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Ricci-Yamabe Solitons in $f(R)$-gravity

Krishnendu De and Uday Chand De*

(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

The main objective of this paper is to describe the perfect fluid spacetimes fulfilling $f(R)$-gravity, when Ricci-Yamabe, gradient Ricci-Yamabe and η-Ricci-Yamabe solitons are its metrics. We acquire conditions for which the Ricci-Yamabe and the gradient Ricci-Yamabe solitons are expanding, steady or shrinking. Furthermore, we investigate η-Ricci-Yamabe solitons and deduce a Poisson equation and with the help of this equation, we acquire some significant results.

Keywords: *Perfect fluids; $f(R)$-gravity; Ricci-Yamabe solitons; η-Ricci-Yamabe solitons.*
AMS Subject Classification (2020): *Primary: 83D05; Secondary: 83C05; 53C50.*

1. Introduction

Einstein's field equations are insufficient to describe the universe late-time inflation without presuming that there are some unobserved components that could explain the origin of dark energy and dark matter. It is the main inspiration behind the extension to find field equations of gravity in higher order. It is noteworthy that the concept of $f(R)$-gravity emerges as an inherent extension of Einstein's theory of gravity. Here, the Hilbert-Einstein action term is changed by the function $f(R)$, in which R stands for the Ricci scalar. Buchdahl proposed the aforementioned hypothesis [6] and Starobinsky [27] has accomplished viability with the study of cosmic inflation. In earlier works, a number of useful functional forms of $f(R)$ were presented in ([7], [8], [9], [14], [15], [18]).

Spacetime is a time-oriented 4-dimensional Lorentzian manifold $\mathcal{M}$ which is a specific class of semi-Riemannian manifold endowed with a Lorentzian metric g of signature $(-,+,+,+)$.

The Einstein's field equation is described in the general theory of relativity by

$$Ric - \frac{R}{2}g = \kappa^2 T, \tag{1.1}$$

in which $\kappa^2 = 8\pi G$, G denotes Newton's gravitational constant and $'Ric'$ stands for the Ricci tensor. The preceding equation entails that the energy momentum tensor T has vanishing divergence.

In a perfect fluid-spacetime the energy momentum tensor T is described by,

$$T = pg + (\nu + p)\mathcal{D} \otimes \mathcal{D}, \tag{1.2}$$

in which $\mathcal{D}$ is a 1-form, p stands for the isotropic pressure, ν denotes the energy density of the perfect fluid-spacetime. Also, $g(\rho,\rho) = -1$, since here ρ is a unit timelike vector field, named the velocity vector, defined by $g(E,\rho) = \mathcal{D}(E)$ for any E.

The far more fascinating mathematical techniques for illustrating the geometric structures in differential geometry throughout the most recent years are those based on the theory of geometric flows. Due to the fact that they emerge as convincing singularity models, the analysis of flow singularities is significantly affected by a certain portion of solutions where the metric changes through diffeomorphisms. These are commonly known as soliton solutions.

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** Corresponding author*

Hamilton concurrently introduced the Yamabe and the Ricci flow in [24]. The particular solutions of the Yamabe and Ricci flow, respectively, are the Ricci soliton and the Yamabe soliton. Many geometers have recently become interested in the theoretical idea of geometric flows, such as the Ricci flow and the Yamabe flow. The study of a new geometric flow called the Ricci-Yamabe map was introduced in [22] in 2019. Just a scalar combination of Ricci flow and the Yamabe flow makes up this map. Such a flow represents an improvement for the metrics on the semi-Riemannian manifold $\mathcal{M}$, described by [22]

$$\frac{\partial}{\partial t}g(t) = \beta_2 r(t)g(t) - 2\beta_1 Ric(t), \quad g_0 = g(0). \tag{1.3}$$

Anybody can think of the Ricci-Yamabe flow as a Riemannian, semi-Riemannian, or singular Riemannian flow depending on the signs of the associated scalars, β_1 and β_2. This variety of options are important in some mathematical or physical models. One more strong motivation for began the investigation of Ricci-Yamabe solitons is that, despite the way that Ricci solitons and Yamabe solitons are equivalent in two dimension, in higher dimensions they are fundamentally unique.

