd-Metzuyanim et. al., 2019; Heyd-Metzuyanim, Tabach & Nachlieli, 2016; Nachlieli & Katz, 2017; Nisa, Lukito & Masriyah, 2021; Roberts & le Roux, 2019; Sfard, 2017). While most of these studies (Heyd-Metzuyanim & Graven, 2019; Heyd-Metzuyanim & Shabtay, 2019; Heyd-Metzuyanim et. al., 2019; Heyd-Metzuyanim et. al., 2016; Nachlieli & Katz, 2017; Sfard, 2017) investigate the mathematical discourses of teachers and teacher candidates, a limited number of them (Baccaglini-Frank, 2021; Nisa et. al., 2021; Roberts & le Roux, 2019) focus on students' mathematical discourse. According to Sfard (2008), learners learn by imitating. Some studies focusing on teacher discourse also draw attention to the effect of teacher discourse on students' discourse (Heyd-Metzuyanim & Graven, 2016; Tabach & Nachlieli, 2012; Sfard, 2017). Researchers have also analyzed the discourses of pre-service teachers in learning environments that offer explorative learning opportunities (Heyd-Metzuyanim & Graven, 2019; Heyd-Metzuyanim et al., 2019; Heyd-Metzuyanim et al., 2016; Nachlieli & Katz, 2017). In addition to analyzing pre-service teachers' discourse, these studies aimed to identify learning opportunities that foster exploratory engagement. According to Naclieli and Katz (2017), prospective teachers should both participate exploratively in the discourse and master the characteristics of explorative participation to make the learning opportunities they provide to their future students more explorative.

In this context, Roberts and le Roux (2019) suggest that the nuances revealed in the analysis of student discourses may be appropriate tools to encourage students from ritual to explorative discourse. Previous studies analyzing students' discourse have examined their participation in various contexts and topics. The notions of function (Baccaglini-Frank, 2021), linear equation (Roberts & le Roux, 2019), and absolute value of a real number were used to investigate students' mathematical discourses (Nisa et. al., 2021). In their interviews, Roberts and le Roux (2019) examined the discourses of fifteen 8th and 9th-grade students who were solving linear equations. According to finding of the study, all students engaged in ritualized rather than explorative discourse. Instead of using relationships between mathematical structures when solving equations, students manipulated symbols instrumentally without knowing the reason for the operation. In a similar research, Nisa et al. (2021) examined the discourses of two high-achieving 10th-grade students learning the absolute value of a real number. According to the research findings that characterized student discourses as ritual or explorative, the students' discourses were at the explorative level. Baccaglini-Frank (2021) studied how digital learning impacted discourse in two low-achieving high school students, unlike Nisa et al. (2021). The study showed that digital learning environments support lower-achieving students' explorative participation in mathematical discourse. This study analyzed student discourse in a classroom setting, in contrast to studies that analyze students' discourse in outside classroom learning environments. In addition, this research examines students' participation in the discourse on inequality and characterizes it as ritual or explorative, similar to other studies. The research is expected to help teachers and prospective teachers through the teaching process of the relevant concept and contribute to the literature on this subject. This study's problem, which investigates 11th-grade students' solutions of polynomial inequalities from a commognitive perspective, is as follows:

- What is the mathematical discourse (ritual or exploration) of high school students working on task situations involving polynomial inequalities?

Commognitive Theory

Commognitive theory (Sfard, 2008, 2020) accepts mathematics as a discourse and defines discourse as a specific type of personal or interpersonal communication. In this context, mathematical thinking means that an individual communicates "mathematically" with others (Sfard, 2017). Sfard (2008) states that cognition and communication are various forms of the same phenomenon. The commognitive framework addresses both the subject matter of mathematical conversation in the classroom and students' involvement in this conversation (Sfard, 2008, 2020). According to Sfard (2008, 2020), verbal or nonverbal discourse is a community-specific communicative activity, and mathematical discourse is distinguished by four elements: keywords, visual mediators, narratives, and routines. Keywords in mathematical discourse primarily express amount and shape (numbers, geometric objects), as well as the relationships among them (equality, inequality, similarity, equivalence, etc.). Visual mediators are visible objects that act to communicate relationships and operations with mathematical objects. Numbers, algebraic and logical representations, graphs, algebraic formulas, geometrical drawings, diagrams, etc. are some of the most typical instances of visual mediators. Written or spoken texts that are "framed as a description of objects, of relations between objects, or of processes with or by objects" are considered narratives (Sfard, 2008, p.

