
IMPLEMENTATIONS OF CONSERVATION THEOREM AND THE EXPONENTIAL RATIONAL FUNCTION METHOD TO VISCOUS BURGERS EQUATION

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ABSTRACT

In this study, we dealt with conservation laws and exact travelling wave solutions of viscous Burgers equation. We used conservation theorem for finding conservation laws for this equation. All founded conservation laws are trivial conservation laws. Then, we found the exact solutions of viscous Burgers equation using the exponential rational function method. The results obtained confirm that the proposed method is an efficient technique for analytic treatment of a wide variety of nonlinear partial differential equations. Also the obtained results are shown graphically.

Keywords: Exact solutions, Symbolic computation, Viscous Burgers equation, Conservation laws, Lie symmetries

1. INTRODUCTION

It is known that all conservation laws of partial differential equations (PDEs) have not physical meanings, but they are necessary for studying the integrability and reduction of PDEs [1], [2]. There are lots of powerful methods obtaining conservation laws for PDEs for example characteristic method, variational approach, symmetry and conservation law relation, multiplier approach, direct construction method for conservation laws, partial Noether approach, Noether approach, conservation theorem [3]-[7]. In this study we will deal with conservation theorem. This theorem was introduced by Ibragimov. This theorem is associated with Lie symmetries, formal Lagrangian equations and adjoint equations of PDEs. Every symmetry of PDEs always provides a conservation law. These conservation laws can be trivial or non-trivial [8], [9].

Nonlinear partial differential equations (NPDEs) have common usage for describing complex phenomena in miscellaneous sciences, especially in physics and engineering. The investigation of analytical exact solutions of NPDEs plays a crucial role in the study of nonlinear physical phenomena. Because new exact solutions may provide more physical information for understanding the physical phenomena and thus lead to further applications in plasma physics, solid state physics, chemical physics, chemical kinematics, biology and so on. In recent years, a variety of powerful methods for obtaining explicit travelling and solitary wave solutions of NPDEs have been proposed. For example, Hirota bilinear transformation [10], the inverse scattering transformation [11], Bäcklund and Darboux transformation [12], the transformed rational function method [13], the tanh function method [14], the exp-function method [15], the sine-cosine method [16], the modified simple equation method [17], the extended trial equation method [18], the ansatz approach [19] and so on.

The rest of this paper is organized as follows. In Section 2, the algorithm of the conservation theorem and exponential rational function method are given. In Section 3, conservation laws and the exact travelling wave solutions of viscous Burgers equation are obtained and the results are shown graphically.

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2. PRELIMINARY

2.1. Algorithm of the Conservation Theorem

We assume that the given NPDE for $u(x, t)$ to be in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \tag{1}$$

where F is a polynomial in its arguments.

Suppose that, the Lie point symmetry generators for Eq.(1) is

$$X = \xi^x(x, t, u) \frac{\partial}{\partial x} + \xi^t(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u} \tag{2}$$

where $\xi^x(x, t, u)$, $\xi^t(x, t, u)$, $\eta(x, t, u)$ are the infinitesimals and the k – th prolongation of (2) is given by

$$X^{(k)} = X + \eta_i^{(1)} \frac{\partial}{\partial u_i^\alpha} + \dots + \eta_{i_1 i_2 \dots i_k}^{(k)} \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}}, \quad k \geq 1$$

Where

$$\begin{aligned} \eta_i^{(1)} &= D_i \eta - (D_i \xi^j) u_j \\ \eta_{i_1 i_2 \dots i_k}^{(k)} &= D_{i_k} \eta_{i_1 i_2 \dots i_{k-1}}^{(k-1)} - (D_{i_k} \xi^j) u_{i_1 i_2 \dots i_{k-1} j} \end{aligned}$$

here $i, j = 1, 2$ and $i_l = 1$ for $l = 1, 2, \dots, k$ and D_i is the total derivative operator.