A data $(g, X, \beta_3, \beta_1, \beta_2)$ obeying

$$\pounds_X g = -2\beta_1 Ric - (2\beta_3 - \beta_2 R)g, \tag{1.4}$$

where $\pounds_X$ is the Lie-derivative operator and $\beta_3, \beta_1, \beta_2 \in \mathbb{R}$ is called a Ricci-Yamabe soliton on $(\mathcal{M}, g)$. It is also called (β_1, β_2) type Ricci-Yamabe soliton. If X is the gradient of the smooth function ψ on $\mathcal{M}$, then the preceding concept is called a gradient Ricci-Yamabe soliton and then the previous equation (1.4) transforms to

$$\nabla^2\psi = -\beta_1 Ric - (\beta_3 - \frac{1}{2}\beta_2 R)g, \tag{1.5}$$

where $\nabla^2\psi$ being the Hessian of ψ.

Here, the Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is steady if $\beta_3 = 0$, expanding if $\beta_3 > 0$, and shrinking for $\beta_3 < 0$. If β_3, β_2 and β_1 are smooth functions on $\mathcal{M}$, then a Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) recovers an almost Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton). If $\beta_2 = 0$, $\beta_1 = 1$, then Ricci-Yamabe soliton or gradient Ricci-Yamabe soliton turns into Ricci soliton or gradient Ricci soliton [24]. Similarly, it produces Yamabe soliton or gradient Yamabe soliton[23] if $\beta_2 = 1$, $\beta_1 = 0$. Again, if $\beta_2 = -1$, $\beta_1 = 1$, it turns into an Einstein or gradient Einstein soliton[11]. The foregoing soliton is named proper if $\beta_1 \neq 0, 1$.

Ricci solitons and Yamabe solitons have been studied by ([12], [16], [17], [20], [21], [28], [29]) and many others.

Although most study of solitons has been conducted in the Riemannian category, solitons have also been examined in the Lorentzian context in ([1], [2], [5]). Few intriguing findings on Ricci solitons, gradient Ricci solitons, and gradient Yamabe soliton in perfect fluid-spacetime on general relativity theory have so far been explored in the literature (see; [3], [19]). In this article, inspired by the above studies, we intend to investigate the Ricci-Yamabe soliton and gradient Ricci-Yamabe soliton on perfect fluid-spacetime obeying $f(R)$-gravity. Precisely, we establish the following theorems.

Theorem 1.1. *For a constant Ricci scalar, if $(g, \rho, \beta_1, \beta_2, \beta_3)$ is a Ricci-Yamabe soliton in a perfect fluid-spacetime obeying $f(R)$-gravity, then the Ricci-Yamabe soliton is shrinking for $\nu < \frac{f(R) - \frac{\beta_2}{\beta_1} R f_R(R)}{2\kappa^2}$, steady if $\nu = \frac{f(R) - \frac{\beta_2}{\beta_1} R f_R(R)}{2\kappa^2}$ and expanding for $\nu > \frac{f(R) - \frac{\beta_2}{\beta_1} R f_R(R)}{2\kappa^2}$.*

Theorem 1.2. *For a constant Ricci scalar, let the perfect fluid-spacetime satisfying $f(R)$-gravity permit a gradient Ricci-Yamabe soliton. If the scalar α_2 remains invariant under the velocity vector field ρ and ρ has vanishing divergence, then either the energy density is constant, or the soliton is shrinking if $\nu < \frac{\frac{\beta_2}{\beta_1} R f_R(R) + f(R)}{2\kappa^2}$, steady for $\nu = \frac{\frac{\beta_2}{\beta_1} R f_R(R) + f(R)}{2\kappa^2}$ and expanding for $\nu > \frac{\frac{\beta_2}{\beta_1} R f_R(R) + f(R)}{2\kappa^2}$.*