134). Axioms, definitions, claims, and proofs are examples of narrative in mathematical discourse. Routines are repeating communication patterns, including actions on objects. In other words, routines refer to the regularities in the usage of keywords and visual mediators, as well as their use in narratives. Mathematical discourse includes routines such as computation and problem solving, verifying and proving novel narratives (Sfard, 2008, 2020). Discourse-specific routines govern actions with mathematical objects in general and mathematical narratives in particular (Sfard, 2020). For this reason, Lavie, Steiner, and Sfard (2019) suggest that routines can be considered as a unit of analysis in studies based on the discursive approach. Lavie et al. (2019) refined and operationalized the concept of "routine" in their previous work. These researchers describe routine using the notions of task situation and procedure. In other words, the routine produced in a certain task situation is referred to as a "task-procedure" pair (Sfard, 2020). Lavie et al. (2019) describe the concept of "task situation" as the situation in which a person feels the need to act. For example, any mathematical activity the participant is expected to do, such as solving or posing a mathematical problem, or making a mathematical definition or proof, is a task. The task situation can be deliberately created by the task specifier (researcher, expert, teacher, etc.) to elicit a certain type of action. The task is the participant's obligation in any mathematical activity. A procedure is a set of instructions that lay out the steps that participants should follow. Thereby, Lavie et al. (2019) refine previous definitions of routines presented by Sfard (2008), where the routine was defined according to the "how" and "when" of a procedure (Heyd-Metzuyanim et. al., 2019). Lavie et al. (2019) define learning as a process of the routinization of students' actions. Researchers identify two sorts of discursive routines for this purpose: rituals and explorations. According to researchers (Lavie & Sfard, 2019; Lavie et al., 2019; Sfard & Lavie, 2005; Sfard, 2008), ritual is the entrance ticket to a novel discourse and is an indispensable part of any learning process. These ritual routines, which manifest as rigid, imitative acts, procedures, or the use of words, are the initial necessary forms of participation that enable the shift to new discourses via a process of de-ritualization. The main concern of the participant in the ritual is social bonding and acting in harmony with others. For this reason, rituals are often constructed and maintained by imitating what others do. Performing the ritual in a particular task situation, the participant answers the procedure "How do I proceed?" and tries to apply it by looking for an answer to the question. Therefore, rituals are process-oriented routines. However, since rituals are performed by imitating others, there is no place for proof in the procedure process (Lavie et al., 2019; Sfard, 2008). According to Sfard (2017), a participant must imitate others (expert participants, teachers, etc.) to participate in a new discourse. However, imitation in this context does not mean imitation without thought. On the contrary, when attempting to replicate the expert's activity, learners must constantly ask themselves which aspects of the action should be kept and which should be adjusted to meet the demands of a new circumstance. Answering this question necessitates understanding the rationale underlying the expert's discourse. Sfard (2008) calls this type of imitation "thoughtful imitation". The existence of such contemplative imitation allows ritual routines to gradually de-ritualize and evolve into exploration routines. Explorations are routines from which a mathematical narrative is produced. Mathematical discourses such as numerical calculations, solving equations, and defining or proving the results in the production of a narrative are examples of exploration. In the case of exploration, the participant can apply the procedure independently of others in a given task situation. Performing the exploration in a particular task situation, the participant answers the procedure "What do I want to achieve?" and tries to apply it by looking for an answer to the question. That's why explorations are product-oriented routines. During the implementation of the exploration procedure, mathematical proofs are included (Lavie et al., 2019; Sfard, 2008). In this context, learning is the process of transforming the participant's routines into exploration from rituals (Lavie et al., 2019). Pure rituals or sheer explorations are uncommon in mathematics classrooms. According to research (Sfard & Lavie, 2005; Lavie & Sfard, 2019; Lavie et al., 2019), the process of de-ritualization (transforming rituals into explorations) can be gradual and slow, and some routines stay rituals forever (Sfard, 2008). This article explores whether the discourse of high school students is more ritualistic or more explorative.

Method

Research Design

This study used a qualitative case study approach to characterize and analyze high school students' mathematical discourse in classroom contexts. A case study is an investigation that is used to analyze a contemporary phenomenon in depth and in its real-world setting (Yin, 2018). A case study draws on a variety of sources of information (e.g., observations, interviews, audio-visual materials, documents, and reports). The researchers offer a description of the circumstance or situation themes (Creswell & Poth, 2016). The mathematical discourses of students in a classroom environment are detailed in this study.

Participants

The participants in this research were five 11th-grade students studying at a public high school. They were members of a classroom of twenty-three students. These participants were chosen using the maximum diversity method of sampling. For this purpose, participants were selected by taking into account the learning principles of the commognitive perspective (Sfard, 2008; Lavie et. al., 2019). All of the participants have high motivation and positive attitudes toward mathematics. The students were coded as S1, S2, S3, S4, and S5. Among the students, while S1 and S3 have high achievements in mathematics at school, the others have moderate achievements. Student S5 was a student who tended to learn by imitating others. S1 and S3 were students who were in leadership roles and seen as authorities in the classroom by their peers. At the same time, these two students were better at performing the procedure than the others. S2 and S4 were more expressive in class than their friends. The first researcher was also the mathematics teacher in this class. The first researcher was coded with the letter R.

Data Collection and Analysis

Participants took six hours of lessons on polynomial inequalities every week for a month. These lessons were appropriate for the school curriculum (Ministry of National Education [MoNE], 2018). The data of this study were obtained from group work which was organized at the end of one of these lessons. The first researcher carried out these lessons and group work. The group work was conducted with four different groups in a classroom of 23. One of these groups consisted of five participants of this study. While the researcher circulated among the groups and provided assistance as needed, her students worked on their task situations. This group study lasted 80 minutes and the discourses of the group, which included only the participants, were videotaped with two cameras. In addition, a voice recorder and student worksheets were used to collect data. Students' discussions about task situations were examined to determine whether their mathematical discourse was more ritualistic or explorative. In this context, the routines of the participants in the task situations in Table 1 are examined. Task situations included the concept of inequality.

Table 1. Task situations

	Task situation
1.	Find the set of real numbers (if any) satisfying the inequality $y^2 - 2y + 5 < 0$.
2.	Describe the real number a when the solution set of the inequality $z^2 - 8z + a - 4 \leq 0$ has only one element, where a is a generalized real number and $z \in \mathbb{R}$.