Formal Lagrangian can be obtain with

$$L = \omega F \tag{3}$$

where $\omega(x, t)$ is the new adjoint variable. And then we can find adjoint equation by

$$F^* = \frac{\delta L}{\delta u} \tag{4}$$

Here $\frac{\delta}{\delta u}$ is the variational derivative. If we find solution of adjoint equation (4), we can obtain finite number of conservation laws.

Theorem 1 Every Lie point, Lie-Bäcklund and nonlocal symmetry of the system of k – th order differential Eq.(1) yields a conservation law. Formulae of conserved vector components are

$$T^i = \xi^i L + \frac{\delta L}{\delta u_i} + \sum_{s \geq 1} D_{i_1} \dots D_{i_s} (W) \frac{\partial L}{\partial u_{i_1 i_2 \dots i_s}} \tag{5}$$

with formal Lagrangian given by Eq.(3) and here $W = \eta - \xi^j u_j$ and ξ^i, η are the functions. The conserved vectors obtained from Eq.(5) involves the arbitrary solutions of the adjoint equation so one obtains an infinite number of conservation laws for Eq.(1) by defining ω [20].

Theorem 2 Obtained conserved vectors from (5) are a conservation laws of Eq.(1) if

$$D_i(T^i) = 0. \tag{6}$$

Here D_i is the total derivative with respect to x and t [21].

2.2. Algorithm of the Exponential Rational Function Method

We can summarize the basic steps of the exponential rational function method as follows

Consider a general nonlinear partial differential equation Eq.(1), say in two independent variables x and t . To find exact solutions of this equation we use the travelling wave transformation

$$u(x, t) = u(\xi), \xi = x - ct, \quad (7)$$

where c is the speed of travelling wave, carries Eq.(1) to a nonlinear ordinary differential equation (ODE) as follows

$$Q(u, u', u'', \dots) = 0. \quad (8)$$

Here prime denotes the derivative with respect to ξ . According to possibility, Eq.(8) can be integrated term by term one or more times. We seek for the solutions of Eq.(8) in the following generalized form

$$u(\xi) = \sum_{n=0}^m \frac{a_n}{(1+e^\xi)^n} \quad (9)$$

in which a_n ($a_m \neq 0$) are constants to be determined later.

Step 1: Firstly, we apply wave transformation (7) to Eq.(1), we obtain reduced Eq.(8).

Step 2: By balancing the highest order linear term with the highest order nonlinear term in Eq.(8), we obtain the balancing number m .

Step 3: We substitute Eq.(9) into Eq.(8) and collect all the coefficients of $e^{n\xi}$ ($n = 0, 1, \dots$) to zero. Afterwards we equate each coefficient of this polynomial to zero and we find a set of algebraic equations.

Step 4: We solve this system with the aid of Maple packet program to find the values of a_n and c .

Step 5: Finally, by substituting these values into Eq.(9), by use of Eq.(7), we find the exact travelling wave solutions of Eq.(1) [22].

3. APPLICATIONS

The viscous Burgers equation

$$u_t + uu_x - vu_{xx} = 0 \quad (10)$$

where v is the viscosity coefficient. Burgers equation is a nonlinear wave propagation model, especially in fluid mechanics. This equation occurs in miscellaneous fields of applied mathematics, for instance modeling of traffic flow and gas dynamics.

3.1. Implementation of the Theorem

In this subsection, we will obtain conservation laws of viscous Burgers equation. Firstly, we can find formal Lagrangian for Eq.(10) in the following form

$$L = \omega(u_t + uu_x - vu_{xx}). \quad (11)$$

If we apply variational derivative to Eq.(10), we obtain adjoint equation with

$$F^* = -w_x u - v w_{xx} - w_t = 0 \tag{12}$$

When we substitute u instead of ω in Eq.(12), we can not obtain Eq.(10); because of this situation Eq.(10) is not self adjoint. We can say that $\omega = 1$ is a solution for Eq.(12). Eq.(10) admits following five Lie point symmetry generators are given by

$$X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial x}, X_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, X_4 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}, X_5 = xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} + (x-ut) \frac{\partial}{\partial u}$$