Cho and Kimura [13] examined real hypersurfaces in a complex space form, expanding the concept of Ricci solitons to η-Ricci solitons. A more general extension, called η-Ricci-Yamabe soliton is introduced in [26] and defined as:

A η-Ricci-Yamabe soliton on $(\mathcal{M}, g)$ is a data $(g, X, \beta_1, \beta_2, \beta_3, \beta_4)$ obeying

$$\pounds_X g = -2\beta_1 Ric - (2\beta_3 - \beta_2 R)g - 2\beta_4 \eta \otimes \eta, \tag{1.6}$$

where $\beta_4 \in \mathbb{R}$.

It is easily seen that η-Ricci-Yamabe soliton of type (1, 0) and type (0, 2) are η-Ricci soliton and η-Yamabe soliton. If we put, $\beta_4 = 0$ in the foregoing equation then it turns into Ricci-Yamabe soliton. η-Ricci soliton and η-Yamabe soliton have been investigated in ([4], [26]).

So far our knowledge goes, the investigation of η- Ricci-Yamabe soliton fulfilling $f(R)$-gravity is still pending. Hence, in this paper we prove the subsequent theorem.

Theorem 1.3. *Let the perfect fluid-spacetime fulfilling $f(R)$-gravity with constant R permit an η- Ricci-Yamabe soliton. If the velocity vector ρ is equivalent to the potential vector and η is the g-dual of the gradient vector field $\rho = grad(\psi)$, then the Poisson equation of ψ is*

$$\Delta(\psi) = -3(\beta_4 + \beta_1 \frac{\kappa^2(p+\nu)}{f_R(R)}).$$

2. $f(R)$-gravity theory

Here we work with a 4-dimensional perfect fluid-spacetime fulfilling $f(R)$-gravity. We set

$$\mathcal{H} = \int \frac{[\mathcal{L}_m + f(R)]\sqrt{(-g)}}{\kappa^2} d^4x, \tag{2.1}$$

in which $\mathcal{H}$ represents modified Einstein-Hilbert action term, $\mathcal{L}_m$ stands for the scalar field's matter Lagrangian density.

Here, T_{ab} is described by

$$T_{ab} = \frac{-2\delta(\sqrt{-g})\mathcal{L}_m}{\sqrt{-g}\delta^{ab}}, \tag{2.2}$$

in which T_{ab} indicates the stress energy tensor of matter.

For a unit time like vector $\mathcal{D}_a$, the perfect fluid type T_{ab} is defined by

$$T_{ab} = pg_{ab} + (\nu + p)\mathcal{D}_a\mathcal{D}_b. \tag{2.3}$$

Letting L_m solely depends on g_{ab}, the variation of action of (2.1) with respect to the metric tensor components g_{ab} yields the field equations of $f(R)$-gravity

$$\begin{aligned} f_R(R)R_{ab} \quad &- \quad \frac{1}{2}f(R)g_{ab} + g_{ab}\Box f_R(R) - \nabla_a\nabla_b f_R(R) \\ &= \kappa^2 T_{ab}, \end{aligned} \tag{2.4}$$

where R_{ab} stand for the local components of Ric and $\Box \equiv \nabla_a\nabla^a$ is named d'Alembert operator, ∇_a indicates the covariant derivative. The previous field equation can be easily weakened by replacing $f(R)$ with R.