According to the detailed analysis, only the descriptions of the students' mathematical expressions of two task situations are included. This study, which focused on students' mathematical discourses from a commognitive approach, analyzed what students said and what they did (what they wrote, drew, etc.). The literature analysis was utilized to create the criterion table and examine the discussions by addressing the criteria in Table 2 (Baccaglini-Frank, 2021; Heyd-Metzuyanim, Cohen, Tabach, 2022; Nachlieli & Katz, 2017; Nachlieli & Tabach, 2022). First, we have included the criteria that we believe can be observed in a task situation. These criteria were reorganized after evaluating the study's raw data. The first researcher coded the data of the study. First, the researcher transcribed student discussions, detailing their spoken and written discourses as well as their actions. Then, the data were reviewed by all authors again based on the criteria presented in Table 2. Through comprehensive data analysis, the researchers distinguished between exploratory and ritual participation in student discourse.

Table 2. Rituals or explorations

Criterion	Routine	
	Ritual	Exploration
Performer's agentivity /External authority	Talking with question marks Verbally or non-verbally seeking approval from others (instructor, friend, etc.) Rarely making independent decisions	Mathematizing with high confidence (no hesitations, question marks, no looking for approval). Tending to propose new actions or outcomes Making independent decisions on the way
Focus on the goal or the procedure	Talking about the actions of the procedure Ending the procedure without relating to the reasonable result Trying to perform a specific task-related procedure	Talking about the result, checking it, or explaining it spontaneously Spontaneously producing articulating mathematical narratives
Flexibility	Showing rigidity as relying on only one procedure Unwilling to use any other procedure	More than one procedure is associated with the main task A non-standard procedure is applied to the task

The performer's agentivity /external authority was coded as explorative if the procedure was fully initiated and enacted by the student, and ritual if some parts of it were mediated by the interviewer or other students. Focus on the goal or the procedure was coded as explorative if students produce narratives, and rituals if students talk about the steps of the procedure. Here, student discourses were coded as a narrative if they aligned with the definition of the concept of inequality (Argün et. al., 2020) and the school curriculum (MoNE, 2018). If the learner did a task in more than one way, flexibility was labeled as explorative, and ritual if the learner performed a rigid procedure.

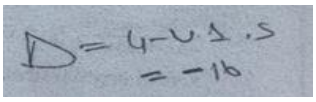
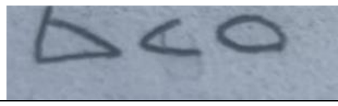
Validity and Reliability

The participants were selected from the school where the first researcher worked to ensure proximity to the research area. Before the group study, the participants were observed directly and in the natural classroom environment where the activity occurred 6 hours a week for a month. Thus, a long-term face-to-face interaction was achieved with the participants. Therefore, the researchers were able to validate the results and gather further details. In addition, two experts in the field examined the task situations. The research data includes various sources such as video and audio recordings, students' worksheets, and unstructured in-class observation forms. Participants were selected to suit the purpose of the research. The role of the researcher, information about the participants, what the study environment was like, and how the participants were selected are explained. The research's data analysis is explained in detail. The findings are presented with direct quotations. It was found that the research data and the results were compatible. All raw data of the research, including student worksheets, and video, and audio recordings, were stored to be examined later.

Findings

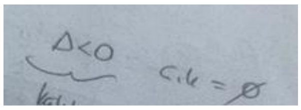
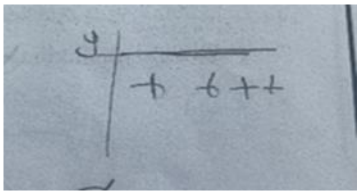
As it is mentioned above, this study examines the discourses of five students during group work. There were four groups in the classroom. Five participants of this study were in the same group. Students discussed task situations related to polynomial inequalities. First, an analysis of the students' discourses on task situation 1 will be presented. The task was given to the students by the instructor. The instructor directed all students to work in groups after providing them with the task worksheets. She provided the students with sufficient time to work. The study's participants read the related task. Following that, as shown in Table 3, the following discussion took place between S2 and S3. The word "turn" is used as a table number and serves as a label for the students' discourse.

Table 3. Students' discourses on task situation 1 (for S2 and S3)

Turn	Speaker	Talk [activity]	Figure
01	S3	Can't it be factorized? Let's look at the delta.[She calculates to find the discriminant on the worksheet.]	
02	S2	I hope the delta is negative. [She waits without doing anything. She looks at her friends.]	
03	S3	[She calculates the value of the discriminant of the equation and shares the result with her friends.] … 'Its delta is minus sixteen'.	
04	S2	Yea! [She is happy that the result is negative and writes Δ <0 on the worksheet after S3.]	

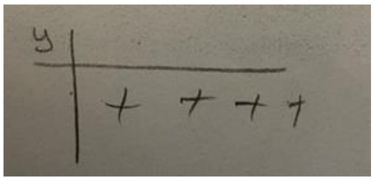
In this excerpt, S3 wondered if the expression $y^2 - 2y + 5$ can be factored, and she said to her friends "… Let's look at the delta." [01] Here, she was aware of how to proceed and made independent proposals about how to proceed. Her routine was therefore more explorative than ritual. "Can't it be factorized?" she asked, announcing that the procedure she chose was to factor the expressions. She concentrated on the algebraic solution of the task and calculated the discriminant of the equation [see the figure in turn 01]. This was sufficient evidence that she saw the procedure's performance as its task. That's why her routine is more ritualistic here. Meanwhile, others started to write something on their worksheets. However, S2 waited for one of her friends to calculate the discriminant, and she said "I hope the delta is negative." [02] After a curious wait, she heard S3 answer "…minus sixteen." [03] She was glad and said "Yea!" [04]. Then she wrote Δ <0 on the worksheet [see the figure in turn 04]. She relied on S3's conclusion without making a solution. This showed that she saw S3 as an authority. Thus, her routine was ritual here. S3's answer "Its delta is minus sixteen." [03] is followed by S1 interrupting "then the solution set is empty, oh well, that's it." [05] as stated in Table 4.