In addition, above Lie-point symmetry generators’s commutator table is

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_2	$2 X_1$	X_4
X_2	0	0	0	X_2	X_3
X_3	$-X_2$	0	0	$-X_3$	0
X_4	$-2 X_1$	$-X_2$	X_3	0	$2 X_5$
X_5	$-X_4$	$-X_3$	0	$-2 X_5$	0

Now, we will obtain conservation laws of given equation for all founded Lie point symmetry generators. We can expand formula (5) for Eq.(10) with

$$T^x = \xi^x L + W \left[\frac{\partial L}{\partial u_x} - D_x \left(\frac{\partial L}{\partial u_{xx}} \right) \right] + D_x(W) \left[\frac{\partial L}{\partial u_{xx}} \right], \tag{13}$$

$$T^t = \xi^t L + W \left[\frac{\partial L}{\partial u_t} \right]$$

Case 1:

When $\omega = 1$ and $X_1 = \frac{\partial}{\partial t}$, we get

$$W = -u_t$$

and then we obtain following conserved vectors

$$T_1^x = -u_t u + v u_{xt}, \tag{14}$$

$$T_1^t = u u_x - v u_{xx}.$$

Conserved vectors (14) are satisfy divergence condition, so they are trivial conservation laws.

Case 2 :

If we consider $X_2 = \frac{\partial}{\partial x}$, we obtain $W = -u_x$. Substituting (11) , $\omega = 1$ and $W = -u_x$ in (13), conserved vectors are found with

$$T_2^x = u_t, \tag{15}$$

$$T_2^t = -u_x.$$

Conserved vectors (15) satisfy (6), so they are trivial conservation laws for Eq.(10).

Case 3:

When $X_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$, we find following conserved vectors

$$T_3^x = t u_t + u, \tag{16}$$

$$T_3^t = 1 - t u_x.$$

If we apply divergence condition to (16), we obtain

$$\begin{aligned} D_x(T_3^x) + D_t(T_3^t) &= tu_{xt} + u_x - tu_{xt} - u_x \\ &= 0, \end{aligned}$$

so they are trivial conservation laws.

Case 4:

Now, let us find conservation laws corresponds to the operator $X_4 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}$. For this operator, the infinitesimals are $\xi^x = x$, $\xi^t = 2t$, $\eta = -u$ and we get

$W = -u - xu_x - 2tu_t$ and the corresponding conserved vector of Eq.(10) as

$$T_4^x = xu_t - u^2 - 2utu_t + 2vu_x + 2vtu_{xt}, \tag{17}$$

$$T_4^t = 2tuu_x - 2tvu_{xx} - u - xu_x.$$

here we used $\omega = 1$. The divergence condition $D_x(T_4^x) + D_t(T_4^t)$ is equal to zero, so conserved vectors (17) are trivial.

Case 5:

Similarly, for operator $X_5 = xt \frac{\partial}{\partial x} + t^2 \frac{\partial}{\partial t} + (x - ut) \frac{\partial}{\partial u}$, when $\omega = 1$, corresponding conserved vector of Eq.(10) as

$$T_5^x = xtu_t + ux - u^2t - ut^2u_t - v + 2vu_xt + vt^2u_{xt}, \tag{18}$$

$$T_5^t = t^2uu_x - t^2vu_{xx} + x - ut - txu_x.$$

The divergence condition (6) is zero, so we can say that founded conserved vectors (18) are trivial conservation laws.

3.1. Implementation of the Exponential Rational Function Method

According the exponential rational function method, introducing the traveling wave transformation Eq.(7), Eq.(10) is transformed to the following ODE

$$-cu' + uu' - vu'' = 0 \tag{19}$$

Integrating Eq.(19) with respect to ξ , we obtain

$$-cu + \frac{u^2}{2} - vu' = 0 \tag{20}$$

Here the integration constant is taken as zero. Balancing the highest order derivative term u' with the highest order nonlinear term u^2 gives $m = 1$. Then the solution turns into