If we choose $R =$ constant, the field equation (2.4) gives the form

$$R_{ab} - \frac{R}{2}g_{ab} = \frac{\kappa^2}{f_R(\mathrm{R})}T^{eff}_{ab}, \tag{2.5}$$

where

$$T^{eff}_{ab} = T_{ab} + \frac{f(R) - Rf_R(R)}{2\kappa^2}g_{ab}.$$

For $R =$ constant, the Ricci tensor in a perfect fluid-spacetime fulfilling $f(R)$-gravity, is described by [14]

$$Ric(E, F) = \alpha_1\mathcal{D}(E)\mathcal{D}(F) + \alpha_2 g(E, F), \tag{2.6}$$

where $\alpha_1 = \frac{\kappa^2(p+\nu)}{f_R(R)}$ and $\alpha_2 = \frac{2\kappa^2 p + f(R)}{2f_R(R)}$. Based on the previous equation Q, the Ricci operator is described by

$$Q(E) = \alpha_1\mathcal{D}(E)\rho + \alpha_2 E, \tag{2.7}$$

in which ρ is the velocity vector field corresponding to the 1-form $\mathcal{D}$ and $g(QE, F) = Ric(E, F)$.

Moreover, p and ν are linked by a state equation of the form $p = p(\nu)$ and the perfect fluid-spacetime is termed isentropic. Furthermore, if $p = \nu$, the perfect fluid-spacetime is referred to as stiff matter. If $p = 0$, the perfect fluid-spacetime is named the dust matter era, if $p = -\nu$, and for $p = \frac{\nu}{3}$, it is called the radiation era [10].

Proof of the Theorem 1.1 :

Let a perfect fluid-spacetime fulfilling $f(R)$-gravity with constant R, admit a Ricci-Yamabe soliton defined by (1.4).

Using the explicit form of Lie derivative, the equation (1.4) takes the form

$$\beta_1 Ric(E,F) = -\frac{1}{2}[g(\nabla_E \rho, F) + g(E, \nabla_F \rho)] - \frac{(2\beta_3 - \beta_2 R)}{2} g(E,F). \tag{2.8}$$

Comparing (2.6) and (2.8), we acquire

$$\begin{aligned} &\beta_1[\alpha_1 \mathcal{D}(E)\mathcal{D}(F) + \alpha_2 g(E,F)] \\ &= -\frac{1}{2}[g(\nabla_E \rho, F) + g(E, \nabla_F \rho)] - \frac{(2\beta_3 - \beta_2 R)}{2} g(E,F). \end{aligned} \tag{2.9}$$

Putting $E = F = \rho$ in the foregoing equation yields

$$\beta_1[\alpha_1 - \alpha_2] = \frac{(2\beta_3 - \beta_2 R)}{2}. \tag{2.10}$$

Now, using the value of α_1 and α_2 in the preceding equation, we lead

$$\beta_1[\frac{2\kappa^2\nu - f(R)}{2f_R(R)}] = \frac{(2\beta_3 - \beta_2 R)}{2}. \tag{2.11}$$

Hence, equation (2.11) reveals that

$$\begin{aligned} \beta_3 &= \beta_1[\frac{2\kappa^2\nu - f(R)}{2f_R(R)}] + \frac{\beta_2 R}{2} \\ &= \frac{\beta_1(2\kappa^2\nu - f(R)) + \beta_2 R f_R(R)}{2f_R(R)}. \end{aligned} \tag{2.12}$$

This finishes the proof.

If we put $\beta_2 = 0$, $\beta_1 = 1$ in equation (2.12), then we get

$$\beta_3 = \frac{2\kappa^2\nu - f(R)}{2f_R(R)}. \tag{2.13}$$

Thus, we can write:

Corollary 2.1. *If (g, ρ, β_3) is a Ricci soliton in a perfect fluid-spacetime fulfilling $f(R)$-gravity with constant R, then the Ricci soliton is steady if $\nu = \frac{f(R)}{2\kappa^2}$, expanding for $\nu > \frac{f(R)}{2\kappa^2}$ and shrinking for $\nu < \frac{f(R)}{2\kappa^2}$.*

Putting $\beta_2 = 1$, $\beta_1 = 0$ in equation (2.12), then we obtain

$$\beta_3 = \frac{R}{2}. \tag{2.14}$$

Hence, we can state:

Corollary 2.2. *If (g, ρ, β_3) is a Yamabe soliton in a perfect fluid-spacetime satisfying $f(R)$-gravity with constant R, then the Yamabe soliton is steady if the Ricci scalar R vanishes, expanding for $R > 0$ and shrinking for $R < 0$.*

If we put $\beta_2 = -1$, $\beta_1 = 1$ in equation (2.12), then we have

$$\beta_3 = \frac{(2\kappa^2\nu - f(R)) - R f_R(R)}{2f_R(R)}. \tag{2.15}$$

Therefore, the subsequent result can be written as:

Corollary 2.3. *For $R =$ constant, if (g, ρ, β_3) is an Einstein soliton in a perfect fluid-spacetime fulfilling $f(R)$-gravity, then the Einstein soliton is expanding for $\nu > \frac{f(R) - R f_R(R)}{2\kappa^2}$, steady if $\nu = \frac{f(R) - R f_R(R)}{2\kappa^2}$, and shrinking for $\nu < \frac{f(R) - R f_R(R)}{2\kappa^2}$.*

Proof of the Theorem 1.2 :
Let the soliton vector field X of the Ricci-Yamabe soliton in a perfect fluid-spacetime fulfilling $f(R)$-gravity with constant R be equal to $D\psi$, where D is the gradient operator. Then equation (1.5) reduces to

$$\nabla_E D\psi = -\beta_1 QE - (\beta_3 - \frac{1}{2}\beta_2 R)E \tag{2.16}$$

for all $E \in \mathfrak{X}(\mathcal{M})$. From the last equation and the relation

$$R(E,F)D\psi = \nabla_E\nabla_F D\psi - \nabla_F\nabla_E D\psi - \nabla_{[E,F]}D\psi, \tag{2.17}$$

we acquire

$$R(E,F)D\psi = -\beta_1[(\nabla_E Q)(F) - (\nabla_F Q)(E)]. \tag{2.18}$$

Now using (2.7), we infer

$$(\nabla_E Q)(F) = E(\alpha_2)F + E(\alpha_1)\mathcal{D}(F)\rho + \alpha_1\{(\nabla_E\mathcal{D})F\rho + \mathcal{D}(F)\nabla_E\rho\}. \tag{2.19}$$

Utilizing (2.19) in (2.18), we obtain

$$\begin{aligned} R(E,F)D\psi &= -\beta_1[E(\alpha_2)F - F(\alpha_2)E + E(\alpha_1)\mathcal{D}(F)\rho \\ &-F(\alpha_1)\mathcal{D}(E)\rho + \alpha_1\{(\nabla_E\mathcal{D})F\rho + \mathcal{D}(F)\nabla_E\rho \\ &-(\nabla_F\mathcal{D})E\rho - \mathcal{D}(E)\nabla_F\rho\}]. \end{aligned} \tag{2.20}$$

Contracting the preceding equation, we find

$$S(F,D\psi) = -\beta_1[F(\alpha_1) + \rho(\alpha_1)\mathcal{D}(F) - 3F(\alpha_2) + \alpha_1\{(\nabla_\rho\mathcal{D})F - \mathcal{D}(F)div\rho\}], \tag{2.21}$$

in which 'div' indicates divergence.
Also, we deduce from the equation (2.6)

$$S(F,D\psi) = \alpha_1\mathcal{D}(F)\mathcal{D}(D\psi) + \alpha_2 g(F,D\psi). \tag{2.22}$$

Putting $F = \rho$ in equations (2.21) and (2.22) and then comparing, we acquire

$$(\alpha_2 - \alpha_1)\rho(\psi) = \beta_1[3\rho(\alpha_2) + \alpha_1 div\rho]. \tag{2.23}$$