Table 4. Students' discourses on task situation 1 (for S1, S2 and S4)

Turn	Speaker	Talk [activity]	Figure
05	S1	Then the solution set is empty, oh well, that's it. [She is looking at her friends and smiling. She is writing on the worksheet "Δ<0, $Ç.K = \emptyset$"]	
06	S4	But no! We cannot directly see the solution set empty (set), no! [She looks at S1 and warns her friend.]	
07	S2	Ah… yes. [After S4's warning, she approves of her friend and explains by looking at S1.] First, we need to create a table, because we found the discriminant less than zero and the sign in the table must be plus. [She draws a table and points to her drawing.]	
08	S4	Yes, we should create a table.	

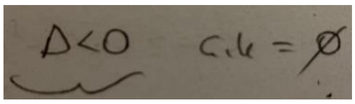
Here, she inferred that "[…] the solution set is the empty set. […]" [05]. S4 reacted quickly to S1 "[…] We cannot directly see the solution set empty (set), no!" [06]. S2 grasped this and asserted her own sense of the "[…] we need to create a table, because we found the discriminant less than zero and the sign in the table must be plus." [07]. Then S4 confirmed what she said by saying "Yes, we should create a table." [08]. In this excerpt, S1 ended the task without relating to the reasonable result. She also didn't calculate the delta, relying on S3 saying that "[…] delta is minus sixteen." [03]. Her routine for solving this task was ritualistic. S4, on the other hand, realized S1's mistake and proposed a new action ([06], [08]). Her routine here served as an exploration in making sense of the task situation. S2's self-confident demeanor was also striking here. Without any hesitation, she explained to S1 how the table will look if Δ<0. Thus, her routine was exploration. In this discussion, S1 seemed to be confused by S4's warning, and she didn't connect their utterances [06], [07], [08] logically. Therefore, S1 was not satisfied by their utterance. S3 provided S1 with yet another word to help herself "The signs in the table will be plus, plus. We call the solution set the empty set because a solution that satisfies the condition of the expression being less than zero cannot be obtained." [09]. Then the following discussion in Table 5 took place between S3 and S1.

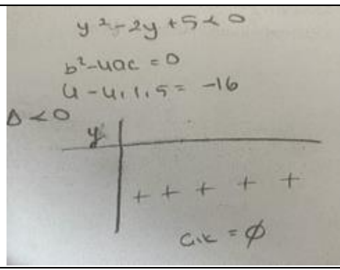
Table 5. Students' discourses on task situation 1 (for S1 and S3)

Turn	Speaker	Talk [activity]	Figure
09	S3	The signs in the table will be plus, plus. We call the solution set the empty set because a solution that satisfies the condition of the expression being less than zero cannot be obtained. [She points to the table on S1's worksheet.]	
10	S1	What? [Again, she hesitates.]	
11	S3	Look now, what's the coefficient of y^2? It is positive. Then won't all the signs in the table be positive? [She tries to explain the solution to S1 over the drawing on S1's worksheet.]	
12	S1	Yes, because we said no solution.	

Here, S3 asked her friend questions like "... what's the coefficient of y^2?" and "…won't all the signs in the table be positive? [11]. Her phrases "...because…" [09], "Look now," and "Then..." demonstrated that her routine was exploration. S1 grasped her explanations and asserted her own sense of the "…because we said no solution." [12]. S1's routine was ritual because she didn't make any new proposals. On the other hand, although S3 made independent decisions, she talked about the procedure in her explanations: "The signs in the table will be plus, plus … because a solution that satisfies the condition of the expression being less than zero cannot be obtained." [09]; "…all the signs in the table be positive?" [11]. Her focus on procedure rather than goal showed that her routine was ritual. At this point, S2 starts talking about "the coefficient of , y^2", and then the following discussion occurs as in Table 6.

Table 6. Students' discourses on task situation 1 (for all students)

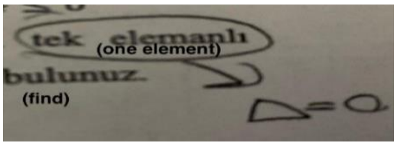
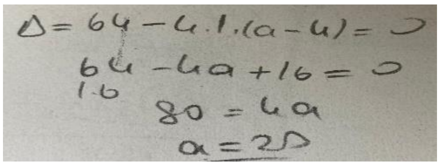
Turn	Speaker	Talk [activity]	Figure
13	S2	If the sign of y was negative, we would say the solution set is real numbers. [She makes a statement to S1.]	
14	S1	If the discriminant was positive… [She is looking at S3].	
15	S2	No, it has nothing to do with discriminant, if the sign of y was negative, we would say the solution set is real numbers.	
16	S3	If it was negative, you've already asked about the smaller one, then we would say the solution set is real numbers. [She shows the smaller symbol on the S1 paper and explains.]	
17	S1	Ok, so it has nothing to do with discriminant, the only reason we check at the discriminant here is to find the solutions of the equation. [She points on the worksheet Δ<0, $Ç.K = \emptyset$.]	
18	S3	Yes so, are there solutions or not? [She confirms S1.]	
19	S4	Yes so, we have no reason to change the sign. [She makes a statement on S1's worksheet.]	
20	S1	Ok.	