$$u(\xi) = a_0 + a_1 e^\xi, \tag{21}$$

from Eq.(9). Substituting Eq.(21) into Eq.(20) and collecting the coefficients of $e^{2\xi}$, e^ξ , $e^{2\xi}$ to zero, we get a set of algebraic equations

$$\begin{aligned} e^{2\xi} &: -ca_0 + \frac{a_0^2}{2} = 0, \\ e^\xi &: -2ca_0 + a_0^2 + va_1 - ca_1 + a_0a_1 = 0, \\ e^{0\xi} &: -ca_0 + \frac{a_0^2}{2} + \frac{a_1^2}{2} - ca_1 + a_0a_1 = 0. \end{aligned}$$

From the solutions of the system above with the aid of Maple, we have two different cases. We can present these cases as follows.

Case 1:

$$a_0 = 0, a_1 = 2v, c = v, \tag{22}$$

By substituting the values Eq.(22) into the exact solution Eq.(21), by use of Eq.(7), we obtain the exact travelling wave solutions of viscous Burgers equation as follows:

$$u(x, t) = \frac{2v}{1 + \cosh(x - vt) + \sinh(x - vt)}$$

When the parameters are given special values, the solitary waves are derived from the travelling waves. We plot this solution of the viscous Burgers equation when $v = 1$ within the interval $0 \leq x, t \leq 10$.

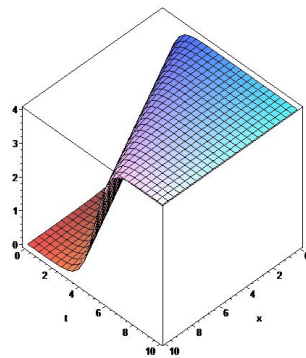


Figure 1. Graphical representation of Case 1 for viscous Burgers equation

Case 2:

$$a_0 = -2v, a_1 = 2v, c = -v \tag{23}$$

When we substitute Eq.(23) into the exact solution Eq.(21), by use of Eq.(7), we find the exact travelling wave solutions of viscous Burgers equation as follows:

$$u(x, t) = -2v \left(\frac{\cosh(x + vt) + \sinh(x + vt)}{1 + \cosh(x + vt) + \sinh(x + vt)} \right)$$

When the parameters are given special values, the solitary waves are derived from the travelling waves. We plot this solution of the viscous Burgers equation when $v = 1$ within the interval $0 \leq x, t \leq 10$.

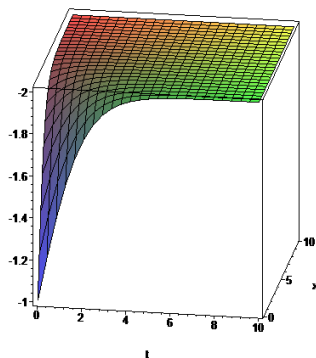


Figure 2. Graphical representation of Case 2 for viscous Burgers equation

4. CONCLUSION

In this study, we considered viscous Burgers equation for finding conservation laws and exact travelling wave solutions. Firstly, we gave needed informations about conservation theorem and exponential rational function method in section 2. Then we applied this methods to our equation in section 3. We obtained trivial conservation laws in subsection 3.1. We gave five Lie point symmetry generators which are admitted by viscous Burgers equation and commutator table is given. Then, we obtained formal Lagrangian and adjoint equation for given equation, we said that $\omega = 1$ is a solution of adjoint equation. We obtained five trivial conservation laws for viscous Burgers equation different from each other. These founded conservation laws are trivial because of divergence condition is satisfied.

In subsection 3.2, the exponential rational function method has been successfully employed to solve viscous Burgers equation. The obtained solutions may be worthwhile for explanation of some physical phenomena accurately. The used method has many advantages: it is straight forward and compendious. Graphical representation is an important instrument for communication and it illustrates obviously the solutions of the problems. So, we gave the graphical representation of the obtained solutions.

Finally, we can note that, when we compare the solutions obtained in this paper with the given ones in, we can remark that our solutions are new. Also since the solution procedure is simple and direct, this work going to be useful for the literature.

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