Let the scalar α_2 remains invariant under ρ, that is, $\rho(\alpha_2) = 0$ and $div\rho = 0$. Hence, equations (2.23) infers that

$$(\alpha_2 - \alpha_1)\rho(\psi) = 0, \tag{2.24}$$

which entails that either $\alpha_1 = \alpha_2(\neq 0)$ or $\rho(\psi) = 0$.
Case I. We assume that $\rho(\psi) \neq 0$ and $\alpha_1 = \alpha_2$. Hence, using the equation (2.6), we acquire

$$\nu = \frac{f(R)}{2\kappa^2} = constant. \tag{2.25}$$

Case II. We suppose that $\alpha_1 \neq \alpha_2$ and $\rho(\psi) = 0$, that is,

$$g(D\psi, \rho) = 0. \tag{2.26}$$

Covariant derivative of the equation (2.26)yields

$$g(\nabla_E\rho, D\psi) = [-\beta_1(\alpha_1 - \alpha_2) + (\beta_3 - \frac{\beta_2 R}{2})]\mathcal{D}(E). \tag{2.27}$$

Replacing E by ρ and putting the value of α_1, α_2 in the foregoing equation gives

$$\beta_3 = \frac{\beta_1(2\kappa^2\nu - f(R)) - \beta_2 R f_R(R)}{2f_R(R)}. \tag{2.28}$$

This ends the proof.
It is known [25] that for a velocity vector field V, the energy equation for a perfect fluid is described by

$$(V\nu) = -(p+\nu)divV. \tag{2.29}$$

Hence, the equation (2.25) reveals that ν = constant. Thus, either $p + \nu = 0$ or $divV = 0$. But the expansion scalar is represented by $divV$.
Therefore, we state:

Corollary 2.4. *Let the perfect fluid-spacetime with constant R and $(\rho\psi) \neq 0$ fulfilling $f(R)$-gravity admit a gradient Ricci-Yamabe soliton. If $div\rho = 0$ and $\rho(\alpha_2) = 0$, then either the perfect fluid has vanishing expansion scalar, or the spacetime represents a dark matter era.*

If we put $\beta_2 = 0$, $\beta_1 = 1$ in equation (2.28), then we get

$$\beta_3 = \frac{2\kappa^2\nu - f(R)}{2f_R(R)}. \tag{2.30}$$

Thus, we can write:

Corollary 2.5. *For a constant Ricci scalar, let the perfect fluid-spacetime satisfying $f(R)$-gravity permit a gradient Ricci soliton. If $div\rho = 0$ and $\rho(\alpha_2) = 0$, then either the energy density is constant, or the soliton is steady if $\nu = \frac{f(R)}{2\kappa^2}$, expanding for $\nu > \frac{f(R)}{2\kappa^2}$ and shrinking for $\nu < \frac{f(R)}{2\kappa^2}$.*

If we put $\beta_2 = 1$, $\beta_1 = 0$ in equation (2.28), then we obtain

$$\beta_3 = -\frac{R}{2}. \tag{2.31}$$

Hence, we write:

Corollary 2.6. *For a constant Ricci scalar, let the perfect fluid-spacetime with constant R obeying $f(R)$-gravity permit a gradient Yamabe soliton. If $div\rho = 0$ and $\rho(\alpha_2) = 0$, then either the energy density is constant, or the soliton is steady if the Ricci scalar R vanishes, expanding for $R < 0$ and shrinking for $R > 0$.*

If we put $\beta_2 = -1$, $\beta_1 = 1$ in equation (2.28), then we have

$$\beta_3 = \frac{(2\kappa^2\nu - f(R)) + Rf_R(R)}{2f_R(R)}. \tag{2.32}$$

Hence, we have

Corollary 2.7. *For a constant Ricci scalar, let the perfect fluid-spacetime satisfying $f(R)$-gravity admit a gradient Einstein soliton. If $div\rho = 0$ and $\rho(\alpha_2) = 0$, then either the energy density is constant, or the soliton is steady if $\nu = \frac{f(R)-Rf_R(R)}{2\kappa^2}$, expanding for $\nu > \frac{f(R)-Rf_R(R)}{2\kappa^2}$ and shrinking for $\nu < \frac{f(R)-Rf_R(R)}{2\kappa^2}$..*