21	S5	Right now, the solution set is the empty set, if the discriminant was positive then we would say solutions are real numbers, right? [She listens to her friends' all discussions. She asks for approval from S4. S4 nods in agreement.]	

Here, S2 proposed a new outcome to S1: "… if the sign of y [the coefficient of y^2] was negative, we would say the solution set is real numbers." [13]. Her proposal was a narrative. S1 focused on the goal rather than the procedure in making sense of the task situation. Thus, her routines were more explorative rather than ritual. Meanwhile, S1 did not listen to S2's narrative, but to S3's discourses, who talked about the procedure, she said: "If it was negative, you've already asked about the smaller one, then we would say the solution set is real numbers." ([14],[16]). S1 couldn't make independent decisions in this task situation and saw S3 as an authority. S1 corrected her mistake at this point and asserted "…the only reason we check at the discriminant here is to find the solutions of the equation. (She means the equation of $y^2 - 2y + 5 = 0$)." [17]. S1 completed the task after her friends approved of her ([18], [19], [20]). In contrast to her relatively limited contributions to the discussion, S4's proposals were a shred of evidence that her routines were explorative ([6], [8], [19]). On the other hand, S5 was one of the students in this group who appeared to be the slowest to grasp mathematical concepts. She listened to the all discussions of her friends in silence. She was looking at S4 and S3's worksheets from time to time, and according to her friends, she wrote and deleted something on the worksheet. At the end of the discussion, she looked at S4 as if asking for approval and whispered to her friend: "…the solution set is the empty set, if the discriminant was positive then we would say solutions are real numbers, right?" [21]. After her friend nodded, she wrote something on her worksheet [see the figure in turn 21]. In this task situation, S4 couldn't decide how to proceed independently. S4 saw her friends as authorities. S4 made no new proposals or produces a narrative. There were question marks in her gaze, demeanor, and speech: "...right?" [21]. To conclude, she didn't act as a problem solver and didn't engage in discussions to make sense of the task situation. For this reason, all her routines were ritualistic.

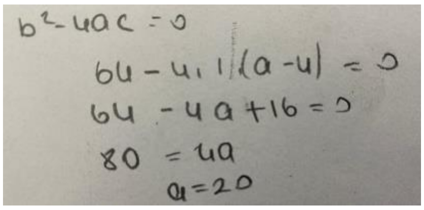
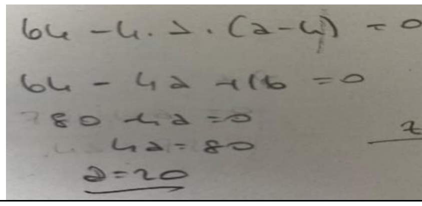
Right after the students completed task situation 1, they immediately read task situation 2. S5 found this task situation challenging, and then the following discussion took place as in Table 7.

Table 7. Students' discourses on task situation 2 (for all students)

Turn	Speaker	Talk [activity]	Figure
22	S5	It looks hard. [She reads the question and looks at her friends. She performs operations by equating delta to zero.]	
23	S2	If it says the solution set has only one element, the discriminant is zero. [She circles "one element" and writes delta equals zero that is "Δ=0".]	
24	S4	Then the discriminant equals zero, right? [She looks at S1 and asks for approval. And then S4 performs operations by equating delta to zero.]	
25	S3	Yes. [She nods to S4.]	
26	S1	80 equals $4a$, so a equals 20, right? [S1 performs her operations audibly.]	
27	S4	Just a sec!	
28	S1	Does it say to find the real number of a, is that all?	
29	S2	Exactly. [She states that she agrees with S1 that they have reached the solution.]	
30	S3	Yes, let's continue.	
31	S4 and S5	How can you do it so quickly, what's your hurry? [They get angry with their friends.]	
32	S1	We just do the operation.	
33	S5	Ok, but if we do it wrong, we have to solve it again.	
34	S1 and S3	Then we are waiting.	

In this excerpt, S1, S2, and S3 quickly grasped the task, and they found the value of a. S4 and S5, on the other hand, had difficulty grasping the task and got angry with their friends: "How can you do it so quickly, what's your hurry?" [31]. S2 acted as a problem solver and produced a mathematical narrative: "If […] the solution set has only one element, the discriminant is zero. [23] She didn't turn to talk about the procedure for a while. Rather she concentrated on the tasks' goal. That's why her routine was exploration. Unlike S2, S1 and S3 focused on the procedure ([28], [30], [32]) and used the discriminant to determine the number a[see the figure turn in 26]. The students' solution here was accurate, but they didn't monitor the appropriate procedure according to the task situation because they concentrated on the formula $\Delta = 0$ and accepted it as a task without thinking about the reason ([28], [30], [32]). Although they made independent decisions, their routine was here ritualistic in making sense of the task. S4 had failed to calculate the number a correctly and told her friends to wait: "Just a sec!" [27]. And S5 confirmed her "… if we do it wrong, we have to solve it again." [33]. While their friends are waiting for them to reach the correct solution, the following interaction in Table 8 between S4 and S5 occurs.

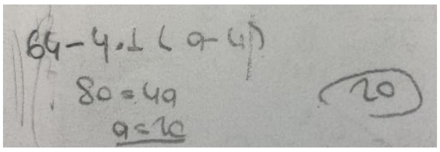
Table 8. Students' discourses on task situation 2 (for S1, S3, S4 and S5)

Turn	Speaker	Talk [activity]	Figure
35	S4	72 divided by 4.	
36	S5	Wait a minute! I did something wrong again. [She calculates the real number a correctly, but she is affected by the fact that S4's answer is wrong. Thus, she is not sure of her solution. She takes S4's worksheet and compares the solutions.]	
37	S4	[She asks S1.] Is the result 72 divided by 4? [She looks at the S5's worksheet, and realizes her mistake.]	
38	S3	Yes. [She nods to S4.]	
39	S1	Ok, that's it, let's move on.	
40	S5	Okay, go ahead, it drove me crazy. [She looks confused. She isn't sure of her solution.]	
41	S4	One second! The answer is 72 over 4, isn't it? [She looks at S3 and S1 to confirm the result.]	
42	S3	No, 80 divided by 4.	
43	S5	So, 80 is equal $4a$, and a is equal to 20. [After S3's approval, she confirms the correctness of her own solution.]	
44	S4	I made a mistake somewhere, I found 4 times 16. [She corrects the error by looking at the S5's solution.]	

Now, the most crucial moment in this interaction was that S3 was telling the correct solution. At this point, S5 actually already got the right solution, but she wasn't sure about it herself because she considered S4 as an authority and was affected by S4's wrong solution until S3 told "…80 divided by 4." [42]. Here, she was encouraged by S5's response and asserted her own sense of "So, 80 is equal $4a$, and a is equal to 20." [43]. In all these discussions, because she was often trying to get approval from his friends and considered their guidance ([36], [39], [40]), she had failed to make independent decisions on how to proceed. Here, S4 also needed guidance from her friends and asked "Is the result 72 divided by 4? [37] and "[…] The answer is 72 over 4, isn't it? [41]. Thus, all interactions between the students (S4 and S5) were ritualistic.

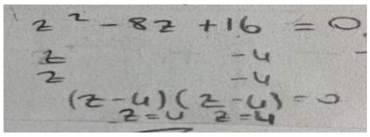
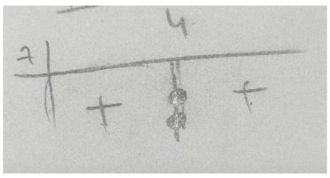
When the students began to solve the next task situation, the teacher (researcher) arrived and interrupted their discussion, pointing out the task situation 2: "How did you solve it and interpret it?" [45]. Then the following discussion takes place as stated in Table 9.

Table 9. Students' discourses on task situation 2 and their discussions with the researcher

Turn	Speaker	Talk [activity]	Figure
45	R	How did you solve it, how did you interpret it? [The researcher comes near the group and examines the solutions.]	
46	S2	Since it says the solution set has only one element, the discriminant is equal to zero, we looked at the discriminant, and when we do it this way, 80 is equal to $4a$, and a came from the division of 80 by 4, here. [She explains the steps in the procedure with a paper-pencil.]	
47	R	There is inequality there, right? [She indicates the task situation.]	
48	S2 and S3	Yes.	
49	R	In the sign table, how did you relate the discriminant being zero to the solution set having only one element?	
50	S2	When finding the solution set of the inequality, we would look at it as an equation. Not to the sign table. We would look at the inequality as an equation to find solutions, and then move on to the sign table.	
51	S4	Yes, since the discriminant is zero, the solution set of the equation has only one solution.	
52	S3	The fact that the discriminant is zero means that the graph of the equation is tangent to the x-axis; thus, the solution set has only one solution, or the equation has two equal solutions.	
53	S1 and S5	Yes. [They confirm what S3 said.]	

S2 first chose to concentrate on talking about the procedure for the researcher's question. [46] The researcher was not satisfied with this answer. Therefore, the researcher provided S2 other questions to help herself with: "There is inequality there, right? […], how did you relate the discriminant being zero to the solution set having only one element? ([47], [49]). This time S2 talked about her experiences in the lesson and continued to talk about the procedure. [50] S2 was likely thinking of the questions as "how is the solution process?". The researcher encouraged students again to provide a more explorative answer. At this point, S4 grasped this and asserted her own sense of the task situation: "Yes, since the discriminant is zero, the solution set of the equation has only one solution." [51]. She produced a mathematical narrative, so makes a new proposal for her friend. Right after that, S5 constructed her routine on narrative: "[…] the graph of the equation is tangent to the x-axis; thus, the solution set has only one solution, or the equation has two equal solutions." [52] S3, who chose to talk about the procedure in her previous speeches, produced a mathematical narrative this time. At the same time, this routine served as an example of explorative flexibility. Here S3 solved the graph of the algebraic expression, and she proposed applying to the task a non-standard procedure. Here, students found the value of the number a when the discriminant is zero. However, none of them thought to determine the real numbers that confirm the inequality. The first researcher wanted students to make them think about the other possible answers: "How did you associate it with the sign table, is there a solution in that case? [54] Then the following discussion takes place as stated in Table 10.