Proof of the Theorem 1.3 :

Let a perfect fluid-spacetime with constant Ricci scalar satisfying $f(R)$-gravity, permit a η-Ricci-Yamabe soliton defined by (1.6), where the 1-form η is identical with the 1-form $\mathcal{D}$ of the perfect fluid-spacetime, that is, $\eta(E_1) = \mathcal{D}(E_1) = g(E_1, \rho)$.
Using the Lie derivative's explicit form, the equation (1.6) transforms to

$$\begin{aligned}\beta_1 Ric(E,F) &= -\frac{1}{2}[g(\nabla_E\rho, F) + g(E, \nabla_F\rho)] \\ &-\frac{(2\beta_3 - \beta_2 R)}{2}g(E,F) - \beta_4\mathcal{D}(E)\mathcal{D}(F).\end{aligned} \tag{2.33}$$

Comparing (2.6) and (2.33), we acquire

$$\begin{aligned}&\beta_1[\alpha_1\mathcal{D}(E)\mathcal{D}(F) + \alpha_2 g(E,F)] \\ &= -\frac{1}{2}[g(\nabla_E\rho, F) + g(E, \nabla_F\rho)] \\ &-\frac{(2\beta_3 - \beta_2 R)}{2}g(E,F) - \beta_4\mathcal{D}(E)\mathcal{D}(F).\end{aligned} \tag{2.34}$$

Putting $E = F = \rho$ in the foregoing equation yields

$$\beta_1[\alpha_1 - \alpha_2] = \frac{(2\beta_3 - \beta_2 R)}{2} - \beta_4. \tag{2.35}$$

Contracting the equation (2.33), we obtain

$$\beta_1 R = -div\rho - 2(2\beta_3 - \beta_2 R) + \beta_4. \tag{2.36}$$

Again, contracting the equation (2.6), we get

$$R = -\alpha_1 + 4\alpha_2. \tag{2.37}$$

Based on the last two equations, we may conclude that

$$\beta_1(-\alpha_1 + 4\alpha_2) = -div\rho - 2(2\beta_3 - \beta_2 R) + \beta_4. \tag{2.38}$$

The equations (2.35) and (2.38) can be rewritten as

$$\beta_4 - \beta_3 = -\beta_1[\alpha_1 - \alpha_2] - \frac{\beta_2}{2}(-\alpha_1 + 4\alpha_2), \tag{2.39}$$

and

$$4\beta_3 - \beta_4 = -div\rho + (2\beta_2 - \beta_1)(-\alpha_1 + 4\alpha_2). \tag{2.40}$$

Solving the last two equations and putting the value of α_1, α_2, we acquire

$$\begin{cases} \beta_3 = -\frac{div\rho}{3} - \frac{\beta_2}{2}\frac{\kappa^2(p+\nu)}{f_R(R)} + (2\beta_2 - \beta_1)\frac{2\kappa^2 p + f(R)}{2f_R(R)} \\ \beta_4 = -\frac{div\rho}{3} - \beta_1\frac{\kappa^2(p+\nu)}{f_R(R)}. \end{cases} \tag{2.41}$$

Consequently, we can deduce from the previous equation that

$$\Delta(\psi) = div(grad\psi) = div\rho = -3(\beta_4 + \beta_1\frac{\kappa^2(p+\nu)}{f_R(R)}).$$

This finishes the proof.