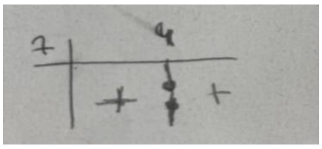
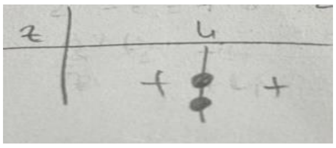
Table 10. Students' discourses on task situation 2 and their discussions with the researcher

Turn	Speaker	Talk [activity]	Figure
54	R	How did you associate it with the sign table, is there a solution in that case?	
55	S1	In this case, there is actually only one solution.	
56	R	Can you draw the sign table? [All of the students work on their worksheets.]	
57	S5	The sign doesn't change anyway because the graph of the equation is tangent to the x −axis, right? [She is looking at her friends.]	
58	S4	Ah, we'll do it, wait for a second! [She starts to write something on her worksheet.]	
59	R	So, what real number equals the element of the solution set here? [All of the students work on their worksheets.]	
60	S5	Let's substitute a and find.	
61	S1	Four, teacher.	
62	S2	If a graph of an equation is tangent to the x-axis, the equation has two equal solutions. Thus, here 4 is two equal solutions.	
63	S4	Ok, let's give it a try.	
64	S1 and S3	Let's try.	
65	S1	We said for z is 4, and then we wrote in the table plus and minus. [She's trying to draw the sign table.]	
66	S2	You can't say plus or minus! 4 is two equal solutions of the equation. [She warns by pointing out her friend's mistake with her pen.]	
67	S1	Ok, ok. [She shakes her head]	

In this excerpt, students wrote the real number a, in the equation of $z^2 - 8z + a - 4 = 0$ and found the solutions of the equation. S5 grasped how she would find the solutions of the equation and said: "Let's substitute a and find." [60]. Here, although S5 acted to solve the problem, her previous discourse was full of question marks. She was not sure of herself in her previous discourse: "The sign doesn't change anyway because the graph of the equation is tangent to the

x −axis, right? [57] S5 rarely made independent decisions in tasks. Thus, in both task situations, her routine was ritualistic. This time, S2 did not turn to talk about the procedure itself; rather she produced a narrative like: "If a graph of an equation is tangent to the x-axis, the equation has two equal solutions. Thus, here 4 is two equal solutions." [62] In order to make sense of both task situations, S2 mostly focused on the goal rather than choosing to talk about the procedure. Here, her routine served as explorative. After all, students found the solution and tried to draw the sign table. At this point, S1 asserted her own performed procedure: "[...]plus and minus." [65] However, she failed to make the correct connection between the solution and the sign table. S2 quickly corrected her friend and said: "…can't say plus or minus! […]." [66] S1's routines were rituals because, a task for her, the procedure was something to do, not to make sense of. The researcher again asked the students to come up with new proposals or to produce any narrative: "Why two equal solutions of the equation is 4?" [68] The discussion is displayed in Table 11.

Table 11. Students' discourses on task situation 2 and their discussions with the researcher (for S1, S2 and S4)

Turn	Speaker	Talk [activity]	Figure
68	R	Why two equal solutions of the equation is 4?	
69	S2	Because here, the graph of the equation is tangent to the *x*-axis.	
70	R	So, why did you equal the discriminant to zero?	
71	S4	Since there is only one solution, the discriminant is equal to zero. If the equation had two different solutions, the discriminant would be greater than zero.	
72	R	What would be the solution set for the inequality greater than zero here?	
73	S1	The solution would still be four.	
74	S2	No, if the discriminant here were greater than or equal to zero, the solution set would be real numbers.	
75	S1	No, wouldn't it be four again? [She looks at what her friends have written.]	
76	S2	Since there is a plus before and after four in the sign table, the solution set is real numbers.	
77	S4	Yes, the solution set is real numbers, because 4 is also an element of the solution set. [She draws the sign table.]	
78	S2	Yes, the solution set would be real numbers. If the inequality symbol were greater than zero,	

		the solution set would be the real numbers different 4.
79	R	Here the algebraic expression is less than or equal to zero. How did you determine the solution set by looking at the sign table?
80	S2	Since the inequality symbol is less than or equal to zero, there should only be 4 in the solution set.
81	S1	But find that a real number... [She thinks about what S2 says during the whole discussion. She looks confused.]
82	R	It means that one element in the solution set provides the state of inequality.
83	S1	Yes.

Here, S2 produced a narrative about the graph of the equation the $z^2 - 8z + 16 = 0$ and asserted: "[...] the graph of the equation is tangent to the x-axis." [69] What the researcher wanted was whether the students saw the algebraic expression $z^2 - 8z + 16$ as a whole to represent a real number. Thus, the researcher encouraged students yet again to provide more explorative answers: "So, why did you equal the discriminant to zero?".[70] At this point, S4 made a new proposal and said: "[...] If the equation had two different solutions, the discriminant would be greater than zero." [71] Thus, referring to the inequality symbol in the task situation, the researcher asked the students: "What would be the solution set for the inequality greater than zero here?". [72] S1 reacted quickly: "The solution would still be four.". [73] S2 grasped her friend's error and offered: "[…] if the discriminant here were greater than or equal to zero, the solution set would be real numbers.". [74] This answer was useless for S1, and she seemed to be confused with the answer. [75] At this point, the friends of S1 started helping out her. ([76], [77], [78]). In this excerpt, S2 continued to answer the researcher's questions by producing narratives. [80] Nevertheless, S1's confusion didn't end until her researcher said: "[...] one element in the solution set provides the state of inequality." [82] Here, S1 had failed to connect her friend's utterances logically. She was likely not thinking of them as an authority. She considered the researcher as an authority. Because although the researcher said the same thing friendly, she was convinced of what the researcher said and asserted: "Yes." [83]. To conclude, here S2's and S4's routines were explorative, but S1's routines were rituals.

Conclusion and Discussion

The purpose of this study was to determine whether students' routines in a specific task situation are more ritual or explorative. The criteria for analyzing the routines were as follows: performer's agentivity /external authority, focus on the goal or the procedure, and flexibility. The findings were obtained from a course in which students were asked to work in small groups on inequality task situations. The study has shown that students' routines were not pure rituals or sheer explorations (Lavie & Sfard, 2019; Lavie et al., 2019; Sfard & Lavie, 2005; Sfard, 2008, 2020). For example, although the S3 acted as a problem solver ([01], [18], [30]), she mostly chose to talk about the procedure ([09], [11], [16]) and rarely produced narrative [52]. In terms of performer's agentivity, her routines were explorative because S3 made independent decisions on the way. She performed all procedure with high confidence (no hesitations, question marks, no looking for approval) ([01], [11], [25], [30], [42]). There was ample evidence in the findings that her friends viewed S3 as an authority ([04], [14], [41], [43], [53]). Despite this, S3 saw the performance of the procedure as her task and mostly focused on the procedure she relied on. Therefore, these routines were rituals rather than exploration. Furthermore, while S5 appeared to be the slowest to grasp among the group members, not all of her routines were purely ritualistic. There was sufficient evidence in the findings that S5's routines were ritualistic in terms of the performer's agentivity. S5's mathematizing was full of hesitations, and question marks and she was constantly trying to get approval from his friends ([21], [57], [64]). S5 was not even sure of her correct solution until her friend approved her. [43] However again, she was reflecting on what had been done and asking meaningful questions about the task. As a result, her routines did not include imitating others, keeping up with others, memorizing, and practicing. However, the findings do not provide enough evidence to argue that her routines are more explorative.

This research also supports the studies that claim the importance of teachers' role in making students' mathematical discourses more explorative (Heyd-Metzuyanim & Graven, 2016; Nachlieli & Tabach, 2012; Sfard, 2017). The striking aspect of the findings was that all students' routines were ritualistic in terms of flexibility. The students preferred to use the algebraic approach commonly used in the books [like as figure in turn 21]. However, the school curriculum also uses other approaches, for example, using the graph of the quadratic equation. Although this approach was also taught in the lessons, the students relied on the algebraic approach. The researcher also dealt with other groups to maintain the classroom environment's naturalness. For this reason, she did not have the opportunity to intervene in the students' discourse in task 1. However, in task situation 2, the researcher provided students prompts for more explorative answers ([49], [68], [70], [72], [79]). Using endorsed narratives about mathematical objects, S2, S3, and S4 made sense of their routines in response to prompts. Especially S3's narrative played a prompting role in producing new narratives for her friends [52]. This narrative demonstrated how S3's flexibility can be explorative when opportunities arise. On the other hand, the findings revealed that S2 focused on the goal rather than talking about the procedure in both task situations (see discourses [15], [23], [62], [69], [74], [78], [80]). In terms of focus on the procedure or the goal, we see that her routines are more explorative. However, S2 produced articulated narratives when the teacher encouraged students to think more explorative with her questions (see discourses [62], [69], [74], [78], [80]). Here, prompts were a lever for shifting students toward explorative discourse. Therefore, prompts that encourage students to explain their thinking, revisit their solutions, and invite alternative approaches may be used in whole-class and small-group classroom interactions (Roberts & le Roux, 2019).

An interesting case that caught our attention in the findings was observed in the routines of S4. S4 mostly tended to propose new actions or outcomes for her friends. Also, S4 knew how to proceed on the way ([06], [08], [51], [71], [77]). However, when S4 was performing the procedure, she found the solutions inaccurate and looked at S3 for approval ([24], [35], [37], [41], [44], [53], [58]). Despite knowing how to move forward, wrong solutions made her prevented from making independent decisions. Our study does not provide sufficient evidence that the causes of S4's procedural errors in her performed procedure. However, observations showed that S4's routines were more exploratory in terms of performer's agentivity. Increasing agentivity means the student can make decisions independently of external authorities (Heyd-Metzuyanim et al., 2022; Lavie et al., 2019). Exploratory routines involve acting independently, making choices independently, and adapting strategies flexibly. (Nachlieli & Tabach, 2022). Besides, S4 noticed the mistakes of her friends in making sense of the task and made new proposals to them ([06], [08], [19]). It sometimes happens that students choose a routine as a solution path and follow it blindly and ritually, so that they lose their focus (Nachlieli & Katz, 2017). Thus, S4's warnings served as important discourses for the correct completion of the task performed. Unlike the S4, the S1 was quite successful in the performance of the procedure. S1 mathematized her task with high confidence in both task situations ([05], [26], [28], [32], [39], [61], [64]). Her routines were explorative. Nonetheless, S1 chose to focus on the procedure in all task situations and proceeded without making sense of the task. Thus, in terms of focus on the procedure or the goal, all her routines were ritualistic. When we compare this situation with S4 and S1, we see S4's routines as more explorative in general because in both task situations, S4's routines were more explorative than S1's routines in making sense of the task.

In this study, we do not describe the reason for the students' ritualized or exploratory discourse. In this way, the weaknesses of this study can be examined again by other researchers. Besides, by analyzing the mathematical discourses of teachers about inequality in the classroom, it can be investigated to what extent these discourses affect the students' discourse. There is a study (Sfard, 2017) about the relationship between mathematics teachers' discourse and their student's discourse on the topic of inequality. According to Sfard (2017), students' discourse is a mirror that analyzes the discourse to which they are exposed. However, more research is needed to better understand how mathematics teachers' discourses and their student's discourses are related. The contribution of our study is that it provides input for future research about students' discourse in solving inequality.

Note

This study has been developed from the doctoral thesis conducted by the first author under the supervision of the second and third author.

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