In (1.6), if we put $\beta_1 = 1$ and $\beta_2 = 0$, then it yields η-Ricci soliton. Thus, we can write:

Corollary 2.8. *For $R =$ constant, let the perfect fluid-spacetime obeying $f(R)$-gravity permit a η-Ricci soliton. If the 1-form η is identical with the 1-form $\mathcal{D}$ of the perfect fluid-spacetime and $\mathcal{D}$ is the g-dual of the gradient vector field $\rho = grad(\psi)$, then the Poisson equation satisfies by ψ is*

$$\Delta(\psi) = -3(\beta_4 + \frac{\kappa^2(p+\nu)}{f_R(R)}).$$

In (1.6), if we put $\beta_1 = 0$ and $\beta_2 = 1$, then it recovers η-Yamabe soliton. Hence, we can write the subsequent:

Corollary 2.9. *For $R =$ constant, let the perfect fluid-spacetime satisfying $f(R)$-gravity permit a η-Yamabe soliton. If the 1-form η is identical with the 1-form $\mathcal{D}$ of the perfect fluid-spacetime and $\mathcal{D}$ is the g-dual of the gradient vector field $\rho = grad(\psi)$, then the Poisson equation satisfies by ψ is*

$$\Delta(\psi) = -3\beta_4.$$

In (1.6), if we put $\beta_1 = 1$ and $\beta_2 = -1$, then it gives η-Einstein soliton. Therefore, we can write the subsequent:

Corollary 2.10. *For a constant Ricci scalar, let the perfect fluid-spacetime satisfying $f(R)$-gravity permit an η-Einstein soliton. If the 1-form η is identical with the 1-form $\mathcal{D}$ of the perfect fluid-spacetime and $\mathcal{D}$ is the g-dual of the gradient vector field $\rho = grad(\psi)$, then the Poisson equation satisfies by ψ is*

$$\Delta(\psi) = -3(\beta_4 + \frac{\kappa^2(p+\nu)}{f_R(R)}).$$

Example 2.1. For a constant Ricci scalar, in a dark fluid fulfilling $f(R)$-gravity, the η-Ricci-Yamabe soliton $(g, \rho, \beta_1, \beta_2, \beta_3, \beta_4)$ is given by

$$\begin{cases} \beta_3 = -\frac{div\rho}{3} + (2\beta_2 - \beta_1)\frac{2\kappa^2 p + f(R)}{2f_R(R)} \\ \beta_4 = -\frac{div\rho}{3}. \end{cases}$$

Example 2.2. For a constant Ricci scalar, in a stiff fluid satisfying $f(R)$-gravity, the η-Ricci-Yamabe soliton $(g, \rho, \beta_1, \beta_2, \beta_3, \beta_4)$ is given by

$$\begin{cases} \beta_3 = -\frac{div\rho}{3} - \beta_2 \frac{\kappa^2 p}{f_R(R)} + (2\beta_2 - \beta_1)\frac{2\kappa^2 p + f(R)}{2f_R(R)} \\ \beta_4 = -\frac{div\rho}{3} - \beta_1 \frac{2\kappa^2 p}{f_R(R)}. \end{cases}$$

Example 2.3. For $R =$ constant, in a radiation fluid obeying $f(R)$-gravity, the η-Ricci-Yamabe soliton $(g, \rho, \beta_1, \beta_2, \beta_3, \beta_4)$ is given by

$$\begin{cases} \beta_3 = -\frac{div\rho}{3} - \beta_2 \frac{2\kappa^2 p}{f_R(R)} + (2\beta_2 - \beta_1)\frac{2\kappa^2 p + f(R)}{2f_R(R)} \\ \beta_4 = -\frac{div\rho}{3} - \beta_1 \frac{4\kappa^2 p}{f_R(R)}. \end{cases}$$

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Author's contributions

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Affiliations

KRISHNENDU DE
ADDRESS: Department of Mathematics, Kabi Sukanta Mahavidyalaya, The University of Burdwan. Bhadreswar, P.O.-Angus, Hooghly, Pin 712221, West Bengal, India.
E-MAIL: krishnendu.de@outlook.in
ORCID ID:https://orcid.org/0000-0001-6520-4520

UDAY CHAND DE
ADDRESS: Department of Pure Mathematics, University of Calcutta, West Bengal, India.
E-MAIL: uc_de@yahoo.com
ORCID ID:https://orcid.org/0000-0002-8990-4609